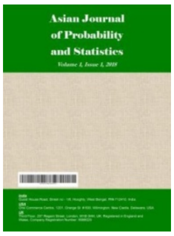

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A Comparison between Maximum Likelihood and Bayesian Estimation Methods for a Shape Parameter of the Weibull-Exponential Distribution

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Authors' contributions

This work was carried out in collaboration between all authors. Author TGI conceived the idea and wrote the first draft of the manuscript. Author PEO managed the literature searches and checked through the analysis for correctness. All authors read and approved the final manuscript.

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ABSTRACT

We considered the Bayesian analysis of a shape parameter of the Weibull-Exponential distribution in this paper. We assumed a class of non-informative priors in deriving the corresponding posterior distributions. In particular, the Bayes estimators and associated risks were calculated under three different loss functions. The performance of the Bayes estimators was evaluated and compared to the method of maximum likelihood under a comprehensive simulation study. It was discovered that for the said parameters to be estimated, the quadratic loss function under both uniform and Jeffrey's priors should be used for decreasing parameter values while the use of precautionary loss function can be preferred for increasing parameter values irrespective of the variations in sample size.

Keywords: Weibull exponential; Bayesian estimation; mathematical statistics; maximum likelihood estimation; simulation.

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1. INTRODUCTION

The Exponential distribution is regarded as being *memoryless* and has a constant failure rate; this latter property makes the distribution unsuitable for real-life problems and hence there is need to generalize the Exponential distribution in order to increase its flexibility and capability to model some other real-life problems, [1]. Some of the recent generalizations of the exponential distribution include the transmuted exponential distribution [2], transmuted inverse exponential distribution [3] and the Weibull-Exponential distribution *(WED)* [4].

According to [4], if *X* is a Weibull-exponential random variable, then the probability density function (*pdf*) and the cumulative distribution function (cdf) of *X* are respectively given by;

$$f(x) = \alpha\beta\lambda e^{\lambda\beta x}\left(1-e^{-\lambda x}\right)^{\beta-1} e^{-\alpha\left(e^{\lambda x}-1\right)^{\beta}} \quad (1)$$

and

$$F\left(x\right) = 1 - e^{-\alpha\left(e^{\lambda x}-1\right)^{\beta}} \quad (2)$$

where; α and β are shape parameters while λ is scale parameter from the exponential distribution.

The classical and non-classical or Bayesian methods of estimation have gained wider applications in statistical theory and analysis. In classical scenario, the parameters are considered to be fixed while in the Bayesian concept, the parameters are viewed as unknown random variables. It is true that in many real-life phenomena which are represented by lifetime models, the parameters cannot be treated as constant throughout the life testing period [5-7] hence the need for Bayesian estimation for lifetime models.

[8] has considered Bayesian estimation for the extreme value distribution using progressive censored data and asymmetric loss, [9] estimated the shape parameter of the Generalized Pareto Distribution (GPD) using quasi, inverted gamma and uniform prior distributions under the LINEX, precautionary and entropy loss functions, [10] estimated the shape parameter of Generalized Exponential distribution using extended Jeffrey's prior under the quadratic loss function, squared error loss function and general entropy loss function, [11] also estimated the parameters of Rayleigh distribution using Bayesian approach, [12] estimated the scale parameter of Laplace model using different asymmetric loss functions comprising precautionary, weighted squared, modified (quadratic) squared loss functions and [13] considered Bayesian Survival Estimator for Weibull distribution with censored data. In addition, [14] has also considered the Bayesian estimation of Weibull distribution under three loss functions.

One of the important things that cannot be ignored in Bayesian approach is choosing the appropriate prior(s) for the parameters, so also is the choice of loss function. [15-21] and many others have however shown a number of symmetric and asymmetric loss functions to be functional in several applications.

The objective of this study is to introduce a statistical comparison between the Bayesian and Maximum Likelihood estimation (MLE) procedures for estimating a shape parameter of the Weibull-Exponential distribution. The resulting estimators are obtained by using squared error, Quadratic and precautionary loss functions.

The layout of the paper is as follows; in Section 2, Maximum likelihood estimate of the shape parameter are obtained, in Section 3, the posterior distributions are obtained under the two different prior distributions while the Bayes estimates and corresponding risks are obtained in section 4. Finally, comparison between *MLE* and Bayes estimates under the two priors and loss functions are made using simulation study in Section 5. Some concluding remarks are given in Section 6.

2. MAXIMUM LIKELIHOOD ESTIMATION

This section presents the estimation of a shape parameter of the Weibull-Exponential distribution using the method of maximum likelihood estimation. Let $X_1, X_2, ..., X_n$ be a random sample from the *WED* with unknown parameter vector $\theta = \left(\alpha, \beta, \lambda\right)^T$. The total log-likelihood function for **θ** is obtained from *f(x)* as follows:

$$L\left(X_1, X_2, \ldots, X_n \mid \alpha, \beta, \lambda\right) = \left(\alpha\beta\lambda\right)^n e^{\lambda\beta\sum_{i=1}^{n} x_i} \sum_{i=1}^{n}\left(1 - e^{-\lambda x_i}\right)^{\beta-1} \exp\left\{-\alpha\sum_{i=1}^{n}\left(e^{\lambda x_i} - 1\right)^{\beta}\right\} \quad (3)$$

The likelihood function for the shape parameter α is given by;

$$L\left(\alpha \mid \underline{X}\right) \propto \left(\alpha\right)^n e^{-\alpha\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}} \quad (4)$$

Let the log-likelihood function $l = \log L\left(\alpha \mid \underline{X}\right)$, therefore

$$l = n\log\alpha - \alpha\sum_{i=1}^{n}\left(e^{\lambda x_i} - 1\right)^{\beta} \quad (5)$$

Differentiating l partially with respect to α , equating to zero and solving for $\hat{\alpha}$ gives;

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\left(e^{\lambda x_i} - 1\right)^{\beta}$$

$$\Rightarrow \hat{\alpha} = \frac{n}{\sum_{i=1}^{n}\left(e^{\lambda x_i} - 1\right)^{\beta}} \quad (6)$$

Hence, equation (6) is the estimator for a shape parameter of the Weibull-Exponential distribution obtained by the method of Maximum Likelihood estimation.

3. POSTERIOR DISTRIBUTIONS

To obtain the posterior distribution of a parameter once the data has been observed, we apply Bayes' Theorem which is given as:

$$p\left(\alpha \mid \underline{X}\right) = \frac{p\left(\alpha\right) L\left(\alpha \mid \underline{X}\right)}{\int_0^{\infty} p\left(\alpha\right) L\left(\alpha \mid \underline{X}\right) d\alpha} \quad (7)$$

where $p\left(\alpha\right)$ and $L\left(\alpha \mid \underline{X}\right)$ are the prior distribution and the Likelihood function respectively.

Here, Posterior distributions are derived by using uniform and Jeffrey's prior.

3.1 Posterior Distributions under the Assumption of Uniform Prior

The uniform prior as a non-informative prior relating to parameter α is defined as:

$$p\left(\alpha\right) \propto 1 \qquad ; \qquad 0 < \alpha < \infty \quad (8)$$

The posterior distribution of parameter α for a given data under uniform prior is obtained from equation (7) using integration by substitution method as:

$$p\left(\alpha \mid \underline{X}\right)=\frac{\alpha^{n} e^{-\alpha \sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}}}{\left(\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}\right)^{-(n+1)} \Gamma(n+1)} \tag{9}$$

3.2 Posterior Distributions under the Assumption of Uniform Prior

Also, the Jeffrey's prior as a non-informative prior relating to parameter α of the *WED* distribution is defined as:

$$p(\alpha) \propto \frac{1}{\alpha} \quad ; \quad \alpha \geq 1 \tag{10}$$

The posterior distribution of parameter α for a given data under Jeffrey prior is obtained from equation (7) using integration by substitution method as:

$$p\left(\alpha \mid \underline{X}\right)=\frac{\alpha^{n-1}\left(\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}\right)^{n} e^{-\alpha \sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}}}{\Gamma(n)} \tag{11}$$

4. BAYES ESTIMATORS AND THEIR RESPECTIVE CORRESPONDING RISKS

Here, we estimate a shape parameter of the *WED* using three loss functions under the posterior distributions obtained from both the uniform and Jeffrey's priors.

The Bayes estimators and their corresponding Bayes posterior risks using uniform prior are as follows:

4.1 Using Squared Error Loss Function (*SELF*) under Uniform Prior

The squared error loss function relating to the parameter α is defined as:

$$L\left(\alpha, \alpha_{SELF}\right)=\left(\alpha-\alpha_{SELF}\right)^{2} \tag{12}$$

where α_{SELF} is the estimator of the parameter α under *SELF*

The derivation of Bayes estimator using *SELF* under uniform prior is given below:

$$\alpha_{SELF}=E(\alpha)=E\left(\alpha \mid \underline{X}\right)$$

$$E\left(\alpha \mid \underline{X}\right)=\int_{0}^{\infty} \alpha\, p\left(\alpha \mid \underline{X}\right) d\alpha \tag{13}$$

Substituting for $p\left(\alpha \mid \underline{X}\right)$ in equation (11), we have:

$$E\left(\alpha \mid \underline{X}\right)=\frac{\left(\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}\right)^{n+1}}{\Gamma(n+1)} \int_{0}^{\infty} \alpha^{n+1} e^{-\alpha \sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}} d\alpha \tag{14}$$

Now, using integration by substitution method in equation (14) and simplification, we obtained the Bayes estimator using *SELF* under the uniform prior as:

$$\alpha_{SELF} = \frac{\Gamma(n+2)}{\Gamma(n+1)\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}} \tag{15}$$

Using the Squared error loss function (*SELF*), the following risk $p(\alpha_{SELF})$ is defined as:

$$P(\alpha_{SELF}) = E\left(\alpha^2 \mid \underline{X}\right) - \left\{E\left(\alpha \mid \underline{X}\right)\right\}^2 \tag{16}$$

And it is obtained as

$$P(\alpha_{SELF}) = \frac{\Gamma(n+3)\Gamma(n+1) - \left(\Gamma(n+2)\right)^2}{\left(\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}\right)^2 \left(\Gamma(n+1)\right)^2} \tag{17}$$

4.2 Using Quadratic Loss Function (QLF) under Uniform Prior

The Quadratic loss function (*QLF*) is defined as

$$L(\alpha, \alpha_{QLF}) = \left(\frac{\alpha - \alpha_{QLF}}{\alpha}\right)^2 \tag{18}$$

where α_{QLF} is the estimator of the parameter α under *QLF*

The derivation of Bayes estimator using *QLF* under uniform prior is as follows:

$$\alpha_{QLF} = \frac{E\left(\alpha^{-1}\right)}{E\left(\alpha^{-2}\right)} = \frac{E\left(\alpha^{-1} \mid \underline{X}\right)}{E\left(\alpha^{-2} \mid \underline{X}\right)}$$

$$E\left(\alpha^{-1} \mid \underline{X}\right) = \int_0^{\infty} \alpha^{-1} p\left(\alpha \mid \underline{X}\right) d\alpha \tag{19}$$

Substituting for $p\left(\alpha \mid \underline{X}\right)$ in equation (19), we have:

$$E\left(\alpha^{-1} \mid \underline{X}\right) = \frac{\left(\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}\right)^{n+1}}{\Gamma(n+1)} \int_0^{\infty} \alpha^{n-1} e^{-\alpha\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}} d\alpha \tag{20}$$

Using integration by substitution method in Equation (20) and simplifying, we obtained the Bayes estimator using *QLF* under the uniform prior as:

$$\alpha_{QLF} = \frac{\Gamma(n)}{\left(\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}\right)\Gamma(n-1)} \tag{21}$$

Using the Quadratic loss function (*QLF*), the following risk $p\left(\alpha_{QLF}\right)$ is defined as:

$$P\left(\alpha_{QLF}\right) = 1-\frac{\left\{E\left(\alpha^{-1} \mid \underline{X}\right)\right\}^{2}}{E\left(\alpha^{-2} \mid \underline{X}\right)} \tag{22}$$

Therefore, the following risk under the uniform prior using the Quadratic loss function is given as:

$$P\left(\alpha_{QLF}\right) = \frac{\Gamma(n+1)\Gamma(n-1)-\left[\Gamma(n)\right]^{2}}{\Gamma(n+1)\Gamma(n-1)} \tag{23}$$

4.3 Using Precautionary Loss Function (*PLF*) under the Uniform Prior

The precautionary loss function (*PLF*) introduced by [22] is an asymmetric loss function and is defined as

$$L\left(\alpha_{PLF},\alpha\right) = \frac{\left(\alpha_{PLF}-\alpha\right)^{2}}{\alpha} \tag{24}$$

where α_{PLF} is the estimator of the parameter α under *PLF*

Similarly, the derivation of Bayes estimator under *PLF* using uniform prior is given below:

$$\alpha_{PLF} = \left\{E\left(\alpha^{2}\right)\right\}^{\frac{1}{2}} = \left\{E\left(\alpha^{2} \mid \underline{X}\right)\right\}^{\frac{1}{2}} = \sqrt{E\left(\alpha^{2} \mid \underline{X}\right)}$$

$$E\left(\alpha^{2} \mid \underline{X}\right) = \int_{0}^{\infty}\alpha^{2} p\left(\alpha \mid \underline{X}\right) d\alpha \tag{25}$$

Substituting for $p\left(\alpha \mid \underline{X}\right)$ in equation (25);, we have:

$$E\left(\alpha^{2} \mid \underline{X}\right) = \frac{\left(\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}\right)^{n+1}}{\Gamma(n+1)}\int_{0}^{\infty}\alpha^{n+2} e^{-\alpha\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}} d\alpha \tag{26}$$

Again, using integration by substitution method in Equation (26) and simplifying, we obtained the Bayes estimator using *PLF* under the uniform prior as:

$$\alpha_{PLF} = \sqrt{\frac{\Gamma(n+3)}{\left(\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}\right)^{2}\Gamma(n+1)}} \tag{27}$$

Applying the Precautionary loss function (*PLF*), the following risk $p\left(\alpha_{PLF}\right)$ is defined as:

$$P\left(\alpha_{PLF}\right)=2\left\{\alpha_{PLF}-E\left(\alpha \mid \underline{X}\right)\right\} \tag{28}$$

and derived as:

$$P\left(\alpha_{PLF}\right)=2\left\{\frac{\left\{\Gamma(n+3)\Gamma(n+1)\right\}^{\frac{1}{2}}-\Gamma(n+2)}{\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}\Gamma(n+1)}\right\} \tag{29}$$

Similarly, the Bayes estimators and posterior risks of a shape parameter of the *WED* using three loss functions under the posterior distribution obtained from Jeffrey's prior are as follows:

4.4 Using Squared Error Loss Function (*SELF*) under Jeffrey's Prior

The derivation of Bayes estimator under *SELF*-using Jeffrey's prior is given below:

$$\alpha_{SELF}=E(\alpha)=E\left(\alpha \mid \underline{X}\right)$$

$$E\left(\alpha \mid \underline{X}\right)=\int_0^{\infty}\alpha\, p\left(\alpha \mid \underline{X}\right)d\alpha \tag{30}$$

Substituting for $p\left(\alpha \mid \underline{X}\right)$ in equation (30), we have:

$$E\left(\alpha \mid \underline{X}\right)=\frac{\left(\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}\right)^{n}}{\Gamma(n)}\int_0^{\infty}\alpha^{n}e^{-\alpha\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}}d\alpha \tag{31}$$

Using integration by substitution method in Equation (31) and simplifying, we obtained the Bayes estimator using *SELF* under Jeffrey prior as:

$$\alpha_{SELF}=\frac{\Gamma(n+1)}{\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}\Gamma(n)} \tag{32}$$

Using the Squared error loss function (*SELF*), the following risk $p\left(\alpha_{SELF}\right)$ is defined as:

$$P\left(\alpha_{SELF}\right)=E\left(\alpha^2 \mid \underline{X}\right)-\left\{E\left(\alpha \mid \underline{X}\right)\right\}^2 \tag{33}$$

Therefore, the following risk under Jeffrey's prior using the squared error loss function is:

$$P\left(\alpha_{SELF}\right)=\frac{\Gamma(n+2)\Gamma(n)-\left(\Gamma(n+1)\right)^2}{\left(\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}\right)^2\left(\Gamma(n)\right)^2} \quad (34)$$

4.5 Using Quadratic Loss Function (*QLF*) under Jeffrey's prior

The derivation of Bayes estimator under *QLF* using Jeffrey's prior is given below:

$$\alpha_{QLF}=\frac{E\left(\alpha^{-1}\right)}{E\left(\alpha^{-2}\right)}=\frac{E\left(\alpha^{-1}\mid \underline{X}\right)}{E\left(\alpha^{-2}\mid \underline{X}\right)}$$

$$E\left(\alpha^{-1}\mid \underline{X}\right)=\int_0^{\infty}\alpha^{-1}p\left(\alpha\mid \underline{X}\right)d\alpha \quad (35)$$

Substituting for $p\left(\alpha\mid \underline{X}\right)$ in equation (35), we have:

$$E\left(\alpha^{-1}\mid \underline{X}\right)=\frac{\left(\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}\right)^n}{\Gamma(n)}\int_0^{\infty}\alpha^{n-2}e^{-\alpha\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}}d\alpha \quad (36)$$

Using integration by substitution method in Equation (36) and simplifying, we obtained the Bayes estimator using *QLF* under Jeffrey prior as:

$$\alpha_{QLF}=\frac{\Gamma(n-1)}{\left(\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}\right)\Gamma(n-2)} \quad (37)$$

Using the Quadratic loss function (QLF), the following risk $p\left(\alpha_{QLF}\right)$ is defined as:

$$P\left(\alpha_{QLF}\right)=1-\frac{\left\{E\left(\alpha^{-1}\mid \underline{X}\right)\right\}^2}{E\left(\alpha^{-2}\mid \underline{X}\right)} \quad (38)$$

Hence, it is obtained as:

$$P\left(\alpha_{QLF}\right)=\frac{\Gamma(n)\Gamma(n-2)-\left[\Gamma(n-1)\right]^2}{\Gamma(n)\Gamma(n-2)} \quad (39)$$

4.6 Using Precautionary Loss Function (*PLF*) under Jeffrey's Prior

Similarly, the derivation of Bayes estimator under *PLF* using Jeffrey's prior is given below:

$$\alpha_{PLF}=\left\{E\left(\alpha^2\right)\right\}^{\frac{1}{2}}=\left\{E\left(\alpha^2\mid \underline{X}\right)\right\}^{\frac{1}{2}}=\sqrt{E\left(\alpha^2\mid \underline{X}\right)}$$

$$E\left(\alpha^2 \mid \underline{X}\right)=\int_0^\infty \alpha^2 p\left(\alpha \mid \underline{X}\right) d\alpha \tag{40}$$

Substituting for $p\left(\alpha \mid \underline{X}\right)$ in equation (40); we have:

$$E\left(\alpha^2 \mid \underline{X}\right)=\frac{\left(\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}\right)^{n}}{\Gamma(n)} \int_0^\infty \alpha^{n+1} e^{-\alpha \sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}} d\alpha \tag{41}$$

Using integration by substitution method in Equation (41) and simplifying, we obtained the Bayes estimator using *PLF* under Jeffrey prior as:

$$\alpha_{PLF}=\sqrt{\frac{\Gamma(n+2)}{\left(\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}\right)^{2}\Gamma(n)}} \tag{42}$$

Applying the Precautionary loss function (PLF), the following risk $p\left(\alpha_{PLF}\right)$ is defined as:

$$P\left(\alpha_{PLF}\right)=2\left\{\alpha_{PLF}-E\left(\alpha \mid \underline{X}\right)\right\} \tag{43}$$

Hence, obtained as:

$$P\left(\alpha_{PLF}\right)=2\left\{\frac{\left\{\Gamma(n+2)\Gamma(n)\right\}^{\frac{1}{2}}-\Gamma(n+1)}{\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}\Gamma(n)}\right\} \tag{44}$$

5. SIMULATION STUDY

We used a package in R software to generate random sample**s** of size n = (25, 35, 75, 125) from *WED* by using different values of α and assuming that the exponential parameter λ and shape parameter β are known, that is $(\lambda=\beta=1)$. Tables 1 to 3 present the results of our simulation study by listing the estimates of the shape parameter under the appropriate estimation methods such as the Maximum Likelihood Estimation (*MLE*), Squared Error Loss Function (*SELF*), Quadratic Loss Function (*QLF*), and Precautionary Loss Function (*PLF*) under both Uniform and Jeffrey priors. The Maximum Likelihood estimates and Bayesian estimates were obtained under a reasonably high number of replications; different sample sizes were used to investigate the performance of the estimators about their biases and mean squared errors as well as the sample sizes.

The performance of the two methods was evaluated using the following performance measures:
Bias:

$$Bias=\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\alpha}_i-\alpha\right)$$

and ***Mean Square Error***,

$$MSE=\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\alpha}_i-\alpha\right)^2$$

Where $\hat{\alpha}$ is the estimate of α from the *ith* simulated data and N is the number of Monte Carlo samples.

Note: the estimate with the minimum bias and MSE will be considered as the best.

Table 1. Simulation of bayes estimates, their biases and mean squared errors based on the replications and sample sizes where $\alpha = 0.5$ assuming that λ and β are known $(\lambda = \beta = 1)$

Sample sizes	Measures	MLE	Uniform prior			Jeffrey's prior		
			SELF	QLF	PLF	SELF	QLF	PLF
25	Estimate	0.4437	0.4615	0.4260	0.4703	0.4437	0.4082	0.4525
	BIAS	0.0840	0.0899	0.0813	0.0940	0.0840	0.0818	0.0866
	MSE	0.0123	0.0145	0.0109	0.0160	0.0123	0.0105	0.0133
35	Estimate	0.4611	0.4743	0.4479	0.4808	0.4611	0.4348	0.4676
	BIAS	0.0699	0.0732	0.0686	0.0756	0.0699	0.0690	0.0713
	MSE	0.0083	0.0094	0.0076	0.0101	0.0083	0.0074	0.0087
75	Estimate	0.4645	0.4707	0.4583	0.4738	0.4645	0.4521	0.4676
	BIAS	0.0464	0.0475	0.0460	0.0482	0.0464	0.0462	0.0469
	MSE	0.0035	0.0037	0.0034	0.0039	0.0035	0.0034	0.0036
125	Estimate	0.4644	0.4681	0.4607	0.4699	0.4644	0.4570	0.4663
	BIAS	0.0360	0.0364	0.0358	0.0368	0.0360	0.0358	0.0362
	MSE	0.0021	0.0021	0.0020	0.0022	0.0021	0.0020	0.0021

Table 2. Simulation of bayes estimates, their biases and mean squared errors based on the replications and sample sizes where $\alpha = 1.5$ assuming that λ and β are known $(\lambda = \beta = 1)$

Sample sizes	Measures	MLE	Uniform prior			Jeffrey's prior		
			SELF	QLF	PLF	SELF	QLF	PLF
25	Estimate	1.3311	1.3844	1.2779	1.4108	1.3311	1.2247	1.3575
	BIAS	0.2521	0.2699	0.2438	0.2820	0.2521	0.2454	0.2598
	MSE	0.1106	0.1309	0.0985	0.1440	0.1106	0.0945	0.1197
35	Estimate	1.3833	1.4228	1.3438	1.4425	1.3833	1.3043	1.4029
	BIAS	0.2099	0.2197	0.2057	0.2267	0.2098	0.2071	0.2140
	MSE	0.0743	0.0842	0.0685	0.0905	0.0743	0.0667	0.0787
75	Estimate	1.3935	1.4121	1.3749	1.4213	1.3935	1.3563	1.4027
	BIAS	0.1393	0.1424	0.1380	0.1446	0.1393	0.1386	0.1406
	MSE	0.0317	0.0337	0.0305	0.0350	0.0317	0.0302	0.0326
125	Estimate	1.3932	1.4044	1.3821	1.4099	1.3932	1.3709	1.3988
	BIAS	0.1079	0.1093	0.1073	0.1104	0.1079	0.1075	0.1085
	MSE	0.0186	0.0193	0.0182	0.0198	0.0186	0.0180	0.0189

Table 3. Simulation of bayes estimates, their biases and mean squared errors based on the replications and sample sizes where $\alpha = 3.5$ assuming that λ and β are known $(\lambda = \beta = 1)$

Sample sizes	Measures	MLE	Uniform prior			Jeffrey's prior		
			SELF	QLF	PLF	SELF	QLF	PLF
25	Estimate	3.1060	3.2302	2.9818	3.2918	3.1060	2.8575	3.1675
	BIAS	0.5882	0.6298	0.5688	0.6579	0.5882	0.5726	0.6062
	MSE	0.6024	0.7127	0.5363	0.7838	0.6024	0.5146	0.6515
35	Estimate	3.2277	3.3199	3.1355	3.3657	3.2277	3.0433	3.2735
	BIAS	0.4897	0.5126	0.4799	0.5291	0.4897	0.4833	0.4994
	MSE	0.4048	0.4582	0.3731	0.4928	0.4048	0.3632	0.4286
75	Estimate	3.2515	3.2948	3.2081	3.3164	3.2515	3.1648	3.2731
	BIAS	0.3249	0.3323	0.3219	0.3375	0.3249	0.3233	0.3281
	MSE	0.1724	0.1833	0.1660	0.1904	0.1724	0.1642	0.1773
125	Estimate	3.2509	3.2769	3.2249	3.2898	3.2509	3.1989	3.2638
	BIAS	0.2518	0.2551	0.2503	0.2575	0.2518	0.2509	0.2532
	MSE	0.1012	0.1051	0.0989	0.1076	0.1012	0.0983	0.1030

6. CONCLUSIONS

We have obtained the Bayesian estimators of a shape parameter of Weibull-Exponential distribution in this study, the Posterior distributions of this parameter are derived by using Uniform and Jeffrey's priors. Bayes estimators and their associated risks have been obtained by using three loss functions under the two prior distributions. The three loss functions used in this study are the Squared Error Loss Function (*SELF*), Quadratic Loss Function (*QLF*) and Precautionary Loss Function (*PLF*). The biases and mean square errors based on all the priors and for all the loss functions relating to the shapee parameter of the Weibull-Exponential distribution expectedly decrease with increase in sample size.

For $\alpha = 0.5, \lambda = \beta = 1$, the result indicates that using *QLF* under Jeffrey prior produces the best estimator with minimum bias and *MSE* followed by *QLF* under Uniform prior and these performances are found to be consistent irrespective of the different sample sizes used.

For $\alpha = 1.5, \lambda = \beta = 1$, the result indicates that using *QLF* under Jeffrey's prior produces the best estimator with minimum bias and *MSE* followed by *QLF* under Uniform prior and these performances are found to be consistent irrespective of the different sample sizes used and this generally means that increasing the exponential or the other shape parameter has a huge impact on the estimate of the shape parameter of the Weibull-Exponential distribution under consideration.

Also, for $\alpha = 3.5, \lambda = \beta = 1$, our result shows that using *QLF* under Jeffrey's prior produces the best estimator with minimum bias and *MSE* followed by *QLF* under Uniform prior and again these performances are found to be consistent irrespective of the different sample sizes considered, and this proves that increasing the shape parameter under study alone, keeping the other shape and exponential parameter constant has a very little or no effect on the estimate of the shape parameter of the Weibull-Exponential distribution under study. Most importantly, we found that using the *QLF* under both priors is a better approach for estimating the shape parameter of the *WED* irrespective of the values of the exponential and shape parameters and the sample sizes.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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