Chapter 10
Mathematical Models of Video-Sequences of Digital Half-Tone Images

E.P. Petrov
Vyatka State University, Russia

E.V. Medvedeva
Vyatka State University, Russia

I.S. Trubin
Vyatka State University, Russia

S.M. Smolskiy
Moscow Power Engineering Institute, Russia

ABSTRACT

This chapter is devoted to Mathematical Models (MM) of Digital Half-Tone Images (DHTI) and their video-sequences presented as causal multi-dimensional Markov Processes (MP) on discrete meshes. The difficulties of MM development for DHTI video-sequences of Markov type are shown. These difficulties are related to the enormous volume of computational operations required for their realization. The method of MM-DHTI construction and their statistically correlated video-sequences on the basis of the causal multi-dimensional multi-value MM is described in detail. Realization of such operations is not computationally intensive; Markov models from the second to fourth order demonstrate this. The proposed method is especially effective when DHTI is represented by low-bit (4-8 bits) binary numbers.

INTRODUCTION

As at this writing, the intensification of scientific research and increased complexity of solving scientific and technological problems require the investigation of not only one-dimensional random processes, but also the investigation of the multi-dimensional ones, for example, different types of fields presented in the form of images or video-sequences. Image processing is of great interest to researchers and engineers in various fields of practice for example: engineers in the area of flaw inspection and the non-destructive testing, developers of industrial robots and systems for the visual inspection of technological processes, experts in automation of scientific research, in TV technologies, in security systems, in remote sensing of natural resources, in space investiga-

DOI: 10.4018/978-1-4666-2208-1.ch010

tions, biologists, medical experts, specialists in forensic crime detection, physicists, astronomers, meteorologists, geologists, cartographers, and so forth (Bykov, 1971; Pisarevsky & Chernyavsky, 1988; Vasiliev, 1995; Ablameiko & Lagunovskiy, 2000; Berchtold, 1999; Vasiliev, 2002; Elfeki, 2001; Shalizi, 2003; Bondur, 2003). It is difficult to find a scientific or technological area, in which applied problems of image processing is not present in one form or the other.

The transition to digital image processing using small-bit numbers (4-8 bits) has sharply extended the possibilities of image application as the most capacious carrier of various types of information. In this connection, digital image processing, because of its importance, has been distinguished as an independent scientific and communication area, involving a great number of highly qualified experts. There is every reason to believe that in the nearest future, there will be a great extension of the practical implementation of image processing methods from Medicare to other various types of technological processes.

The development and investigation of image processing algorithms are based on mathematical models (MM), which adequately represent real images. To date, a variety of MM for two-dimensional images are already developed, on the basis of which whole series of effective processing algorithms offered has been reported in the literature by Jine (1981) as well as Derin and Kelly (1989). Most of these algorithms however require enormous computational resources. Approximation of digital half-tone images (DHTI) by random Markov processes (MP) allows for the achievement of significant progress in the area of MM development and algorithms of image processing. Important contributions in the development of Markov type MM have been introduced by Russian researchers like Berchtold (1999), Bondur (2003), Krasheninnikov (2003), Vasiliev (1995), Vasiukov (2002), Furman (2003), Soifer (2003) as well as other experts such as Jine (1981), Abend (1965), Woods (1972), Besag (1974), Kashyap (1981), Vinkler (2002), Modestino (1993), Politis (1994), Chellapa (1982, 1985). The most interest for practical application is generated by multi-dimensional mathematical models of DHTI video-sequences. The number of publications devoted to such MM are few. Notable among them are Bykov (1971), Vasiliev (1995, 2002), Jine (1981), Derin and Kelly (1989), Spector (1985), Dagion and Mercero (1988), Politis (1994), Petrov (2003), Trubin (2004a, 2004b), Trubin and Butorin (2005).

The MM of DHTI video-sequences based on the multi-dimensional discrete-time and continuous-values Markov process are the most studied by researchers like Vasiliev (1995), Spector (1985), Dagion and Mercero (1988). Two-dimensional MM of DHTI presented by Jine in Jine (1981) and constructed on the basis two-dimensional Gaussian Markov process was developed by Krasheninnikov, Vasiliev, and Spector in Krasheninnikov (2003), Vasiliev (1995), Spector (1985) up to multi-dimensional image MM based on the multi-dimensional Gaussian MP. The structure of the algorithm for generating these processes is rather simple and clear, however, the MM proposed in Jine (1981) based on the causal two-dimensional Gaussian MP has found the widest application (see Box 1).

To realize the MM of equation (1) it is necessary to use four multiplications and three additions, which is fully acceptable for medium sized images.

Krasheninnikov (2003), Vasiliev (1995), Spector (1985), suggested on the analogy of equation (1), MMs of processes of larger dimensions. Thus, for the description of the image frame sequence with two spatial coordinates defining the location of the image element in the frame and the third coordinate: the number of the frame or the discrete time in the frame sequence, the MM will be of the form shown in Box 2.

The computational effectiveness defined by the required computer memory usage and the number of computational operations is one of the most important features of MM. We should con-

Box 1.

$$\mu_{i,j} = r_1\mu_{i-1,j} + r_2\mu_{i,j-1} + r_1 r_2\mu_{i-1,j-1} - \sigma_\xi^2\sqrt{\left(1-r_1^2\right)\left(1-r_2^2\right)}\ \xi\left(i,j\right), \tag{1}$$

where μ_{ij} is the image element with spatial coordinates $\left(i \in m,\ j \in n\right)$;

r_1, r_2 are horizontal and vertical correlation coefficients respectively;

$\xi\left(i,j\right)$ is sample of white Gaussian noise with zero mean and the unit variance σ_ξ^2.

sider as most effective the MMs, in which the necessary number of required calculation operations per the image element does not depend on the image size. Most of the known MMs require a number of computational operations proportional to $\log N,\ N^2$ and even larger powers for realization of the image with sizes $N \times N$. For example, in spite of the simple MM structure for the multi-dimensional Gaussian processes offered in Spector (1985), the number of calculation operations at its realization quickly increases with the growth of the dimension of the generating process. For instance, to generate one element of the three-dimensional Gaussian MP (equation 2), which is adequate for the video-sequence of Gaussian Markov images, it is necessary to have seven multiplications and six additions, which makes application of the method of MM construction for processes with large number of measurements and elements for each measurement offered in Spector (1985) problematic.

It is envisaged that difficulties in MM development will significantly increase when required to develop and examine algorithms for the DHTI processing and statistically couple video-sequences, which represent random processes with more than two dimensions. Random processes become not only multi-dimensional but multi-valued as well, taking $Q = 2^g$ discrete values where g is the number of bits of DHTI elements presentation. Therefore, we devote the main attention to the MM of DHTI construction and their video-sequences required for realizing minimal computation resources.

The problem of MM construction for DHTI video-sequences on the basis of multi-dimensional and multi-valued random processes require non-traditional approach to its solution because of the great computational complexity. In Petrov and Chasikov (2001) and Petrov, Trubin and Butorin (2005a), the validity of multi-dimensional discrete-value MP selection as the MM of the

Box 2.

$$\begin{aligned}\mu_{i,j,k} = r_1\mu_{i-1,j,k} + r_2\mu_{i,j-1,k} + r_3\mu_{i,j,k-1} - r_1 r_2\mu_{i-1,j-1,k} - r_1 r_3\mu_{i-1,j,k-1} - \\ -r_2 r_3\mu_{i,j-1,k-1} + \sigma_\xi^2\sqrt{\prod_{l=1}^{3}\left(1-r_l^2\right)}\ \xi\left(i,j\right),\end{aligned} \tag{2}$$

where $r_i\left(i \in 3\right)$ are correlation coefficients between the image elements in horizontal, vertical and in time, accordingly; $\mu_{i,j,k}$ $\left(i \in m,\ j \in n, k = 1,2,...\right)$ is the image element with sizes $m \times n$ in the k-th frame. On the analogy of Equation (2) we can construct MM of higher orders as shown in Vasiliev (1995).

Box 10.

$$\alpha_1 = \pi_{iiii} = \pi(\nu_4 = M_1 \mid \nu_1 = M_1; \nu_2 = M_1; \nu_3 = M_1) = 1 - \frac{{}^1\pi_{ij} \; {}^2\pi_{ij}}{{}^3\pi_{ii}},$$
$$\alpha_2 = \pi_{iiji} = \pi(\nu_4 = M_1 \mid \nu_1 = M_1; \nu_2 = M_2; \nu_3 = M_1) = 1 - \frac{{}^1\pi_{ij} \; {}^2\pi_{ii}}{{}^3\pi_{ij}},$$
$$\alpha_3 = \pi_{ijii} = \pi(\nu_4 = M_1 \mid \nu_1 = M_2; \nu_2 = M_1; \nu_3 = M_1) = 1 - \frac{{}^1\pi_{ii} \; {}^2\pi_{ij}}{{}^3\pi_{ij}}, \tag{19}$$
$$\alpha_4 = \pi_{ijjj} = \pi(\nu_4 = M_2 \mid \nu_1 = M_2; \nu_2 = M_2; \nu_3 = M_2) = 1 - \frac{{}^1\pi_{ii} \; {}^2\pi_{ii}}{{}^3\pi_{ii}},$$

where ${}^3\pi_{ij}$ $(i, j = \overline{1,2};\ i \neq j)$ is the element of the matrix

$${}^3\Pi = {}^1\Pi \cdot {}^2\Pi = \begin{Vmatrix} {}^3\pi_{11} & {}^3\pi_{12} \\ {}^3\pi_{21} & {}^3\pi_{22} \end{Vmatrix}. \tag{20}$$

ν_4, we can write the transition probability matrix for the complicated Markov chain as shown in Box 9.

Elements of the matrix Π are connected with the elements of matrices (10), (11) by the equations shown in Box 10.

Elements of the matrix (18) satisfy the normalization requirement, i.e.

$$\alpha_l + \alpha_l' = 1, \quad l = \overline{1,4}. \tag{21}$$

For instance, elements α_1 and α_1' of the matrix, where

$$\alpha_1' = 1 - \alpha_1 = \frac{{}^1\pi_{ij} \; {}^2\pi_{ij}}{{}^3\pi_{ii}}, \tag{22}$$

are equal in sum by 1, i.e. $\alpha_1 + \alpha_1' = 1$.

Let us consider the most important issues of the MM operation for the random Markov type BBI confirming its adequacy to real images.

Let the transition probability matrices in horizontal and on vertical be specified and equal, i.e. ${}^1\pi_{ij} = {}^2\pi_{ij}$ $(i, j = \overline{1,2};\ i \neq j)$.

We assume ${}^1\pi_{ii} = {}^2\pi_{ii} = 0,9$. Then, in accordance with (19) we obtain that shown in Box 11.

Values of matrix Π elements $\alpha_3 = \alpha_4 = 0,5$ are the specific checking point for correctness of the MM operation. Really, at equal transition probabilities ${}^1\pi_{ii} = {}^2\pi_{ii}$ (in horizontal and in vertical) and for opposite values of elements ν_1 and ν_2 the appearance of the value M_1 or M_2 in the element ν_4 is equiprobable.

Let us calculate the matrix Π elements using formula (19) for the limit cases of matrices ${}^1\Pi$ and ${}^2\Pi$.

Let

$${}^1\Pi = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}, \quad {}^2\Pi = \begin{Vmatrix} {}^2\pi_{ii} & {}^2\pi_{ij} \\ {}^2\pi_{ji} & {}^2\pi_{jj} \end{Vmatrix}.$$

Box 6.

$$I(\nu_1,\nu_2,\nu_3,\nu_4) - I(\nu_1,\nu_2,\nu_3) = \log \frac{p(\nu_1,\nu_2,\nu_3)w(\nu_4 \mid \nu_1,\nu_2,\nu_3)}{p(\nu_1,\nu_2,\nu_3)p(\nu_4)} = \log \frac{w(\nu_4 \mid \nu_1,\nu_2,\nu_3)}{p(\nu_4)}, \tag{15}$$

where $w(\nu_4 \mid \nu_1,\nu_2,\nu_3)$ is the transition probability density in the three-dimensional Markov chain.

Box 7.

$$I(\nu_1,\nu_2,\nu_4) = \log \frac{w(\nu_4 \mid \nu_1,\nu_2,\nu_3)p(\nu_3)p(\nu_4)}{p(\nu_3)p(\nu_4)w(\nu_4 \mid \nu_3)} = \log \frac{w(\nu_4 \mid \nu_1,\nu_2,\nu_3)}{w(\nu_4 \mid \nu_3)} = \log \frac{w(\nu_4 \mid \nu_1)w(\nu_4 \mid \nu_2)}{w(\nu_4 \mid \nu_3)}. \tag{16}$$

Box 8.

$$w(\nu_4 \mid \Lambda_{ij}) = \sum_{j,q,r=1}^{2} \pi(\nu_4 = M_i \mid \nu_1 = M_j; \nu_2 = M_q; \nu_3 = M_r) \times \delta(\nu_1 - M_j)\delta(\nu_2 - M_q)\delta(\nu_3 - M_r), \tag{17}$$

where $\delta(\cdot)$ is the delta-function., $i = \overline{1,2}$.

Box 9.

$$\Pi = \begin{Vmatrix} \pi_{iiii} & \pi_{jiii} \\ \pi_{iiji} & \pi_{jiji} \\ \pi_{ijii} & \pi_{jjii} \\ \pi_{ijjj} & \pi_{jjjj} \end{Vmatrix} = \begin{Vmatrix} \alpha_1 & \alpha_1' \\ \alpha_2 & \alpha_2' \\ \alpha_3 & \alpha_3 \\ \alpha_4 & \alpha_4' \end{Vmatrix}, \quad i,j = \overline{1,2}; \quad i \neq j. \tag{18}$$

ν_1 and ν_2 i the vicinity $\Lambda_{i,j}$ and subtract it from (13) (see Box 6).

By definition, the BBI MM represents the superposition of two one-dimensional Markov chains, therefore, the information between elements ν_4 and ν_3 can be eliminated from (15). Then, the equation for the information quantity between the element ν_4 and elements ν_1, ν_2 takes the form (see Box 7).

The transition probability density in the complicated Markov chain $w(\nu_4 \mid \nu_1,\nu_2,\nu_3)$ can be expressed in the form (13) (see Box 8).

Using the entropy between mutually independent elements of the vicinity $\Lambda_{i,j}$ and the element

$$\rho(k,l) = \mathrm{E}\left[\mu_{i,j}, \mu_{i+k,j+l}\right] = \sigma^2 \exp\left\{-\alpha_1 |l| - \alpha_2 |k|\right\}, \qquad (8)$$

where $\mathrm{E}[\cdot]$ is the expected value; σ_μ^2 is the image signal variance; α_1, α_2 are multipliers depending upon the width of the power spectral density of the random processes on horizontal and on vertical. The fragment of the two-dimensional BBI corresponding to area F_4 of NSHP (Figure 1) is shown in Figure 2, where the following designation are taken:

$$\mathrm{v}_1 = \mu_{i,j-1};\ \mathrm{v}_2 = \mu_{i-1,j};\ \mathrm{v}_3 = \mu_{i-1,j-1};\ \mathrm{v}_4 = \mu_{i,j}. \qquad (9)$$

Dotted lines in Figure 2 indicate the presence of the statistical correlation between image elements.

We consider the two-dimensional Markov chain on the NSHP with two equiprobable $(p_1 = p_2)$ values of M_1, M_2 and probability matrices of the transition from the value M_i to the adjacent value M_j on image horizontal and vertical, accordingly, as the MM of Markov BBI:

$$^2\Pi = \begin{Vmatrix} ^2\pi_{11} & ^2\pi_{12} \\ ^2\pi_{21} & ^2\pi_{22} \end{Vmatrix}, \qquad (10)$$

$$^2\Pi = \begin{Vmatrix} ^2\pi_{11} & ^2\pi_{12} \\ ^2\pi_{21} & ^2\pi_{22} \end{Vmatrix}. \qquad (11)$$

If we know the correlation coefficients between BBI elements in lines r_{hor} and in columns r_{ver}, the matrix elements of the transition probability (10) can be obtained so:

$$^1\pi_{ii} = \frac{1 + r_{hor}}{2},$$

$$^2\pi_{ii} = \frac{1 + r_{ver}}{2} \qquad \pi_{ii} = 1 - \pi_{ij};\ i \neq j;\ (i,j). \qquad (12)$$

The probability of appearance of the BBI element ν_4 (Figure 2) with the value M_1 or M_2 completely defines by the mutual information quantity between ν_4 and its vicinity $\Lambda_{ij} = \{\nu_1, \nu_2, \nu_3\}$.

According to Dech (1971), let us present the information quantity containing BBI elements ν_1, ν_2, ν_3 with regards to the element ν_4 in the form.

$$I(\nu_1, \nu_2, \nu_3, \nu_4) = \log \frac{p(\nu_1, \nu_2, \nu_3, \nu_4)}{p(\nu_1)p(\nu_2)p(\nu_3)p(\nu_4)}, \qquad (13)$$

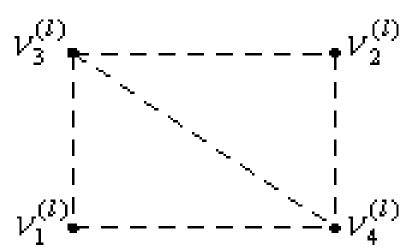
where $p(\nu_i)$, $i = \overline{1,4}$ are *a priori* probability densities values of BBI element; $p(\nu_1, \nu_2, \nu_3, \nu_4)$ is the mutual probability density of values of image element.

The quantity of mutual information between elements falling in the vicinity $\Lambda_{ij} = \{\nu_1, \nu_2, \nu_3\}$ can be written in the form on the analogy of (13):

$$I(\nu_1, \nu_2, \nu_3) = \log \frac{p(\nu_1, \nu_2, \nu_3)}{p(\nu_1)p(\nu_2)p(\nu_3)}. \qquad (14)$$

In the complicated Markov chain, as BBI is, all elements falling in the vicinity $\Lambda_{i,j}$ must be independent. For this, we shall find the mutual information between the element ν_3 with elements

Figure 2. The image fragment with the vicinity of three elements



The most useful property of the causal field is the opportunity to express the mutual distribution $\left\{\mu_{ql},(q,l)\in\Phi_{mn}\right\}$ in the form if the product of the causal conditional distributions as described in Derin and Kelly (1989) (see Box 4).

The property is similar to those of the one-dimensional Markov chains and allows for the construction of processing algorithm for two-dimensional signals on the analogy of the one-dimensional signals.

It should be noted that on the field boundary, i.e. for $i = 1$ or $j = 1$ the vicinity $\Lambda_{i,j}$ has a configuration different from that of the internal points. Since the values of elements lying over an upper line of more left of the initial column are unknown (or not defined), the vicinity for the boundary elements is taken in the form of the intersection of the general carrier with the existing mesh. Thus, for these elements, only '*abbreviated'* vicinities are obtained. In other words, in the field boundary elements the conditional probability $w\left(\mu_{i,j}\middle|\mu_{q,l};\left(q,l\in\Lambda_{i,j}\right)\right)$ is given as depending upon only those parts of $\Lambda_{i,j}$, that fall in $\Psi_{i,j}$.

Thus, and in line with Derin and Kelly (1989), all UMRF area with the vicinity of type (5) can be conditionally divided into four parts, each of which has its own view of $\Lambda_{i,j}$ (Figure 1) (see Box 5).

This circumstance will be further considered investigated for obtaining the algorithm of UMRF element formation.

It has been shown by Petrov (2003), Trubin (2004a, 2004b), Trubin and Butorin (2005) and Petrov et al. (2006a, 2006b) that DHTI representation by the set of *g* bit binary images reduces the problem of constructing mathematical models for DHTI to one of the creation of mathematical models of BBI. This represents the stationary two-dimensional Markov chain with two equiprobable values of M_1 and M_2.

MATHEMATICAL MODEL OF THE TWO-DIMENSIONAL BINARY MARKOV IMAGE

Let us specify the vicinity $\Lambda_{i,j}$ of the element ν_4 in the form given in expression (5) and let us assume that BBI represents the stationary field of the Markov type with the autocorrelation function:

Box 4.

$$p\left(\mu_{q,l};\left(q,l\right)\in\Phi_{m,n}\right)=\prod_{i=1}^{m}\prod_{j=1}^{n}w\left(\mu_{i,j}\middle|\mu_{q,l};\left(q,l\right)\in\Lambda_{i,j}\right) \quad (6)$$

Box 5.

$$\Lambda_{i,j}=\begin{cases}\varnothing, & if\ \ (i,j)\in F_1\\ \left\{(i-1,j)\right\}, & if\ \ (i,j)\in F_2\\ \left\{(i,j-1)\right\}, & if\ \ (i,j)\in F_3\\ \left\{(i,j-1),(i-1,j),(i-1,j-1)\right\}, & if\ \ (i,j)\in F_4\end{cases} \quad (7)$$

it from the communication channel, only earlier received image elements can fall in the aperture of this filter. Filtering can be executed repelling from the causally located element's variety only. The DHTI MM should therefore be in this case the causal random field as well. Based on the above-mentioned considerations, the unilateral Markov random field (UMRF) discussed in Jine (1981), Derin and Kelly (1989) were chosen as the digital half-tone image mathematical model.

We adopt the definition of the unilateral Markov field also called the two-dimensional Markov chain on the non-symmetric half-plane (NSHP) given in Derin and Kelly (1989):

Let $\mu = \left\{\mu_{i,j}\right\}$ be the random field specified on the rectangular mesh $L = \left\{\left(i,j\right): 1 \le i \le m, 1 \le j \le n\right\}$ with sizes $m \times n$ elements. Let us assume that:

$$\Phi_{i,j} = \left\{\left(q,l\right); 1 \le q \le i, 1 \le l \le j\right\}, \left(i,j\right) \in L,$$

$$\Psi_{i,j} = \left\{\left(q,\, l\right)\right\} \in L;\;\; q \le i \;\;\; or \;\;\; \left(q = i, l < j\right) \tag{3}$$

$$\Lambda_{i,j} \subset \Psi_{i,j}.$$

These subsets are shown in Figure 1.

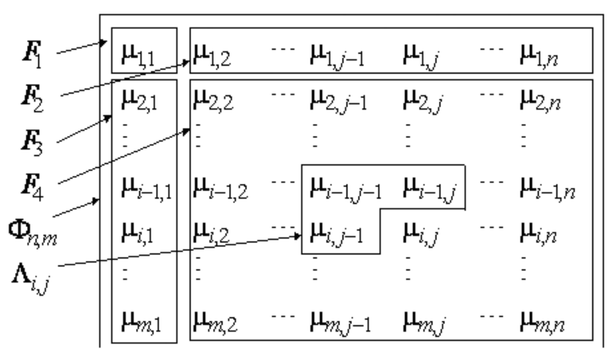
In order for μ to be the unilateral Markov random field (a.k.a. the Markov chain on the non-symmetric half-plane) it is necessary to fulfill the condition shown in Box 3.

The main property of the UMRF is that, if the conditional dependence is defined starting from the upper left fragment, then $\mu_{i,j}$ depends on the random variables only from some subset $\Lambda_{i,j}$ of this fragment; this subset is called a neighborhood.

The key UMRF property consists in the fact that if the conditional function is defined from the left upper segment, value of $\mu_{i,j}$ depends upon the random variables from some subset $\Lambda_{i,j}$ of this segment called the vicinity. The vicinity $\Lambda_{i,j}$ may be any subset $\Psi_{i,j}$, but it usually has a fixed configuration with respect to $\mu_{i,j}$. The following vicinity configuration offered in Derin and Kelly (1989) best satisfies the causality condition:

$$\Lambda_{i,j} = \left\{\mu_{i,j-1}, \mu_{i-1,j}, \mu_{i-1,j-1}\right\}. \tag{5}$$

Figure 1. OSMRF areas with the vicinity of three elements



Box 3.

$$w\left(\mu_{i,j} \middle| \mu_{q,l}; \left(q,l\right) \in \Psi_{i,j}\right) = w\left(\mu_{i,j} \middle| \mu_{q,l}; \left(q,l\right) \in \Lambda_{i,j}\right). \tag{4}$$

Box 11.

$$\alpha_1 = \pi(\nu_4 = M_1 \mid \nu_1 = M_1; \nu_2 = M_1; \nu_3 = M_1) = 1 - \frac{{}^1\pi_{ij} \; {}^2\pi_{ij}}{{}^3\pi_{ii}} = 1 - \frac{0,1 \cdot 0,1}{0,82} = 0,9878$$

$$\alpha_2 = \pi(\nu_4 = M_1 \mid \nu_1 = M_1; \nu_2 = M_2; \nu_3 = M_1) = 1 - \frac{{}^1\pi_{ij} \; {}^2\pi_{ii}}{{}^3\pi_{ij}} = 1 - \frac{0,1 \cdot 0,9}{0,18} = 0,5$$

$$\alpha_3 = \pi(\nu_4 = M_1 \mid \nu_1 = M_2; \nu_2 = M_1; \nu_3 = M_1) = 1 - \frac{{}^1\pi_{ii} \; {}^2\pi_{ij}}{{}^3\pi_{ij}} = 1 - \frac{0,9 \cdot 0,1}{0,18} = 0,5$$

$$\alpha_4 = \pi(\nu_4 = M_2 \mid \nu_1 = M_2; \nu_2 = M_2; \nu_3 = M_2) = 1 - \frac{{}^1\pi_{ij} \; {}^2\pi_{ij}}{{}^3\pi_{ii}} = 1 - \frac{0,1 \cdot 0,1}{0,82} = 0,9878.$$

Then

$$\pi_{iiii} = 1 - \frac{{}^1\pi_{ij} \cdot {}^2\pi_{ij}}{{}^3\pi_{ii}} = 1 - 0 = 1\,, \quad i \neq j.$$

If

$${}^1\Pi = \begin{Vmatrix} 0,5 & 0,5 \\ 0,5 & 0,5 \end{Vmatrix}, \quad {}^2\Pi = \begin{Vmatrix} {}^2\pi_{ii} & {}^2\pi_{ij} \\ {}^2\pi_{ji} & {}^2\pi_{jj} \end{Vmatrix},$$

then

$$\pi_{iiii} = 1 - \frac{{}^1\pi_{ij} \cdot {}^2\pi_{ij}}{{}^3\pi_{ii}} = 1 - \frac{0,5 \cdot {}^2\pi_{ij}}{0,5} = {}^2\pi_{ii}\,, i \neq j\,.$$

Let us check the normalization requirement for the matrix Π on the example of the first line. For this we calculate the element located in the matrix right column (see Box 12).

Let us sum the values of the first line elements

$$\pi_{iiii} + \pi_{jiii} = 0,9878 + 0,0122 = 1, \quad i \neq j.$$

As we see, the normalization requirements are fulfilled and for other lines as well.

BBI MATHEMATICAL MODELS WITH VICINITY OF FOUR ELEMENTS

Let the vicinity of element ν_4 consist of four BBI elements (Figure 3) located at the upper-left, according to Krasheninnikov (2003). At that, the condition of the strict causality peculiar to the vicinity of (5) type is something disturbed but this disturbance is not critical as to define the causal properties or UMRF it is not required any addi-

Box 12.

$$\alpha_1' = \pi(\nu_4 = M_2 \mid \nu_1 = M_1; \nu_2 = M_1; \nu_3 = M_1) = \frac{{}^1\pi_{ij} \; {}^2\pi_{ij}}{{}^3\pi_{ii}} = \frac{0,1 \cdot 0,1}{0,82} = 0,0122.$$

tional element sets besides the left segment Φ_{ij} or accordingly NSHP Ψ_{ij} (Figure 1).

The probability of appearance of the BBI element ν_4 with the value M_1 or M_2 is completely defined by the mutual information quantity between elements of the vicinity

$$\Lambda'_{i,j} = \{v_1, v_2, v_3, v_5\}, \tag{23}$$

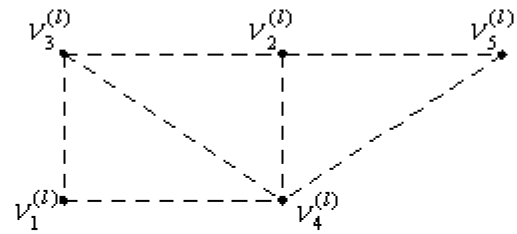
and the image element ν_4 (Figure 3).

The information quantity between Λ'_{ij} and the element ν_4 can be determined as shown in Box 13.

At mutual independence of elements of the vicinity $\Lambda'_{i,j}$ the information quantity between the element ν_4 and $\Lambda'_{i,j}$ can be determined on the analogy of (15) (see Box 14).

Because of v_4 is the element of the two-dimensional Markov chain, the information between

Figure 3. The image fragment with the vicinity of four elements



elements v_3, v_4 and v_5, v_4 is redundant disturbing the information balance between elements v_1, v_2 and the element v_4.

Let us subtract the redundant information caused by elements ν_3 and ν_5 from equation (25) (see Box 15).

Having compared (26) and (16), we can conclude that the use of the vicinity (5) or (23) does not change the probability of the element ν_4 value.

Box 13.

$$I(v_1,v_2,v_3,v_4,v_5) = \log \frac{p(v_1,v_2,v_3,v_4,v_5)}{p(v_1)\,p(v_2)\,p(v_3)\,p(v_4)\,p(v_5)}, \tag{24}$$

where $p(v_i),\ i = \overline{1,5}$ are *a priori* probability densities of values of image elements; $p(v_1,v_2,v_3,v_4,v_5)$ is the mutual probability density.

Box 14.

$$\begin{aligned} I'(v_1,v_2,v_3,v_4,v_5) &= I(v_1,v_2,v_3,v_4,v_5) - I(v_1,v_2,v_3,v_5) \\ &= \log \frac{p(v_1,v_2,v_3,v_5)\,w(v_4 \mid v_1,v_2,v_3,v_5)}{p(v_1,v_2,v_3,v_5)\,p(v_4)} = \log \frac{w(v_4 \mid v_1,v_2,v_3,v_5)}{p(v_4)}, \end{aligned} \tag{25}$$

where $w(v_4 \mid v_1,v_2,v_3,v_5)$ is the transition probability density in the complicated Markov chain.

Box 15.

$$I\left(v_1, v_2, v_4\right) = \log \frac{w\left(\mathrm{v}_4 \left|\mathrm{v}_1,\mathrm{v}_2,\mathrm{v}_3,\mathrm{v}_5\right.\right) p\left(\mathrm{v}_3\right) p\left(\mathrm{v}_4\right) p\left(\mathrm{v}_5\right)}{p\left(v_3, v_4, v_5\right) p\left(v_4\right)}$$
$$= \log \frac{w\left(\mathrm{v}_4 \left|\mathrm{v}_1,\mathrm{v}_2,\mathrm{v}_3,\mathrm{v}_5\right.\right) p\left(\mathrm{v}_3\right) p\left(\mathrm{v}_5\right)}{p\left(\mathrm{v}_3,\mathrm{v}_5\right) w\left(\mathrm{v}_4 \left|\mathrm{v}_3,\mathrm{v}_5\right.\right)} = \log \frac{w\left(\mathrm{v}_4 \left|\mathrm{v}_1,\mathrm{v}_2,\mathrm{v}_3,\mathrm{v}_5\right.\right)}{w\left(\mathrm{v}_4 \left|\mathrm{v}_3\right.\right) w\left(\mathrm{v}_4 \left|\mathrm{v}_5\right.\right)} \tag{26}$$
$$= \log \frac{w\left(\mathrm{v}_4 \left|\mathrm{v}_1,\mathrm{v}_2,\mathrm{v}_3\right.\right) w\left(\mathrm{v}_4 \left|\mathrm{v}_5\right.\right)}{w\left(\mathrm{v}_4 \left|\mathrm{v}_3\right.\right) w\left(\mathrm{v}_4 \left|\mathrm{v}_5\right.\right)} = \log \frac{w\left(\mathrm{v}_4 \left|\mathrm{v}_1,\mathrm{v}_2,\mathrm{v}_3\right.\right)}{w\left(\mathrm{v}_4 \left|\mathrm{v}_3\right.\right)}.$$

THE ALGORITHM OF MARKOV BBI FORMATION

To construct the artificial BBI representing the two-dimensional Markov chain with two equiprobable values it is necessary to have the *a priori* known matrices of one-step transition probabilities (13) and the vector of initial probabilities of values $\mathbf{P} = \left[p_1, p_2\right]; \left(p_1 = p_2\right)$.

The BBI modeling includes several stages. The first line of BBI is modeling (areas F_1, F_2 in Figure 1) as thew one-dimensional stationary Markov chain with two equiprobable values and the given matrix ${}^1\Pi$. The length of state's sequences of Markov chain is equal to the length of the line m. The modeling of BBI elements of the area F_3 (Figure 1) is similar to those for elements of the first line.

The modeling of the area F_4 (Figure 1) is the most complicated and it consists in the following.

1. Matrices ${}^3\Pi$ and Π are calculated in the basis of known matrices ${}^1\Pi$ and ${}^2\Pi$;
2. We take an arbitrary number $\xi_l (l \le m \cdot n)$, which is equally distributed on the interval $\left[0, 1\right]$;
3. From the first column of the matrix Π we select the element $\alpha_s (s = \overline{1, 4})$ corresponding to element values of the vicinity Λ_{ij};
4. The number ξ_l is compared to the selected element $\alpha_s (s = \overline{1, 4})$. If $\alpha_s (s = \overline{1, 4})$ and $\xi_l \le \alpha_s$, the image element ν_4 takes the value $M_1 = 0$, otherwise $M_2 = 1$;
5. If $l \le m \cdot n$, we transit to p. 2, otherwise – to point 6;
6. Stop.

To check the correctness of the model operation let us consider the process of the image formation for the extreme cases of the Markov chain along one of coordinates, when matrices ${}^1\Pi$ or ${}^2\Pi$ become either the unitary matrices or all elements in matrices are equal to 0.5.

Let the BBI probability matrix of transitions in horizontal be an unitary matrix:

$${}^1\Pi = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}. \tag{27}$$

For the probability matrix of transition in vertical ${}^2\Pi$ the following condition acts:

$${}^2\Pi \ne {}^1\Pi. \tag{28}$$

The BBI with sizes 256×256 obtained in accordance of the above-described algorithm and the matrix (27) is presented in Figure 4a. The BBI with the unitary matrix ${}^2\Pi$ is shown in Figure

Box 16.

$$r_{l,q,s} = \mathrm{E}\left[\mu_{i,j,t}, \mu_{i+l,j+q,t+s}\right] = \sigma_\mu^2 \exp\left\{-\alpha_1 \left|l\right| - \alpha_2 \left|q\right| - \alpha_3 \left|s\right|\right\}, \quad (29)$$

where $\mathrm{E}\left[\cdot\right]$ has the sense of the mathematical expectation; σ_μ^2 is the image signal variance; $\alpha_1, \alpha_2, \alpha_3$ are coefficients similar to (8). In accordance with (29), the image sequence can be presented as the superposition of three one-dimensional discrete-valued MP with two equiprobable $\left(p_1 = p_2\right)$ values $M_1,\ M_2$ and transition probability matrices from one value to another inside the image frame (10) and between adjacent frames ${}^4\Pi$

$${}^1\Pi = \begin{Vmatrix} {}^1\pi_{11} & {}^1\pi_{12} \\ {}^1\pi_{21} & {}^1\pi_{22} \end{Vmatrix},\ {}^2\Pi = \begin{Vmatrix} {}^2\pi_{11} & {}^2\pi_{12} \\ {}^2\pi_{21} & {}^2\pi_{22} \end{Vmatrix},\ {}^4\Pi = \begin{Vmatrix} {}^4\pi_{11} & {}^4\pi_{12} \\ {}^4\pi_{21} & {}^4\pi_{22} \end{Vmatrix} \quad (30)$$

accordingly.

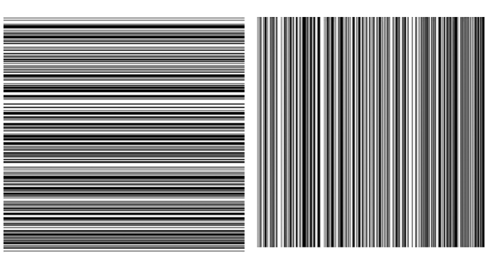
Box 17.

$$\Lambda_{i,j,k} = \begin{cases} \varnothing, & \mathit{if}\ \ (i,j) \in F_{1k} \\ \left\{(i-1,j,k)\right\}, & \mathit{if}\ \ (i,j) \in F_{2k} \\ \left\{(i,j-1),k\right\}, & \mathit{if}\ \ (i,j) \in F_{3k} \\ \left\{(i,j-1,k),(i-1,j,k),(i-1,j-1,k)\right\}, & \mathit{if}\ \ (i,j) \in F_{4k} \end{cases} \quad (32)$$

4b. Figures 5a,b show the binary artificial images with sizes 256x256 obtained for ${}^1\pi_{ii} = {}^2\pi_{ii} = 0,5;\ {}^1\pi_{ii} = {}^2\pi_{ii} = 0,9$ with estimates of the transition probabilities ${}^1\hat{\pi}_{ii}$ и ${}^2\hat{\pi}_{ii},\ i = \overline{1,2}$ calculated on the basis of artificial images and the two-dimensional auto-correlation function.

Estimates ${}^r\hat{\pi}_{ii}\left(r = \overline{1,2}\right)$ of elements of the transfer probabilities if artificial images coincide with the high accuracy (less than 0.2%) with the given values ${}^r\hat{\pi}_{ii} = 0,9$ on the statistic with sizes 512×512.

Figure 4. The artificial BBI with unitary matrices ${}^1\Pi$ and ${}^2\Pi$ a) ${}^1\pi_{ii} = 1,\ {}^2\pi_{ii} = 0,5$; b) ${}^1\pi_{ii} = 0,5,\ {}^2\pi_{ii} = 1$



(a) (b)

THE MATHEMATICAL MODEL OF THE DHTI

The mathematical model of DHTI represented by g-bit binary numbers is formed by the simple bitwise "summation" of g binary images in the register of the binary number. The factual summation is absent as the own bit position in the register with the appropriate weight corresponds to each BBI. It should be noted that proper transi-

Figure 5. The binary artificial images with sizes 256x256 and the two-dimensional auto-correlation function a) ${}^{1}\pi_{ii} = {}^{2}\pi_{ii} = 0,5$; *b)* ${}^{1}\pi_{ii} = {}^{2}\pi_{ii} = 0,9$

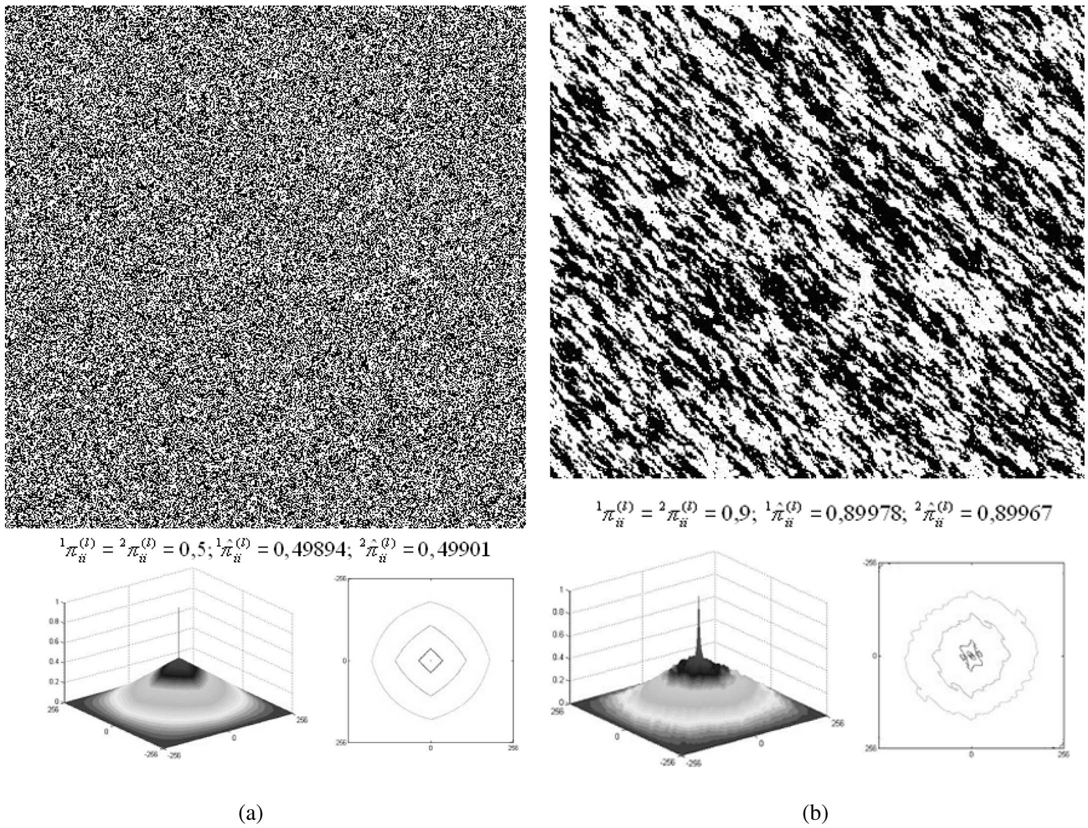
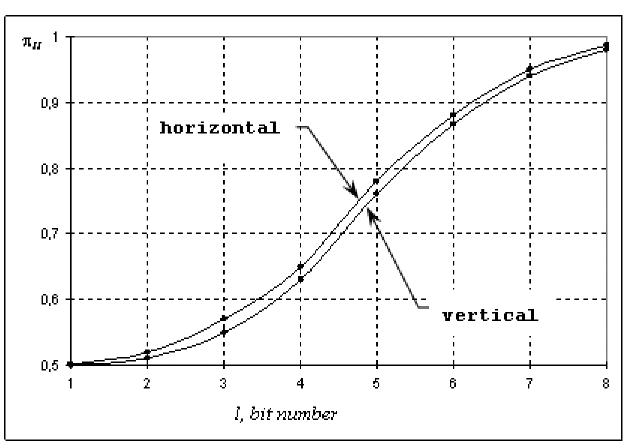


Figure 6. Graphs of a value variation of the transition matrices' (10), (11) elements of DHTI averaged over a great number of the real DHTI



The random Markov process represents a superposition of four one-dimensional Markov chains with two states.

Let us construct the mathematical model for the l-th bit $\left(p_1^{(l)} = p_2^{(l)}\right)$ of DHTI based on the four-dimensional stationary Markov chain with two equiprobable states M_1, M_2 and the matrices of one-step transition probabilities from one value to another inside the frame ${}^1\Pi^{(l)}$, ${}^2\Pi^{(l)}$, from frame to frame ${}^4\Pi^{(l)}$ and from position to position ${}^8\Pi^{(l)}$, accordingly:

$$\begin{aligned} {}^1\Pi &= \begin{Vmatrix} {}^1\pi_{11} & {}^1\pi_{12} \\ {}^1\pi_{21} & {}^1\pi_{22} \end{Vmatrix}, \ {}^2\Pi = \begin{Vmatrix} {}^2\pi_{11} & {}^2\pi_{12} \\ {}^2\pi_{21} & {}^2\pi_{22} \end{Vmatrix}, \\ {}^4\Pi &= \begin{Vmatrix} {}^4\pi_{11} & {}^4\pi_{12} \\ {}^4\pi_{21} & {}^4\pi_{22} \end{Vmatrix}, {}^8\Pi = \begin{Vmatrix} {}^8\pi_{11} & {}^8\pi_{12} \\ {}^8\pi_{21} & {}^8\pi_{22} \end{Vmatrix}. \end{aligned} \tag{45}$$

Let the random Markov process $\mu_{ijkd}^{(l)}$ being the process of l-th $\left(l \in g\right)$ binary bit of DHTI in k-th frame and in position d represent a superposition of four one-dimensional binary Markov processes. We take as the BBI mathematical model in k-th frame in position d, the UMRF in NSHP with the vicinity of type Figure 12 (see Box 26).

Consider the case when the BBI element ν_4 in $k-$th frame in position d belonging to $F_{4,k,d}$ area^w (Figure 12) is the subject for modeling. Modeling of the BBI elements belonging to areas $F_{1,k,d}$, $F_{2,k,d}$ and $F_{3,k,d}$ is simpler than the area $F_{4,k,d}$ and reduces to modeling of one-dimensional, two-dimensional and three-dimensional stationary Markov chains.

Fragments of mathematical model of two statistically correlated sequences (Figure 13) for two adjacent frames and two adjacent positions in the space are presented in Figure 14.

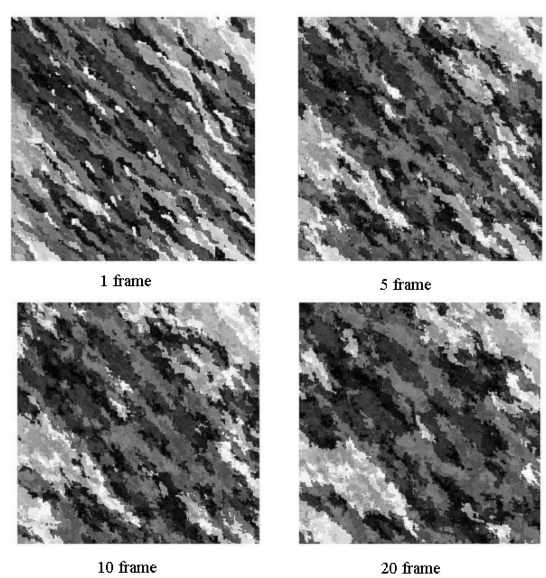
BBI elements in position d (Figure 14) will be designated as: $\nu_1 = \mu_{i,j-1,k,d}$, $\nu_2 = \mu_{i-1,j,k,d}$, $\nu_3 = \mu_{i-1,j-1,k,d}$, $\nu_4 = \mu_{i,j,k,d}$, $\nu_1' = \mu_{i,j-1,k-1,d}$, $\nu_2' = \mu_{i-1,j,k-1,d}$, $\nu_3' = \mu_{i-1,j-1,k-1,d}$, $\nu_4' = \mu_{i,j,k-1,d}$, an as $\varepsilon_1 = \mu_{i,j-1,k,d-1}$, $\varepsilon_2 = \mu_{i-1,j,k,d-1}$, $\varepsilon_3 = \mu_{i-1,j-1,k,d-1}$, $\varepsilon_4 = \mu_{i,j,k,d-1}$, $\varepsilon_1' = \mu_{i,j-1,k-1,d-1}$, $\varepsilon_2' = \mu_{i-1,j,k-1,d-1}$, $\varepsilon_3' = \mu_{i-1,j-1,k-1,d-1}$, $\varepsilon_4' = \mu_{i,j,k-1,d-1}$ are the image elements in position $d-1$.

The vicinity of BBI element ν_4 in position d has 15 adjacent image elements (see Box 27).

The quantity of information contained in elements of the vicinity (48) with regard to element ν_4 without taking into account the statistical correlation between elements of the Λ_{ijkd} vicinity can be represented similar to expression (35) in the form shown in Box 28.

The modeling process is the four-dimensional Markov chain, therefore, the information quantity defining the appearance of this or that value

Figure 12. Frames of the video-sequence of the artificial BBI



information containing in BBI elements $\nu_3, \nu'_1, \nu'_2, \nu'_3$ with regard to the element ν_4 is redundant and we must eliminate it. In this case the equation for information between BBI elements ν_4 and ν_1, ν_2, ν'_4 will take the form shown in Box 21.

Having eliminated information about the element ν_3 in elements ν_3, ν'_1, ν'_2 in the conditional probability density $w\left(\nu_4 \left| \nu'_1, \nu'_2, \nu_3, \nu'_3\right.\right)$, equation (36) can be presented as shown in Box 22.

The transition probability density for the complicated Markov chain, which can be approximated the BBI sequence is completely defined by the transition probability matrix Π, which elements have the form shown in Box 23.

For known matrices ${}^1\Pi, {}^2\Pi, {}^4\Pi,$ in order to calculate the matrix Π elements, it is necessary to calculate preliminarily matrices

$$\begin{aligned} &{}^3\Pi = {}^1\Pi \cdot {}^2\Pi; \\ &{}^5\Pi = {}^1\Pi \cdot {}^4\Pi; \\ &{}^6\Pi = {}^2\Pi \cdot {}^4\Pi; \\ &{}^7\Pi = {}^3\Pi \cdot {}^4\Pi = {}^1\Pi \cdot {}^2\Pi \cdot {}^4\Pi. \end{aligned} \tag{39}$$

Using the entropy approach to statistically connected in-pairs the elements of the BBI video-sequence, we can rewrite the transition probability matrix for the complicated Markov chain in the form shown in Box 24.

Matrix Π element values (36) can be calculated in accordance with (35). For example, expressions for calculation of elements of the first line of the matrix Π have the form:

$$\begin{aligned} \pi_{iiiiiii} &= 1 - \frac{{}^1\pi_{ij} \cdot {}^2\pi_{ij} \cdot {}^4\pi_{ij} \cdot {}^7\pi_{ij}}{{}^3\pi_{ii} \cdot {}^5\pi_{ii} \cdot {}^6\pi_{ii}}, \\ \pi_{jiiiiii} &= \frac{{}^1\pi_{ij} \cdot {}^2\pi_{ij} \cdot {}^4\pi_{ij} \cdot {}^7\pi_{ij}}{{}^3\pi_{ii} \cdot {}^5\pi_{ii} \cdot {}^6\pi_{ii}}, i \neq j. \end{aligned} \tag{41}$$

The determination of other elements of the matrix Π is executed in accordance with the vicinity $\Lambda_{i,j,k}$ element values. For instance, elements of the second line can be calculated as:

$$\begin{aligned} \pi_{iijiiii} &= 1 - \frac{{}^1\pi_{ij} \cdot {}^2\pi_{ij} \cdot {}^4\pi_{ii} \cdot {}^7\pi_{ii}}{{}^3\pi_{ii} \cdot {}^5\pi_{ij} \cdot {}^6\pi_{ij}}, \\ \pi_{jiijiiii} &= \frac{{}^1\pi_{ij} \cdot {}^2\pi_{ij} \cdot {}^4\pi_{ii} \cdot {}^7\pi_{ii}}{{}^3\pi_{ii} \cdot {}^5\pi_{ij} \cdot {}^6\pi_{ij}}, i \neq j. \end{aligned} \tag{42}$$

Box 24.

$$\Pi = \left\| \begin{matrix} \pi_{iiiiiiii} & \pi_{jiiiiiii} \\ \pi_{iiijiiii} & \pi_{jiijiiii} \\ \pi_{iijiiiii} & \pi_{jijiiiii} \\ \pi_{iijjiiii} & \pi_{jijjiiii} \\ \pi_{ijiiiiii} & \pi_{jjiiiiii} \\ \pi_{ijijiiii} & \pi_{jjijiiii} \\ \pi_{ijjiiiii} & \pi_{jjjiiiii} \\ \pi_{ijjjiiii} & \pi_{jjjjiiii} \end{matrix} \right\| = \left\| \begin{matrix} \alpha_1 & \alpha'_1 \\ \alpha_2 & \alpha'_2 \\ \alpha_3 & \alpha'_3 \\ \alpha_4 & \alpha'_4 \\ \alpha_5 & \alpha'_5 \\ \alpha_6 & \alpha'_6 \\ \alpha_7 & \alpha'_7 \\ \alpha_8 & \alpha'_8 \end{matrix} \right\|, \quad i, j = \overline{1, 2}; \quad i \neq j. \tag{40}$$

Box 20.

$$I\left(\nu_1,\nu_2,\nu_3,\nu_4,\nu_1',\nu_2',\nu_3',\nu_4'\right)-I\left(\nu_1,\nu_2,\nu_3,\nu_1',\nu_2',\nu_3',\nu_4'\right)$$
$$=\log\frac{p(\nu_1,\nu_2,\nu_3,\nu_1',\nu_2',\nu_3',\nu_4')\cdot w(\nu_4\mid\nu_1,\nu_2,\nu_3,\nu_1',\nu_2',\nu_3',\nu_4')}{p(\nu_4)\cdot p(\nu_1,\nu_2,\nu_3,\nu_1',\nu_2',\nu_3',\nu_4')} \tag{35}$$
$$=\log\frac{w(\nu_4\mid\nu_1,\nu_2,\nu_3,\nu_1',\nu_2',\nu_3',\nu_4')}{p(\nu_4)}.$$

Box 21.

$$I\left(\nu_1,\nu_2,\nu_4',\nu_4\right)$$
$$=\log\frac{w(\nu_4\mid\nu_1,\nu_2,\nu_3,\nu_1',\nu_2',\nu_3',\nu_4')\cdot p(\nu_1')\cdot p(\nu_2')\cdot p(\nu_3')\cdot p(\nu_3)\cdot p(\nu_4)}{p(\nu_4)\cdot p(\nu_1',\nu_2',\nu_3',\nu_3,\nu_4)} \tag{36}$$
$$=\log\frac{w(\nu_4\mid\nu_1,\nu_2,\nu_3,\nu_1',\nu_2',\nu_3',\nu_4')}{w(\nu_4\mid\nu_1',\nu_2',\nu_3',\nu_3)}.$$

Box 22.

$$I\left(\nu_1,\nu_2,\nu_4',\nu_4\right)$$
$$=\log\frac{w(\nu_4\mid\nu_1)\cdot w(\nu_4\mid\nu_4')\cdot w(\nu_4\mid v_2)\cdot w(\nu_4\mid\nu_3')}{w(\nu_4\mid\nu_3)\cdot w(\nu_4\mid\nu_1')\cdot w(\nu_4\mid\nu_2')}. \tag{37}$$

Box 23.

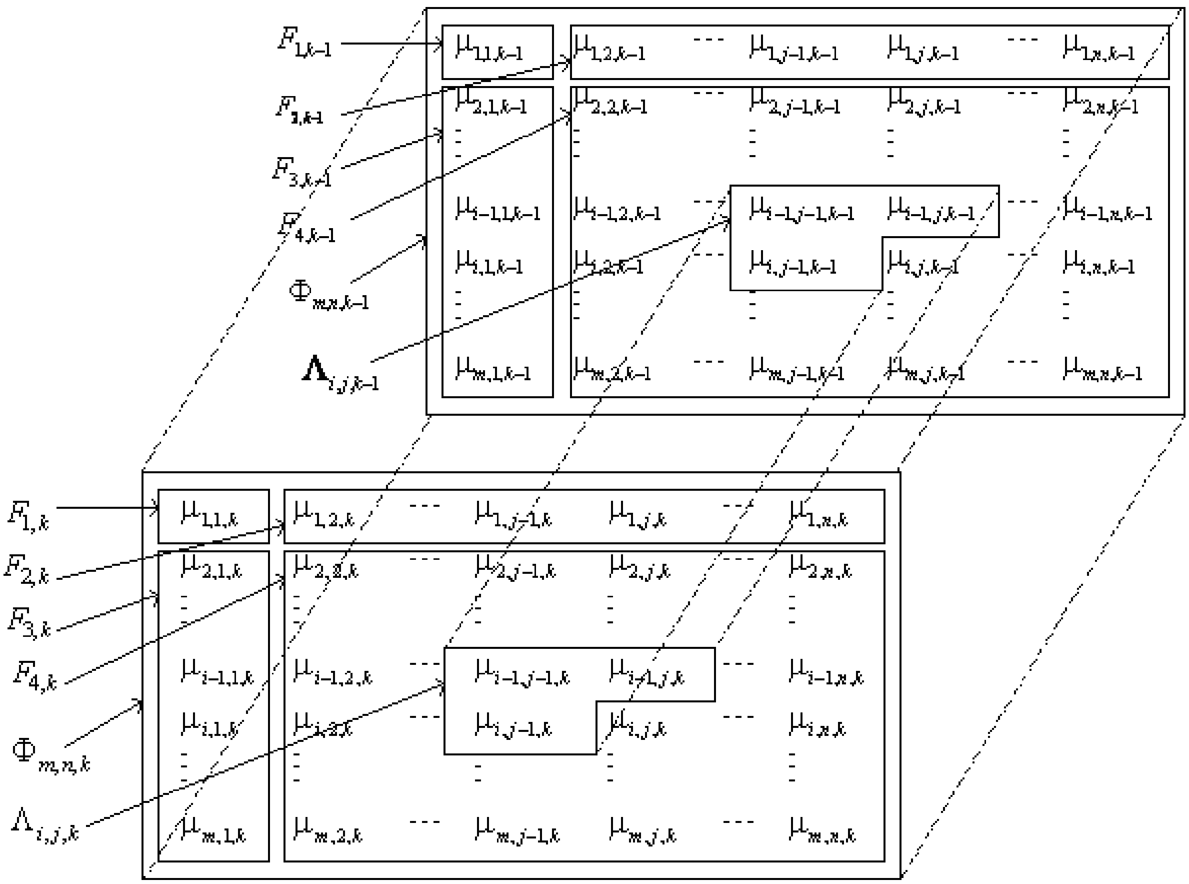
$$\pi_{ijklmnqr}=\pi(\nu_4=M_i\mid\nu_1=M_j,\nu_2=M_k;\nu_3=M_l;\nu_1'=M_m;$$
$$\nu_2'=M_n;\nu_3'=M_q;\nu_4'=M_r),i,j,k,l,m,n,q,r=\overline{1,2}. \tag{38}$$

The information containing between BBI elements falling inside the vicinity $\Lambda_{i,j,k}$ can be calculated similar to (33) (see Box 19).

In the complicated Markov chain representing the video-sequence the image elements inside the vicinity Λ_{ijk} should be independent. To fulfill this condition we shall obtain the difference of expressions (33) and (34) (see Box 20).

By the data, the BBI sequence represents three-dimensional discrete-valued Markov chain formed by the superposition of three one-dimensional independent Markov chains. Therefore, the

Figure 10. The OSMRF area with the vicinity of seven elements



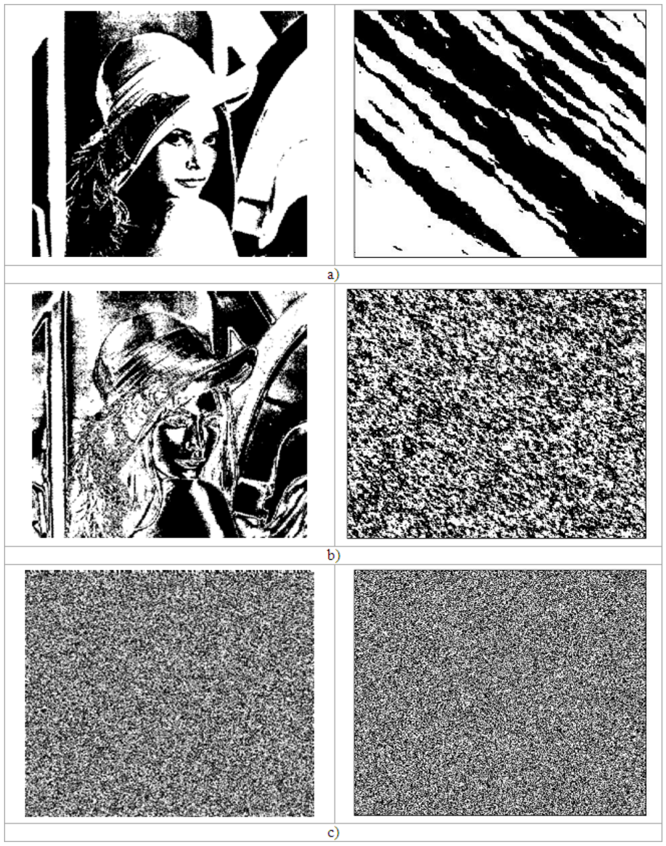
Box 18.

$$I\left(\nu_1,\nu_2,\nu_3,\nu_4,\nu_1',\nu_2',\nu_3',\nu_4'\right) = \log \frac{p(\nu_1,\nu_2,\nu_3,\nu_4,\nu_1',\nu_2',\nu_3',\nu_4')}{\prod_{i=1}^{4} p(\nu_i)\cdot\prod_{i=1}^{4} p(\nu_i')}. \tag{33}$$

Box 19.

$$I\left(\nu_1,\nu_2,\nu_3,\nu_1',\nu_2',\nu_3',\nu_4'\right) = \log \frac{p(\nu_1,\nu_2,\nu_3,\nu_1',\nu_2',\nu_3',\nu_4')}{\prod_{i=1}^{3} p\left(\nu_i\right)\prod_{i=1}^{4} p\left(\nu_i'\right)}. \tag{34}$$

Figure 9. The BBI of the real and artificial DHTI "Lena": a) 8-bit; b) 6-bit; c) 1-bit



The vicinity of the image element ν_4 at modeling of the video-sequence will be increased up to seven adjacent elements

$\Lambda_{i,j,k} = \left\{ \nu_1;\ \nu_2;\ \nu_3;\ \nu'_1;\ \nu'_2;\ \nu'_3,\ \nu'_4 \right\}$ (Figure 10).

The following designations are used in Figure 11:

$$\nu_1 = \mu\left(i, j-1, k\right),$$
$$\nu_2 = \mu\left(i-1, j, k\right),$$
$$\nu_3 = \mu\left(i-1, j-1, k\right),$$
$$\nu_4 = \mu\left(i, j, k\right),$$

$$\nu'_1 = \mu\left(i, j-1, k-1\right),$$
$$\nu'_2 = \mu\left(i-1, j, k-1\right),$$
$$\nu'_3 = \mu\left(i-1, j-1, k-1\right),$$
$$\nu'_4 = \mu\left(i, j, k-1\right).$$

The statistical connections between BBI elements including inside the vicinity $\Lambda_{i,j,k}$ of the element ν_4 are shown by the firm and dotted lines in Figure 11.

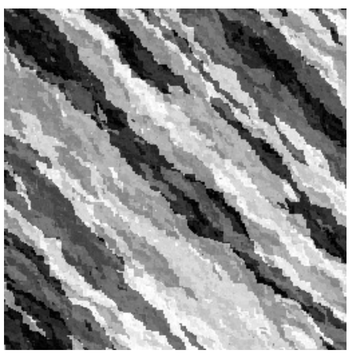
The information quantity containing in elements of the vicinity

$$\Lambda_{ijk} = \left\{\nu_1, \nu_2, \nu_3, \nu_4, \nu'_1, \nu'_2, \nu'_3, \nu'_4\right\}$$

with regards to the element ν_4, can be determine from equation of the view shown in Box 18.

Figure 7. The real DHTI "Lena"

Figure 8. The artificial DHTI obtained with the help of MM at statistical characteristics of the real DHTI "Lena"



tion matrices (10), (11) satisfying the condition (16) correspond to the each l-th $(l \in g)$ binary image. At constructing the DHTI MM adequate to the real situation, it is necessary to know values of transition matrix elements for each BBI. The graphs of value variation of transition matrix (10), (11) elements of DHTI averaged over the great number of the real DHTI, are presented in Figure 6. These graphs concern to presentation in the form of g-bit (g=8) binary numbers and it follows from them that the correlation connection between DHTI elements bitwise is nonlinear in general case and it should be taken into consideration at modeling the BBI adequate to the real situation.

Figures 7 and 8 shows the real DHTI "Lena" and the artificial DHTI obtained with the help of MM at equal statistical characteristics. The BBI of 8,6,1 bits of the real and artificial DHTI "Lena" are presented in Figures 9,a,b,c.

The developed mathematical model of the artificial BBI was used to design nonlinear filtering algorithms for the real DHTI represented by 8-bit binary numbers and it showed high degree of correspondence with the real images which corroborates the findings of Petrov and Chasikov (2001) and Petrov, Trubin and Butorin (2005a).

MATHEMATICAL MODELS OF BBI VIDEO-SEQUENCES

The DHTI representation by the set of g binary sections reduces the problem of MM construction of the DHTI sequence to the construction of MM of the BBI sequence.

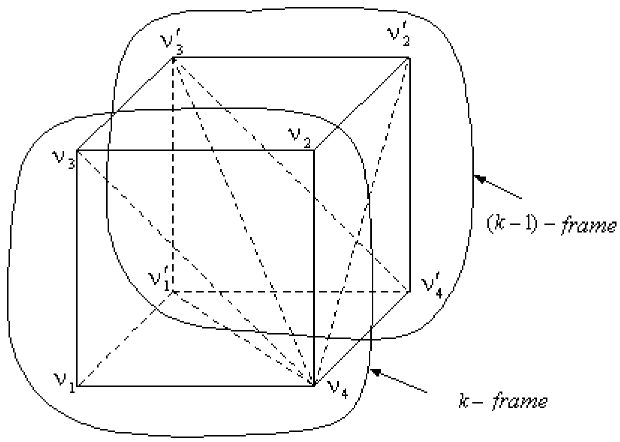
We shall assume that the sequence of BBI frames is the three-dimensional discrete-valued Markov process $\mu_k = \mu\left(i, j, k\right)$ with two spatial coordinates $\left(i, j; i \in m, j \in n\right)$ and time as the third coordinate k = 1,2…, related to the frame number in the image sequence.

We suppose that the correlation function of the BBI frame sequence has the form shown in Box 16.

Let us choose UMRF on NSHP with the vicinity of the form shown in Figure 1 as the mathematical model of BBI (see Box 17).

We consider the case when the BBI random binary element ν_4 in the k-th frame (Figure 10) belonging to the area $F_{4,k}$ should be modeled. The modeling of BBI elements belonging to areas $F_{1,k}$, $F_{2,k}$ and $F_{3,k}$ is simpler than the area $F_{4,k}$ and it can be reduced to modeling of one-dimensional and two-dimensional stationary Markov chains.

Figure 11. Seven elements of vicinity of the image element ν_4



Similarly, we can write formulas for calculating the matrix Π elements for various combinations of values of the vicinity $\Lambda_{i,j,k}$ elements.

If one of the initial matrices ${}^1\Pi, {}^2\Pi$ or ${}^4\Pi$ is the unitary matrix, elements of the matrix Π will contain 1 and 0 only. The element ν_4 value will coincide with the value of the vicinity $\Lambda_{i,j,k}$ element, whose transition probability is defined by the unitary matrix. In the case when one of matrices ${}^1\Pi, {}^2\Pi$ or ${}^4\Pi$ consists of elements with value equaled 0.5 (equiprobable independent transitions), the product of this matrix with the others gives a similar matrix. Computation of the matrix Π element becomes simpler. For instance, if elements of the matrix ${}^1\Pi$ are equal to 0.5, elements of matrices ${}^3\Pi, {}^5\Pi, {}^7\Pi$ also equal to 0.5. The value of the BBI element will be defined by matrices' ${}^2\Pi$ and ${}^4\Pi$ elements only. If elements of matrices ${}^1\Pi$ and ${}^2\Pi$ are equal to 0.5, the appearance of this or that value of the BBI element ν_4 will depend upon values of the matrix ${}^4\Pi$ elements only. For the same and equal to 0.5 elements of ${}^1\Pi, {}^2\Pi, {}^4\Pi$ the appearance of this or that value of the BBI element ν_4 is equiprobable. Matrix Π elements will be equal to 0.5.

Let us consider the case when matrices ${}^1\Pi, {}^2\Pi$ and ${}^4\Pi$ are equal, i.e.

$$ {}^1\Pi = {}^2\Pi = {}^4\Pi = \begin{Vmatrix} 0,9 & 0,1 \\ 0,1 & 0,9 \end{Vmatrix}. \tag{43}$$

Let all elements of the vicinity have the equal values $M_1 = 0$ or $M_2 = 1$. We calculate by formulas (38) probability of appearance and absence the element value $\nu_4 = M_1 = 0$.

$$\pi_{iiiiiiii} = 1 - \frac{0,1 \cdot 0,1 \cdot 0,1 \cdot 0,756}{0,82 \cdot 0,82 \cdot 0,82} = 0,99863,$$
$$\pi_{ijjjjjjj} = \frac{0,1 \cdot 0,1 \cdot 0,1 \cdot 0,756}{0,82 \cdot 0,82 \cdot 0,82} = 0,00137.$$

THE ALGORITHM OF FORMATION OF THE MARKOV BBI SEQUENCE

The basis for the mathematical model construction for Markov BBI sequence is the equation (37). The modeling of the Markov BBI sequence includes several stages.

1. Specify the transition matrices ${}^1\Pi, {}^2\Pi, {}^4\Pi$ and calculate matrices ${}^3\Pi, {}^5\Pi, {}^6\Pi, {}^7\Pi$ and the matrix Π;
2. Take the random number $\xi_l (l \leq m \cdot n)$ equally distributed over the interval $[0,1]$.

3. Select the element $\alpha_s (s = \overline{1,8})$ corresponding to values of the vicinity $\Lambda_{i,j,k}$ elements from the first column of the matrix Π;
4. The number ξ_l is compared with the chosen element $\alpha_s (s = \overline{1,8})$ and if $\alpha_s = \overline{1,8}$ and $\xi_l \leq \alpha_s$, the image elements ν_4 takes the value $\nu_4 = M_1 = 0$, otherwise $\nu_4 = M_2 = 1$;
5. If $i < n; j < m; k < K$, where *K* is the sequence length we pass to point 3, otherwise to point 6;
6. Stop.

Investigations of mathematical models of the BBI sequence for various statistical correlation between adjacent BBI elements in the space (frame) and time (between frames) was carried out.

MATHEMATICAL MODELS OF VIDEO-SEQUENCE OF MARKOV DHTI

The mathematical model of the DHTI video-sequence represents the three-dimensional Markov chain with $q = 2^g$ values and consists of *g* MM sequences of Markov BBI ordered by bits of binary numbers of the DHTI representation. Combination of BBI - DHTI in the each video-sequence frame is fulfilled on the *g*-bits register and does not require calculation operations. At that, the memory volume does not exceed one frame of DHTI. At DHTI modeling we need to take into account that each BBI has its own individual matrices of the transition probabilities of type (30).

Figure 11 shows 1st, 5th, 10th, 20th frames of the video-sequence of the artificial BBI at matrix values

$${}^1\pi_{ii}^{(1)} = {}^2\pi_{ii}^{(1)} = 0,6,$$

$${}^1\pi_{ii}^{(2)} = {}^2\pi_{ii}^{(2)} = 0,65,$$

$${}^1\pi_{ii}^{(3)} = {}^2\pi_{ii}^{(3)} = 0,7,$$

$${}^1\pi_{ii}^{(4)} = {}^2\pi_{ii}^{(4)} = 0,75$$

$${}^1\pi_{ii}^{(5)} = {}^2\pi_{ii}^{(5)} = 0,8,$$

$${}^1\pi_{ii}^{(6)} = {}^2\pi_{ii}^{(6)} = 0,85,$$

$${}^1\pi_{ii}^{(7)} = {}^2\pi_{ii}^{(7)} = 0,9$$

$${}^1\pi_{ii}^{(8)} = {}^2\pi_{ii}^{(8)} = 0,95$$

and

$${}^4\pi_{ii} = 0,9.$$

The auto-correlation function analysis of video-sequences of artificial and real DHTI shows that the MM is adequate to the real process.

Results obtained from the construction of two- and three-dimensional mathematical models allow for assuming that the approximation of the statistically correlated video-sequences of DHTI by the multi-dimensional and multi-valued Markov process is the reasonable approach to solving the problem of construction of multi-dimensional mathematical models realized by means of minimal computation resources.

Let $\mu\left(\Theta_1, \Theta_2, ..., \Theta_h\right)$ (where Θ_i are discrete coordinates) be the multi-dimensional multi-level multi-valued Markov process, this corresponds to statistically correlated video-sequences of DHTI. Let us construct the mathematical models of several statistically correlated video-se-

Box 25.

$$r_{i,\ j,\ l,\dots,h} = \sigma_\mu^2 \exp\{-\alpha_1 \left|i\right| - \alpha_2 \left|j\right| - \alpha_3 \left|l\right| - \dots - \alpha_h \left|h\right|\}, \tag{43}$$

where $\alpha_1,\dots\alpha_h$ are coefficients similar to those in expression (9).

quences of DHTI, whose realization requires the minimal computing resources. In constructing the mathematical models, we shall assume that $\mu\left(\Theta_1,\Theta_2,\dots,\Theta_h\right)$ represents the superposition of h one-dimensional, multi-valued Markov process.

The correlation function of this process has a form shown in Box 25.

For better understanding of the method of multi-dimensional mathematical model construction we will be limited by four-dimensional, multi-valued Markov process, which is adequate for the spread of statistically correlated DHTI video-sequences in the space.

MATHEMATICAL MODELS OF TWO STATISTICALLY CORRELATED BBI SEQUENCES

Let us represent Markov DHTI with sizes $m \times n$ elements as a sum of g BBI. Similar to the previous mathematical model, we first construct the mathematical model of two statistically correlated BBI sequences.

The more the dimensions of the random processes, the more complicated it is to select an example of its physical implementation. We assume that the three-dimensional random binary Markov process described earlier moves discretely in the space with equal intervals, sensing the image of the same object from the different locations. We shall suppose that the sequence of BBI elements from one position to another is the four-dimensional, multi-level, discrete-valued Markov process with the correlation function of the following form:

$$\begin{aligned} r_{f,\ q,\ s,\ p}^{(l)} &= E[\mu_{i,j,t,v}^{(l)}\ \mu_{i+f,j+q,t+s,v+p}^{(l)}] \\ &= \sigma_\mu^2 \exp\{-\alpha_1^{(l)}\left|f\right| - \alpha_2^{(l)}\left|q\right| - \alpha_3^{(l)}\left|s\right| - \alpha_4^{(l)}\left|p\right|\}, \end{aligned} \tag{44}$$

where $E[\mu_{i,j,t,v}^{(l)}\ \mu_{i+f,j+q,t+s,v+p}^{(l)}]$ is the mathematical expectation; σ_μ^2 is a variance of the random process; $\alpha_i^{(l)} (i = \overline{1,4})$ are the scale multipliers related to the process' spectrum on each coordinate.

Box 26.

$$\Lambda_{i,j,k} = \left\{\mu_{i,j-1,k,d}, \mu_{i-1,j,k,d}\mu_{i-1,j-1,k,d}\right\}, \tag{46}$$

$$\Lambda_{i,j,k,d} = \begin{cases} \varnothing, & if \quad (i,j) \in F_{1kd} \\ \left\{\left(i-1,j,k,d\right)\right\}, & if \quad (i,j) \in F_{2kd} \\ \left\{\left(i,j-1\right),k,d\right\}, & if \quad (i,j) \in F_{3kd} \\ \left\{\left(i,j-1,k,d\right),\left(i-1,j,k,d\right),\left(i-1,j-1,k,d\right)\right\}, & if \quad (i,j) \in F_{4kd} \end{cases} \tag{47}$$

Figure 13. The mathematical model of two statistically correlated BBI sequences

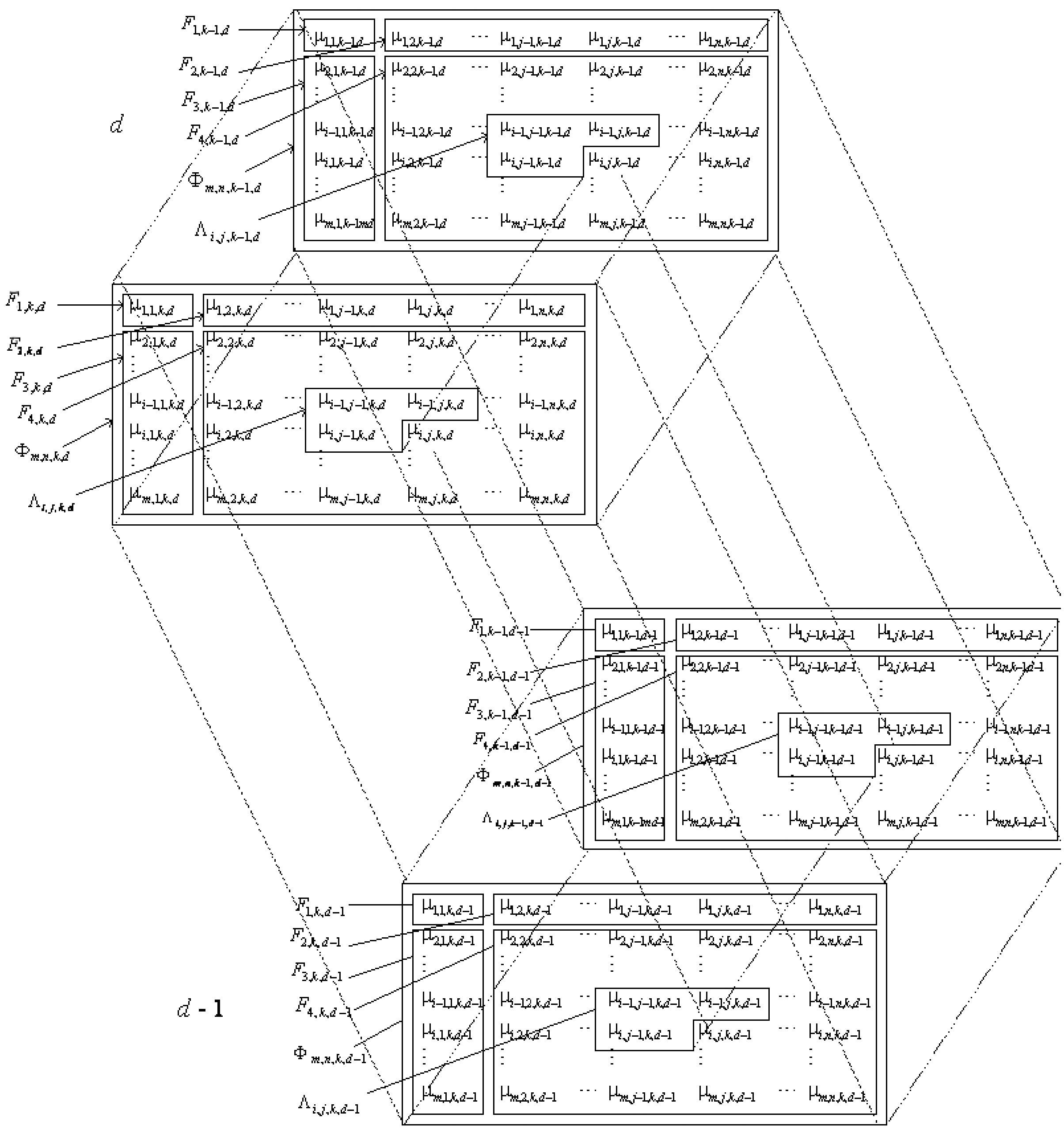
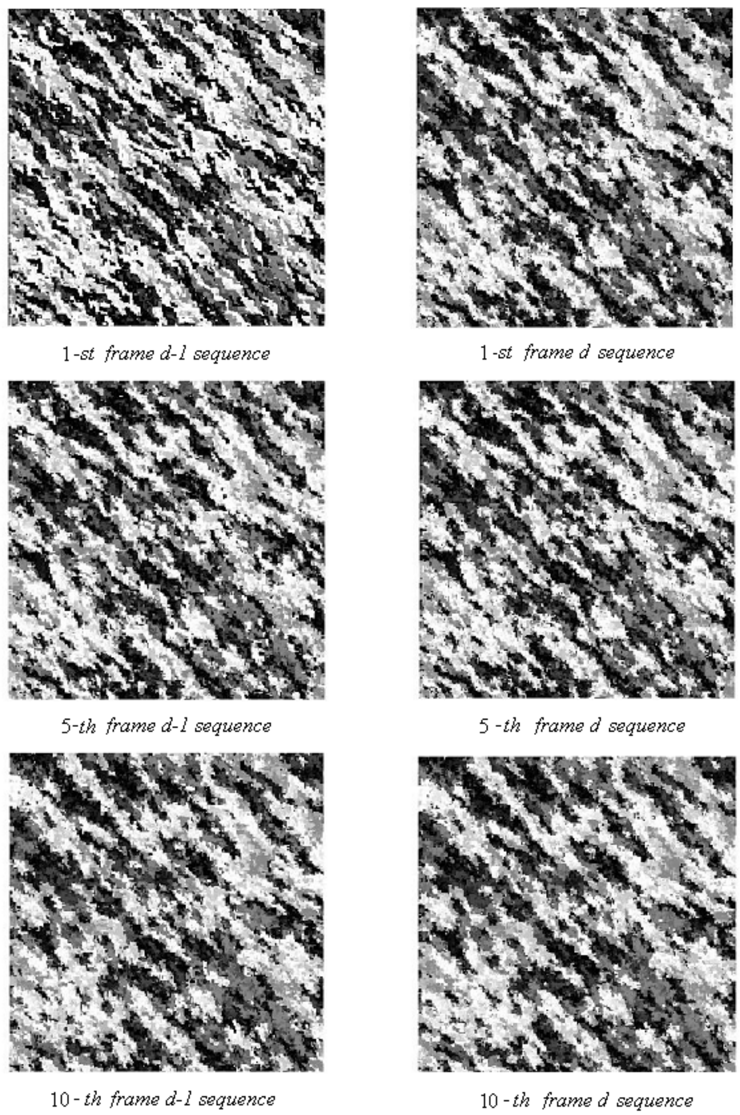


Figure 15. Frames of the statistically correlated video-sequences of the artificial DHTI



MATHEMATICAL MODEL OF TWO STATISTICALLY CORRELATED DHTI VIDEO-SEQUENCES

Mathematical model of statistically correlated DHTI video-sequences represented by g-bit binary numbers are formed by the simple bitwise presentation of g values of binary images into the g-bit register of the binary number representing the sample of the four-dimensional multi-valued Markov process.

Figure 15 shows 1st, 5th, and 10th frames of two statistically correlated video-sequences of the artificial DHTI obtained with the help of the developed mathematical model for

$${}^{1}\pi_{ii}^{(1)} = {}^{2}\pi_{ii}^{(1)} = 0,6,\ {}^{1}\pi_{ii}^{(2)} = {}^{2}\pi_{ii}^{(2)} = 0,65,$$
$${}^{1}\pi_{ii}^{(3)} = {}^{2}\pi_{ii}^{(3)} = 0,7,\ {}^{1}\pi_{ii}^{(4)} = {}^{2}\pi_{ii}^{(4)} = 0,75$$
$${}^{1}\pi_{ii}^{(5)} = {}^{2}\pi_{ii}^{(5)} = 0,8,\ {}^{1}\pi_{ii}^{(6)} = {}^{2}\pi_{ii}^{(6)} = 0,85,$$
$${}^{1}\pi_{ii}^{(7)} = {}^{2}\pi_{ii}^{(7)} = 0,9 \text{ и } {}^{4}\pi_{ii} = {}^{8}\pi_{ii} = 0,9.$$

falling in the vicinity (48) and form the matrix shown in Box 32.

Values of the element of matrix Π may be calculated in accordance with the argument of the logarithm (51) using the entropy between the generating element ν_4 in position *d* and mutually independent elements of the vicinity $\Lambda_{i,j,k,d}$. For example, at known values of the matrices in (53), expressions for element calculation of the first two lines of the matrix Π have the form shown in Box 33.

We calculate the probability $\pi_{ii\ldots i}$ under the condition that

$$^{1}\Pi = {}^{2}\Pi = {}^{4}\Pi = {}^{1}\Pi' = {}^{2}\Pi' = {}^{4}\Pi' = {}^{8}\Pi = \begin{Vmatrix} 0,9 & 0,1 \\ 0,1 & 0,9 \end{Vmatrix}.$$

$$\pi_{jiiii\underbrace{ii\cdots ii}_{11}} = 0,0001522.$$

Other elements are calculated in the similar way depending on combinations of element values in the vicinity Λ_{ijkd}.

From (55) one can easily obtain the matrix Π for the model of the discrete-valued Markov process of the smaller dimension. For example, excepting the element of the transition probability matrix $^{8}\Pi$ and associated elements of transition probability matrices $^{9}\Pi\ldots{}^{15}\Pi,$ we obtain the matrix Π for the model of the three-dimensional discrete-valued Markov process. If matrices associated with matrices $^{4}\Pi$ and $^{8}\Pi$ are excluded, we obtain the matrix Π for two-dimensional discrete-valued Markov process. Using the similarity with the approach of the four-dimensional mathematical model constructions, we can construct mathematical model of several statistically correlated DHTI corresponding to the multidimensional multi-valued Markov process of higher order.

THE ALGORITHM OF FORMATION OF STATISTICALLY CORRELATED SEQUENCES OF MARKOV BBI

The matrix Π (54) and equation (51) are the basis for constructing the mathematical model for statistically correlated DHTI video-sequences representing the four-dimensional discrete Markov process. The algorithm of mathematical model operation consists of the following stages:

Step 1: The size of the random two-dimensional field (image) of $m \times n$ elements, the video-sequence length *K* and the number of positions *D*, matrices of transitions $^{1}\Pi,$ $^{2}\Pi,$ $^{4}\Pi,$ $^{8}\Pi$ are specified and matrices $^{3}\Pi, {}^{5}\Pi, {}^{6}\Pi, {}^{7}\Pi, {}^{9}\Pi, \ldots, {}^{15}\Pi$ and Π are calculated;

Step 2: We take the random number $\xi_l (l \le m \cdot n \cdot K \cdot D)$ uniformly distributed over interval $\left[0,1\right]$;

Step 3: From the first column of the matrix Π we choose the element $\alpha_s (s = \overline{1,16})$ corresponding to element values of the vicinity Λ_{ijkd};

Step 4: The number ξ_l is compared with the chosen element $\alpha_s (s = \overline{1,16})$ and if $\alpha_s \left(s = \overline{1,16}\right)$ and $\xi_l \le \alpha_s,$ then the image element ν_4 takes the value $\nu_4 = M_1 = 0,$ otherwise $\nu_4 = M_2 = 1;$

Step 5: If $i < n; j < m; k < K; d < D,$ where *K* is the video-sequence length, *D* if the number of positions, then we pass to p. 3, otherwise to p. 6;

Step 6: Stop.

MP is stationary on all coordinates. Evidently, in this case there are 15 transition matrices:

$$
\begin{aligned}
&{}^{1}\Pi,\ {}^{1}\Pi',\ {}^{2}\Pi,\ {}^{2}\Pi',\ {}^{4}\Pi,\ {}^{4}\Pi',\\
&{}^{3}\Pi = {}^{1}\Pi' \times {}^{2}\Pi',\\
&{}^{3}\Pi' = {}^{1}\Pi' \times {}^{2}\Pi',\\
&\\
&{}^{5}\Pi,\ {}^{5}\Pi' = {}^{1}\Pi' \times {}^{4}\Pi';\\
&{}^{6}\Pi,\ {}^{6}\Pi' = {}^{2}\Pi' \times {}^{4}\Pi',\\
&{}^{7}\Pi,\ {}^{7}\Pi' = {}^{3}\Pi' \times {}^{4}\Pi',\\
&{}^{9}\Pi = {}^{3}\Pi' \times {}^{8}\Pi;\\
&{}^{10}\Pi = {}^{5}\Pi' \times {}^{8}\Pi,\\
&{}^{11}\Pi = {}^{6}\Pi' \times {}^{8}\Pi,\\
&{}^{12}\Pi = {}^{7}\Pi' \times {}^{8}\Pi,\\
&\\
&{}^{13}\Pi = {}^{1}\Pi' \times {}^{8}\Pi;\\
&{}^{14}\Pi = {}^{2}\Pi' \times {}^{8}\Pi;\\
&{}^{15}\Pi = {}^{4}\Pi' \times {}^{8}\Pi.
\end{aligned}
\tag{53}
$$

Probabilities of appearance of the BBI element ν_4 with the value $\nu_4 = M_1$ or $\nu_4 = M_2$ depend upon combinations of values of BBI elements

Box 32.

$$
\Pi = \begin{Vmatrix}
\pi_{iiiii\underbrace{ii\cdots ii}_{11}} & \pi_{jiiii\underbrace{ii\cdots ii}_{11}} \\
\pi_{iiiij\underbrace{ii\cdots ii}_{11}} & \pi_{jiiij\underbrace{ii\cdots ii}_{11}} \\
\vdots & \vdots \\
\pi_{ijjji\underbrace{jj\cdots jj}_{11}} & \pi_{jjjji\underbrace{jj\cdots jj}_{11}} \\
\pi_{ijjjj\underbrace{jj\cdots jj}_{11}} & \pi_{jjjjj\underbrace{jj\cdots jj}_{11}}
\end{Vmatrix}
= \begin{Vmatrix}
\alpha_1 & \alpha_1' \\
\alpha_2 & \alpha_2' \\
\vdots & \vdots \\
\alpha_{15} & \alpha_{15}' \\
\alpha_{16} & \alpha_{16}'
\end{Vmatrix}, \quad i, j = \overline{1, 2}; \quad i \neq j. \tag{54}
$$

Box 33.

$$
\begin{aligned}
\pi_{iiiii\underbrace{ii\cdots ii}_{11}} &= 1 - \frac{{}^{1}\pi_{ij}\,{}^{2}\pi_{ij}\,{}^{4}\pi_{ij}\,{}^{8}\pi_{ij}\,{}^{7}\pi_{ij}\,{}^{9}\pi_{ij}\,{}^{10}\pi_{ij}\,{}^{11}\pi_{ij}}{{}^{3}\pi_{ii}\,{}^{5}\pi_{ii}\,{}^{6}\pi_{ii}\,{}^{13}\pi_{ii}\,{}^{14}\pi_{ii}\,{}^{15}\pi_{ii}\,{}^{12}\pi_{ii}};\\
\pi_{jiiii\underbrace{ii\cdots ii}_{11}} &= \frac{{}^{1}\pi_{ij}\,{}^{2}\pi_{ij}\,{}^{4}\pi_{ij}\,{}^{8}\pi_{ij}\,{}^{7}\pi_{ij}\,{}^{9}\pi_{ij}\,{}^{10}\pi_{ij}\,{}^{11}\pi_{ij}}{{}^{3}\pi_{ii}\,{}^{5}\pi_{ii}\,{}^{6}\pi_{ii}\,{}^{13}\pi_{ii}\,{}^{14}\pi_{ii}\,{}^{15}\pi_{ii}\,{}^{12}\pi_{ii}}; i \neq j.\\
\pi_{iiiij\underbrace{ii\cdots ii}_{11}} &= 1 - \frac{{}^{1}\pi_{ij}\,{}^{2}\pi_{ij}\,{}^{4}\pi_{ij}\,{}^{8}\pi_{ii}\,{}^{7}\pi_{ii}\,{}^{9}\pi_{ij}\,{}^{10}\pi_{ij}\,{}^{11}\pi_{ij}}{{}^{3}\pi_{ij}\,{}^{5}\pi_{ij}\,{}^{6}\pi_{ii}\,{}^{13}\pi_{ij}\,{}^{14}\pi_{ij}\,{}^{15}\pi_{ij}\,{}^{12}\pi_{ij}};\\
\pi_{jiiij\underbrace{ii\cdots ii}_{11}} &= \frac{{}^{1}\pi_{ij}\,{}^{2}\pi_{ij}\,{}^{4}\pi_{ij}\,{}^{8}\pi_{ii}\,{}^{7}\pi_{ii}\,{}^{9}\pi_{ij}\,{}^{10}\pi_{ij}\,{}^{11}\pi_{ij}}{{}^{3}\pi_{ij}\,{}^{5}\pi_{ij}\,{}^{6}\pi_{ii}\,{}^{13}\pi_{ij}\,{}^{14}\pi_{ij}\,{}^{15}\pi_{ij}\,{}^{12}\pi_{ij}}.
\end{aligned}
\tag{55}
$$

Box 29.

$$I(\nu_1, \nu_2, \nu_4', \nu_4, \varepsilon_4) \\ = \log \frac{w(\nu_4 \mid \nu_1, \nu_2, \nu_3, \nu_1', \nu_2', \nu_3', \nu_4', \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_1', \varepsilon_2', \varepsilon_3', \varepsilon_4')}{w(\nu_4 \mid \nu_3, \nu_1', \nu_2', \nu_3', \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_1', \varepsilon_2', \varepsilon_3', \varepsilon_4')} \\ \times \frac{w\left(v_4 \left| v_3', \varepsilon_1', \varepsilon_2', \varepsilon_3, \varepsilon_3' \right.\right)}{w\left(v_4 \left| \varepsilon_3' \right.\right)}. \tag{50}$$

Box 30.

$$I(\nu_1, \nu_2, \nu_4', \varepsilon_4) \\ = \log \frac{w(\nu_4 \mid \nu_1) w(\nu_4 \mid \nu_2) w(\nu_4 \mid \nu_4') w(\nu_4 \mid \varepsilon_4) w(\nu_4 \mid \varepsilon_1') w(\nu_4 \mid \varepsilon_2')}{w(\nu_4 \mid \nu_3) w(\nu_4 \mid \nu_1') w(\nu_4 \mid \nu_2') w(\nu_4 \mid \varepsilon_1)} \\ \times \frac{w(\nu_4 \mid \varepsilon_3) w(\nu_4 \mid \nu_3')}{w(\nu_4 \mid \varepsilon_2) w(\nu_4 \mid \varepsilon_4') w(\nu_4 \mid \varepsilon_3')} \tag{51}$$

Box 31.

$$\pi_{ijklmnqrtsfhuvpw} = \pi(\nu_4 \mid \nu_1 = M_j; \nu_2 = M_k; \nu_3 = M_l; \nu_4' = M_n; \nu_3' = M_r; \\ \varepsilon_1 = M_t; \varepsilon_2 = M_s; \varepsilon_3 = M_f; \varepsilon_4 = M_h; \varepsilon_1' = M_u; \varepsilon_2' = M_v; \varepsilon_3' = M_p; \varepsilon_4' = M_w), \\ i, j, k, l, m, n,\ q, r, t, s, f, h, u, v, p, w = \overline{1,2}. \tag{52}$$

of the BBI element ν_4, should depend upon the statistical correlation only between the element ν_4 and elements $\nu_1, \nu_2, \nu_4', \varepsilon_4$ of the vicinity (48) (see Box 29).

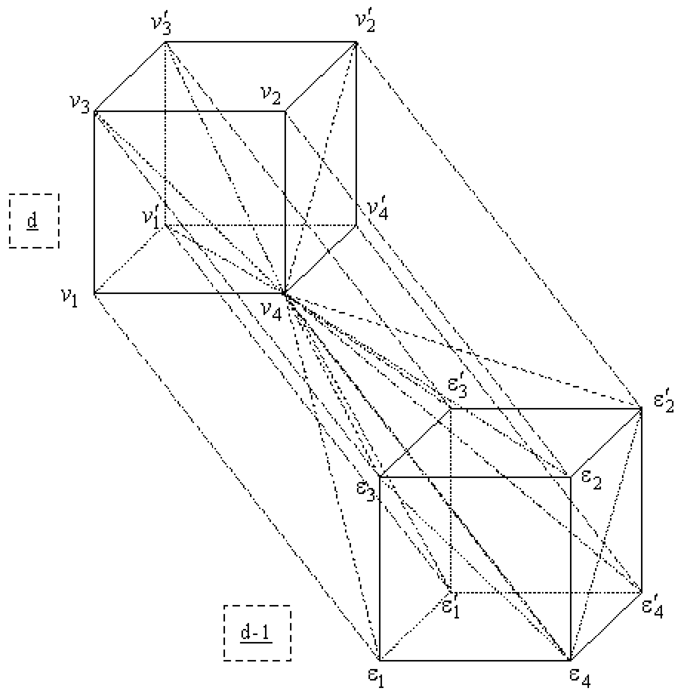
Taking into consideration that conditions of mutual independence of the vicinity Λ_{ijkd} elements are fulfilled, equation (50) can be transformed to the form shown in Box 30.

Equation (51) is the basis of the model construction of the four-dimensional discrete-valued Markov process with two values.

Transition probabilities for the discrete-valued four-dimensional MP are defined by the matrix of transition probabilities Π, with elements of the form shown in Box 31.

Let matrices of the single-step transition probabilities on the four coordinates (dimensions) be specified. For the three-dimensional MP these are ${}^1\Pi$, ${}^2\Pi$, ${}^4\Pi$ in position d and similar matrices of transition probabilities are ${}^1\Pi'$, ${}^2\Pi'$, ${}^4\Pi'$ for three-dimensional MP in position $d-1$. The statistical correlation between the three-dimensional processes in positions d and $d-1$ is characterized by the matrix of transition probabilities ${}^8\Pi$. We shall assume that the multi-dimensional random

Figure 14. The MM fragment of two statistically correlated video-sequences



Box 27.

$$\Lambda_{i,j,k,v} = \left(\nu_1, \nu_2, \nu_3, \nu_1', \nu_2', \nu_3', \nu_4', \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_1', \varepsilon_2', \varepsilon_3', \varepsilon_4'\right). \tag{48}$$

Box 28.

$$I(\nu_1, \nu_2, \nu_4', \nu_4,\ \varepsilon_4) \\ = \log \frac{w(\nu_4 \mid \nu_1, \nu_2, \nu_3, \nu_4, \nu_1', \nu_2', \nu_3', \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_1', \varepsilon_2', \varepsilon_3', \varepsilon_4')}{p(\nu_4)}, \tag{49}$$

where $w\left(\cdot\right)$ is the multi-dimensional probability density in the complicated Markov chain.

The important peculiarity of the mathematical model operation algorithm is the absence of computing operations at creation of the artificial DHTI in the *k*-th frame in position *d*.

From the analysis of results obtained for the three-dimensional and four-dimensional mathematical models, it follows that for the same statistical characteristics of the random process components its statistical redundancy increases with growth of the process dimension. If in the one-dimensional case the probability of appearance of the same image value is equal to $\pi_{ii}^{(l)} = 0,9,$ in the two-dimensional case $\pi_{ii}^{(l)} = 0,987,$ in the three-dimensional case $\pi'_{\underbrace{ii\ldots i}_{7}} = 0,998629,$ in the four-dimensional case $\pi^{(l)}_{\underbrace{ii\ldots i}_{15}} = 0,9998478.$ It follows from this that statistically correlated DHTI video-sequences may have a very large statistical redundancy, which can be expediently used at DHTI processing.

THE APPROACH TO CONSTRUCT THE MM OF STATISTICALLY CORRELATED DHTI VIDEO-SEQUENCES OF THE BASIS OF THE H-DIMENSIONAL MULTI-VALUED MP

To construct the mathematical model of several statistically correlated DHTI video-sequences based on the *h*-dimensional Markov process, it is necessary, first of all, to divide the DHTI into BBI, the number of the latter is equal to digit capacity of the DHTI representation. Then we need to define the vicinity of the BBI element generating in the given moment.

If we succeeded to form the vicinity $\Lambda_{ij\ldots h}$ of the generating element ν_{41} (similar to the developed mathematical model) on the basis of the analysis of the *h*-dimensional discrete-valued MP, the next stage is the rewriting of the equation similar to (50), which defines the mutual information quantity between the vicinity $\Lambda_{ij\ldots h}$ and the element ν_{41} (see Box 34).

The value of the element ν_{41} in mathematical model of *h*-th order should be determined by the statistical correlation only between the generating element ν_{41} and elements of the vicinity belonging to *h* independent coordinates. All other elements of the vicinity $\Lambda_{ij\ldots h}$ have the redundant information, which should be eliminated. We can do it by means of the successive transformation of the multi-dimensional transition probabilities in (56) with the purpose to eliminate the statistical correlation between elements of any group falling in the vicinity $\Lambda_{ij\ldots h}$, which allows transfer from multi-dimensional transition probabilities of the complicated Markov chain to the simple equation for one-dimensional single-step transition probabilities similar to (51). The expression obtained in such a manner is the basis for the structure of construction of elements of the tran-

Box 34.

$$I\left(\nu_{11},\ldots,\nu_{41},\nu'_{11},\ldots,\nu'_{41},\ldots,\gamma_{1k},\ldots,\gamma_{4k},\gamma'_{1k},\ldots,\gamma'_{4k},\ldots,\lambda_{1h},\ldots,\lambda_{4h},\lambda'_{1h},\ldots,\lambda'_{4h}\right)$$
$$= \ln \frac{w\left(\nu_{41}\left|\nu_{11},\ldots,\nu_{31},\nu'_{11},\ldots,\nu'_{41},\ldots,\gamma_{1k},\ldots,\gamma_{4k},\gamma'_{1k},\ldots,\gamma'_{4k},\ldots,\lambda_{1h},\ldots,\lambda_{4h},\lambda'_{1h},\ldots,\lambda'_{4h}\right.\right)}{w\left(\nu_{41}\left|\nu_{31},\nu'_{11},\ldots,\nu'_{31},\ldots,\gamma_{1k},\ldots,\gamma_{3k},\gamma'_{1k},\ldots,\gamma'_{4k},\ldots,\lambda_{1h},\ldots,\lambda_{3h},\lambda'_{1h},\ldots,\lambda'_{4h}\right.\right)}, \tag{56}$$

where $\nu_{11},\nu_{21},\nu'_{41},\ldots,\gamma_{4k},\ldots,\lambda_{4h}$ are elements of the vicinity $\Lambda_{ij\ldots h}$ belonging to *h* independent coordinates of the *h*-dimensional discrete-valued MP. The second index of variables in (56) indicates the number of the DHTI sequence.

Box 35.

$$\Pi = \begin{Vmatrix} \pi_{i\underbrace{ii\cdots ii}_{2^h-1}} & \pi_{j\underbrace{ii\cdots ii}_{2^h-1}} \\ \vdots & \vdots \\ \pi_{i\underbrace{jj\cdots jj}_{2^h-1}} & \pi_{i\underbrace{jj\cdots jj}_{2^h-1}} \end{Vmatrix} = \begin{Vmatrix} \alpha_1 & \alpha_1' \\ \vdots & \vdots \\ \alpha_{2^h} & \alpha_{2^h}' \end{Vmatrix}, \quad i,j=\overline{1,2}; \quad i \neq j, \tag{57}$$

and the number of the transition matrices of type (53) will be $2^h - 1.$

sition matrix Π in *h*-dimensional Markov chain. The matrix Π in this case will have the form (see Box 35).

The algorithm of set formation of statistically correlated DHTI video-sequences consists of the following steps:

1. The initial probabilities of the multi-dimensional binary Markov process p_1 and p_2 and matrices of one-dimensional single-step transition probabilities ${}^1\Pi, {}^2\Pi, \ldots, {}^h\Pi$ are specified and matrices of transition probabilities and the matrix Π (associated with the former) are calculated;
2. We take the random number $\xi_l (l \leq m \cdot n \cdot K \cdot D \cdot T)$ uniformly distributed over the interval $\left[0,1\right]$;
3. From the first column of the matrix Π we choose the element $\alpha_s (s = \overline{1, 2^h})$ corresponding to values of the vicinity $\Lambda_{ij\ldots h}$ elements;
4. The number ξ_l is compared with the chosen element $\alpha_s (s = \overline{1, 2^h})$ and if $\xi_l \leq \alpha_s$, then the image element ν_4 takes the value $\nu_4 = M_1 = 0$, otherwise $\nu_4 = M_2 = 1$;
5. If $i < n; j < m; k < K; d < D; t < T$, where *K* is the video-sequence length, *D* is the number of positions, T is the number of sets, we pass to p. 3, otherwise to p. 6;
6. Stop.

The mathematical model of the set of statistically correlated DHTI sequences represented by *g*-bit binary numbers is formed by the simple bitwise presentation of *g* values of the BBI elements in the *g*-bit register of the binary number representing the sample of the *h*-dimensional multi-valued process similar to p. 6.

Realization of the developed mathematical model does not require computation operations, and the memory volume at modeling of the *h*-dimensional process does not exceed BBI size of $\left(h-2\right) \cdot g$.

CONCLUSION

The main conclusions are the following:

1. The theory of conditional Markov processes is expanded to the static and dynamic DHTI representing the multi-dimensional discrete-valued random processes with several states.
2. The method of DHTI division is offered presented by *g*-bit binary numbers per *g* BBI (binary sections) each of which represents the causal binary Markov field or the binary Markov chain on the non-symmetrical half-plane.
3. On the basis of the Markov type DHTI division method onto bit binary sections and using the entropy approach to probability calculation of each BBI element values, the DHTI mathematical models have been synthesized.

4. The model adequacy to real image is confirmed by element estimations of the transition probability matrices calculated for the artificial and real images.
5. The spatial-time MM of the DHTI video-sequence is synthesized, which is the three-dimensional multi-valued Markov process with the dividable exponential correlation function allowing the presentation of the three-dimensional multi-valued Markov process as a superposition of three one-dimensional multi-valued Markov processes.
6. The approach for MM construction of several statistically correlated DHTI video-sequences is offered, which can be presented by *h-dimensional* multi-valued Markov processes. This approach can be reduced to formal procedures of the sequential elimination of the statistical redundancy between vicinity elements of the simulating image element belonging to *h* independent coordinates and all others.

DIRECTIONS OF FURTHER RESEARCHES AND DEVELOPMENTS

Developed methods for construction of DHTI and video-sequence mathematical models are the effective tool for development of simple, reliable and affective algorithms of multi-dimensional signals allowing approximation by the discrete-valued Markov random processes.

We suppose to apply the DHTI MM and video-sequences synthesized at development of new algorithms on the basis of Markov chains with several states.

Investigation of multi-dimensional non-stationary mathematical models is interesting as well, which have been created of the basis of Markov chains with several states.

The authors in the of this chapter have done extensive research and are widely published in the area of development of DHTI mathematical models and the synthesis on its basis of algorithms of recovering images distorted by the noise.

REFERENCES

Abend, K., Harley, T. J., & Kanal, L. N. (1965). Classification of binary random patterns. *IEEE Transactions on Information Theory*, *11*, 538–544. doi:10.1109/TIT.1965.1053827.

Ablameiko, S. V., & Lagunovskiy, D. M. (2000). *Image processing: Technology, methods, application*. Amalphea Publishing.

Berchtold. (1999). *The double chain Markov model* (Technical Report Nº 348). Seattle, WA: University of Washington, Department of Statistics.

Besag, J. E. (1974). Spatial interaction and statistical analysis of lattice systems. *Journal of the Royal Statistical Society. Series B. Methodological*, *36*, 192–236.

Bondur, V. G. (2003). Modeling of multi-spectral airspace images of the dynamic fields of brightness (in Russian). *Investigation of Earth from Space,* (2), 3-17.

Bykov, V. V. (1971). *Digital modeling in the statistical radio engineering*. Moscow: Sovetskoe Radio Publishing.

Chellappa, R. (1985). Two-dimensional discrete Gaussian Markov random fields for image processing. In *Progress in Pattern Recognition 2*. Amsterdam: Elsevier Science Publishers BV.

Chellappa, R., & Kashyap, R. L. (1982). Digital image restoration using spatial interaction models. *IEEE Transactions on Acoustics, Speech, and Signal Processing*, *30*, 461–472. doi:10.1109/TASSP.1982.1163911.

Dagion, D., & Mercero, R. (1988). *Digital processing of multi-dimensional signals*. Moscow: MIR Publishing.

Dech, G. (1971). *Manual to the practical application of the Laplace transformation and z-transformation*. Moscow: Nauka Publishing.

Derin, H., & Kelly, P. (1989). Random processes of Markov type with discrete arguments. *TIEEE*, *77*(10), 42–71.

Dragan, Y. P. (1993). *Status and development prospects of probabilistic models of random signals and fields*. Kharkiv, Ukraine: HIRE Publishing.

Elfeki, A. A., & Dekking, M. (2001). Markov chain model for subsurface characterization: Theory and applications. *Mathematical Geology*, *33*, 569–589. doi:10.1023/A:1011044812133.

Fano, R. (1965). Statistical theory of communication. In *Transmission of Information*. Moscow: MIR Publishing.

Furman, Y. A. (2003). *Introduction to contour analysis and its application to signal and image processing*. Moscow: FIZMATLIT Publishing.

Jain, A. K. (1974). Noncausal representation for finite discrete signals. In *Proceedings of IEEE Conference on Decision and Control.* IEEE Publishing.

Jain, A. K., & Jain, J. R. (1978). Partial differential equations and finite difference methods in image processing, part II: Image restoration. *IEEE Transactions on Automatic Control*, *23*, 817–834. doi:10.1109/TAC.1978.1101881.

Jain, A. K., & Rangansth, S. (1980). Image coding by autoregressive synthesis. In *Proceedings of ICASSP'80*. IEEE.

Jain, A. K., & Wang, S. H. (1977). *Stochastic image models and hybrid coding*. NOSC.

Jine, A. K. (1981). Achievements in the field of mathematical models for image processing. *TIEEE*, *69*(5), 9–39.

Kashyap, R. L. (n.d.). Analysis and synthesis of image patterns by spatial interaction models. In *Progress in Pattern Recognition*. New York: Elsevier North-Holland.

Krasheninnikov, B. P. (2003). *Fundamentals of the image processing theory*. Ulyaniovsk, Russia: UlGTU Publ.

McGill, W. J. (1954). Multivariate information transmission, transactions PGIT. In *Proceedings of Symposium on Information Theory.* PGIT.

Modestino, J. W., & Zhang, J. (1993). A Markov random field model-based approach to image interpretation. In *Markov Random Fields: Theory and Applications*. Boston: Academic Press, Inc..

Nahi, N. E., & Franco, C. A. (1972). Application of Kalman filtering to image enhancement. In *Proceedings of IEEE Conference on Decision and Control.* IEEE Publishing.

Petrov, E. P. (2006). The spatial-time mathematical model of the sequence of digital half-tone image of Markov type. *Problems of Information Processing*, *1*(6), 46–52.

Petrov, E. P., & Chastikov, A. V. (2001). Method of adaptive filtering of binary pulse correlated signals. *Radio Engineering and Electronics*, *46*(10), 1155–1158.

Petrov, E. P., & Prozorov, D. E. (n.d.). Filtering of Markov processes with several states. *Radar, Navigation, and Communications.*.

Petrov, E. P., Smolskliy, S. M., & Kharina, N. L. (2007). Synthesis of models of multi-dimensional multi-valued Markov processes. *Vestnik of MPEI,* (1), 147-152.

Petrov, E. P., Trubin, I. C., & Butorin, E. L. (2002). The spatial-time model of digital Markov images. In *Proceedings of VIII Conference: Radar Technology, Navigation, and Communications,* (vol. 1, pp. 371-380). Voronezh, Russia: RTNC.

Petrov, E. P., Trubin, I. C., & Butorin, E. L. (2003). The spatial-time model of digital Markov images. In *Proceedings of IX Conference of Radar Technology, Navigation, and Communications*, (vol. 1, pp. 330-337). Voronezh, Russia: RTNC.

Petrov, E. P., Trubin, I. S., & Butorin, E. L. (2005). Nonlinear filtering of the sequence of digital half-tone images. *Radio Engineering and Electronics*, *10*(10), 1265–1272.

Petrov, E. P., Trubin, I. S., & Harina, N. L. (2006). *Modeling of multi-dimensional multi-valued Markov processes*. Radio Engineering.

Petrov, E. P., Trubin, I. S., & Harina, N. L. (2006). Problems of information processing. *Vestnik*, *1*(6), 41–46.

Pisarevsky, A. N., & Chernyavsky, A. F. (1988). *Systems of technical vision (principal fundamentals, hardware and software support)*. Leningrad, Russia: Mashinostroenie Publishing.

Politis, D. N. (1994). Markov chains in many dimensions. *Advances in Applied Probability*. doi:10.2307/1427819.

Shalizi, C. R. (2003). Optimal nonlinear prediction of random fields on networks. In *Proceedings of Center for the Study of Complex Systems*. University of Michigan.

Soifer, V. A. (2003). *Methods of computer image processing*. Moscow: FIZMATLIT Publishing.

Spector, A. A. (1985). Multi-dimensional discrete Markov fields and its filtering at the presence of the non-correlated noise. *Radio Engineering and Electronics*, (5): 512–523.

Spector, A. A. (1987). Two-stage filtering of random fields at interference presence. In *Methods of Processing of Digital Signals and Fields under Condition of Interference*. Novosibirsk, Russia: IEEE Publishing.

Tikhonov, V. I., & Mironov, M. A. (1977). *Markov processes*. Moscow: Sovetskoe Radio Publishing.

Trubin, I. S. (2004). Mathematical model of two statistically correlated video-sequences. []. St. Petersburg, Russia: SP Technical University Publishing.]. *Proceedings of Universities in Communications*, *171*, 90–97.

Trubin, I. S., & Butorin, E. L. (2004). Mathematical model of the digital image sequence. In *Proceedings of Russian NTO Popov Society: Digital Processing of Signals and its Application* (*Vol. 2*, pp. 166–169). Moscow: NTO.

Trubin, I. S., & Butorin, E. L. (2005). Spatial-time Markov model of digital half-tone images. *Radio Engineering,* (10), 10-13.

Vasiliev, K. K. (1995). Digital processing of image sequences in global monitoring problems of the Earth surface, the medicine, the air motion control, radar systems and hydro-location. In *Conversion of Military Complex, Double-Application Technologies*. Moscow: RIA Publishing.

Vasiliev, K. K. (1995). *Applied theory of random processes and fields*. Ulyanovsk, Russia: UlGTU Publishing.

Vasiliev, K. K. (2002). Representation and fast processing of multi-dimensional images. *New Scientific Technologies,* (3), 4-24.

Vasiukov, V.N. (2002). New approaches to solution of the image recognition and processing. *New Scientific Technologies*, (3), 44-51.

Vinkler, G. (2002). Image analysis, random fields, and dynamic methods of Monte-Carlo. In *Mathematical Fundamentals*. Novosibirsk, Russia: Siberian Division of RAS Publishing.

Vishnevsky, V. M. (2003). *Theoretical bases of computer network design*. Moscow: Technosphere Publishing.

Wong, E. (1978). Recursive causal linear filtering for two-dimensional random fields. *IEEE Transactions on Information Theory*, *24*, 50–59. doi:10.1109/TIT.1978.1055818.

Woods, J. W. (1972). Two-dimensional discrete Markov fields. *Information Theory*, *22*, 232–240. doi:10.1109/TIT.1972.1054786.

APPENDIX

List of Abbreviations

- **BBI:** Bit binary image
- **DHTI:** Digital half-tone image
- **MAP:** Maximal *a posteriori* probability
- **MC:** Markov chain
- **MM:** Mathematical model
- **MP:** Markov process
- **MTP:** Matrix of transition probability
- **NF:** Nonlinear filter
- **NSHP:** Non-symmetric half-plain
- **OSMRF:** One-sided Markov random field
- **RRD:** Radio receiving device
- **VS:** Video-sequence
- **WGN:** White Gaussian noise