International Journal of Advanced and Applied Sciences, 5(8) 2018, Pages: 18-23

Contents lists available at Science-Gate

International Journal of Advanced and Applied Sciences

Journal homepage: http://www.science-gate.com/IJAAS.html

Programming codes of block-Milne's device for solving fourth-order ODEs

Oghonyon Jimevwo Godwin *, Bishop Sheila Amina, Eke Stella Kanayo

Department of Mathematics, College of Science, Covenant University, P.M.B. 1023, Ota, Ogun State, Nigeria

ARTICLE INFO

Article history:
Received 11 January 2018
Received in revised form
18 May 2018
Accepted 20 May 2018

Keywords:
Block-Milne's device
Mathematica programming codes
Convergence limit
Max calculated errors
Principal local truncation error

ABSTRACT

Block-Milne's device is an extension of block-predictor-corrector method and specifically developed to design a worthy step size, resolve the convergence criteria and maximize error. In this study, programming codes of block-Milne's device (P-CB-MD) for solving fourth order ODEs are considered. Collocation and interpolation with power series as the basic solution are used to devise P-CB-MD. Analysing the P-CB-MD will give rise to the principal local truncation error (PLTE) after determining the order. The P-CB-MD for solving fourth order ODEs is written using Mathematica which can be utilized to evaluate and produce the mathematical results. The P-CB-MD is very useful to demonstrate speed, efficiency and accuracy compare to manual computation applied. Some selected problems were solved and compared with existing methods. This was made realizable with the support of the named computational benefits.

1. Introduction

The elongation of block-predictor-corrector method is all essential for yielding approximate solution to fourth order ordinary differential equations. This paper is mainly concerned with representing programming codes of block-Milne's device for working out fourth order ODEs of the form (Dormand, 1996; Oghonyon et al., 2015):

$$y'''' = f(u, y, y', y'', y'''), \quad y(\alpha) = \beta_0, \quad y'(\alpha) = \beta_1, \quad y''(\alpha) = \beta_2, \quad y'''(\alpha) = \beta_3$$
for $a \le u \le b$ and $f: R \times R^{\mu} \to R^{\mu}$. (1)

The estimated numeric approximation to (1) is widely seen as

$$\sum_{i=1}^{k} \alpha_i z_{n+i-1} = h^4 \sum_{i=1}^{k} \beta_i f_{n+i-1}, \tag{2}$$

where h is step size, is a determrine quantity unambiguously set implying construction of k order (Akinfenwa et al., 2013; Oghonyon et al., 2015).

It is assumed that $f \in R$ is sufficient to a certain degree on time interval $u \in [a, b]$ and gratifies a planetary worldwide condition, i.e., thither is a unchanging $L \ge 0$ such that $|f(u, y) - f(u, \bar{\bar{y}}| \le L|y - \bar{\bar{y}}|, \quad \forall y, \bar{\bar{y}} \in R$.

Underneath the condition, par (1) checkout the universe and singularity defined on $x \in [a, b]$, as well searched to meet the Weierstrass theorem (Jain et al., 2007; Lambert, 1973).

Since a and b are finite and $y' = [y_1', y_2', \dots, y_n']^T$, $y = [y_1, y_2, \dots, y_n]$ and $f = [f_1, f_2, \dots, f_n]$, comes up in tangible world applications for both scientific research and engineering such as fluid dynamics and movment of rocket as discussed (Mehrkanoon et al., 2010).

Scholars suggested the established method to work out par (1) by reducing to first-order ODEs. This idea of simplifying to a system of first order ODEs have very strong encumbrance which admits waste of human effort, difficulty in programming/coding and consuming implementation time. Bookmen formulated straight forward method for estimating (1) with improve efficiency and accuracy. Such techniques consist of block method, parallel processing predictor-corrector technique, block implicit method, block hybrid method, Backward Differentiation Formula (BFD) and so on. Nevertheless, each has their benefits and shortcomings for executing them. Interested readers are invited to read Adesanya et al. (2012), Anake and Adoghe (2013), Anake et al. (2013), Awoyemi et al. (2014, 2015), Kayode (2008), Kayode et al. (2014), Oghonyon et al. (2015,2016), Olabode (2009), and Olabode and Alabi (2013) for more information.

Authors have proposed block-predictor-corrector method in the form of Adams family as situated in Anake et al. (2013), Awoyemi et al. (2014), Kayode (2008), Kayode et al. (2014), Oghonyon et al.

* Corresponding Author.
Email Address: godwin.oghonyon@covenantuniversity.edu.ng (O. J. Godwin)
https://doi.org/10.21833/ijaas.2018.08.003

(2015,2016), Olabode (2009), and Olabode and Alabi (2013). The backward differentiation formula (BDF) is Gear's method known for implementing stiff ODEs as mentioned by Gear (1971), Ibrahim et al. (2007), Majid and Suleiman (2007, 2008), Mehrkanoon et al. (2010), and Zarina et al. (2007). The objective is to write a programming codes of block-Milne's device for solving fourth order ODEs. This technique of implementing programming codes is geared towards easy computation, speed, efficiency and accuracy apart from the computational gains of block-Milne's device established in research (Abell and Braselton, 2009; Dormand, 1996; Faires and Burden, 2012; Lambert, 1973;1991; Oghonyon et al., 2015;2016).

Definition: z-parallel processing-r point method. Assume r denotes the block size and h is the stepsize, then parallel processing size in time is rh. Let $m = 0,1,2,\ldots$ describe the parallel processing number and let $n = mr$, then the $z - parallel\ processing,\ r - point$ technique can be spelt in the succeeding ecumenical category:

$$Y_\mu = \sum_{s=1}^{b} A_s Y_{\mu-s} + h \sum_{s=0}^{b} B_s F_{\mu-s}. \tag{3}$$

Where

$$Y_\mu = [y_{n+1}, \ldots, y_{n+i}, \ldots, y_{n+r}]^V,$$
$$F_\mu = [f_{n+1}, \ldots, f_{n+i}, \ldots, f_{n+r}]^V.$$

A_s and B_s are $r \times r$ coefficients matrices (Ibrahim et al., 2007).

Thus, from the definition supra, a block method has the numerical benefit that for each virtual application, the result is evaluated to a greater extent or at more than one point simultaneously. The total number of points relies on the formulation of the block method. Therefore, applying these methods can supply faster and more flying results to the problem which can be examined to produce the sought after accuracy (Majid and Suleiman, 2007, 2008; Mehrkanoon et al., 2010; Mohammed and Tech, 2010; Oghonyon et al., 2015;2016). Thence, the need of this composition is to propose programming codes of block-Milne's device that aid the implementation of fourth order ODEs as well assist to realize the vantages of block-Milne's device like designing a suitable step size, determining the convergence criteria and error control.

The residual of this paper is examined as follows: in Section 2 programming codes of the materials and methods. Section 3 programming codes for implementing block Milnes' device. Section 4 Conclusion as cited (Akinfenwa et al., 2013; Oghonyon et al., 2016).

2. Programming Codes of the materials and methods

Under this discussion section, the primary goal to be achieved is to formulate the programming codes of block-Milne's device. Block-Milne's device is a combination of the $q - step$ (predictor) method and $q - 1 - step$ (corrector) technique of the same order. A combination can be of the form

$$y(x) = \sum_{j=0}^{k} \alpha_j y_{n-j} + h^4 \sum_{j=0}^{k} \beta_j f_{n-j}, \tag{4}$$
$$y(x) = \sum_{j=0}^{k} \alpha_j y_{n-j} + h^4 \sum_{j=1}^{k} \boldsymbol{\beta}_j^* f_{n+j}. \tag{5}$$

Pars (4) and (5) forms the class of block-Milne's device with $\beta_j,\ j = 0,1,2,3,4$. Stating that y_{n+j} is the numerical estimate to the analytical results $y(u_{n+j})$ i.e. $y(u_{n+j}) \approx y_{n+j}$, and $f(u_{n+j}, y_{n+j}) \approx f_{n+j}$ having $j = 0,1,2,3,4$. To get par (4) and (5), the basis function approximation is interpolated and collocated at selected intervals. This turns out to become a system of linear equation i.e. Au=b.

$$y(u) = \sum_{j=0}^{k} a_j \left(\frac{u-u_n}{h}\right)^j. \tag{6}$$

Expanding (6) gives birth to the basis function approximation which can presented in programming codes as

$$y[u\_] = v[0] + v[1]\frac{(u-u[n])}{h} + v[2]\frac{(u-u[n])^2}{h^2} + v[3]\frac{(u-u[n])^3}{h^3} + v[4]\frac{(u-u[n])^4}{h^4} + v[5]\frac{(u-u[n])^5}{h^5} + +v[6]\frac{(u-u[n])^6}{h^5} + v[7]\frac{(u-u[n])^7}{h^7} + v[8]\frac{(u-u[n])^8}{h^8}, \tag{7}$$

where $v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7$ and v_8 are unknown constants which is required to be determine in a peculiar way. Assume the condition that method (6) agrees with the analytical result at the time interval u_n, u_{n-j} to become the approximate

$$y(u_n) \approx y_n, \quad y(u_{n-k}) \approx y_{n-k}. \tag{8}$$

Expecting that the estimating function (7) satisfies problem (1) at the points $u_{n+k}, k = 0,1,2$ to formulate the following estimates as

$$y^{iv}(u_{n+k}) \approx f_{n+k}], k = 0,1,2,3,4. \tag{9}$$

Joining the estimates of par (8) and (9) will result to ninefold systems of equation which gives rise to Ax=b. Solving the systems of equation will produce the block-Milne's device in the form of parallel processing predictor-corrector technique constituted as the programming codes

$$matrixa = \left\{\begin{array}{c} \{1,0,0,0,0,0,0,0,0\}, \\ \{1,-1,1,-1,1,-1,1,-1,1\}, \\ \{1,-2,4,-8,16,-32,64,-128,256\}, \\ \{1,-3,9,-27,81,-243,729,-2187,6561\}, \\ \{0,0,0,0,24,0,0,0,0\}, \\ \{0,0,0,0,24,-120,360,-840,1680\}, \\ \{0,0,0,0,24,-240,1440,-6720,26880\}, \\ \{0,0,0,0,24,-360,3240,-226800,136080\}, \\ \{0,0,0,0,24,-480,5760,-53760,430080\} \end{array}\right\};$$

$$b = \{y[n], y[n-1], y[n-2], y[n-3], f[n], f[n-1], f[n-2], f[n-3], f[n-4]\}; \quad \{c,d,l,q,r,s,t,u,v\} = Inverse[matrixa].b \tag{10}$$

$$matrixa=\left\{\begin{array}{c}\{1,0,0,0,0,0,0,0,0\},\\ \{1,-1,1,-1,1,-1,1,-1,1\},\\ \{1,-2,4,-8,16,-32,64,-128,256\},\\ \{1,-3,9,-27,81,-243,729,-2187,6561\},\\ \{0,0,0,0,24,-120,360,-840,1680\},\\ \{0,0,0,024,-360,3240,-226800,136080\},\\ \{0,0,0,0,24,120,360,840,1680\}\\ \{0,0,0,0,24,240,1440,6720,26880\},\\ \{0,0,0,0,24,360,3240,226800,136080\}\end{array}\right\};$$

$b=\{y[n],y[n-1],y[n-2],y[n-3],f[n-1],f[n-3],f[n+1],f[n+2],f[n+3]\};\quad \{c,d,l,q,r,s,t,u,v\}=Inverse[matrixa].b$ (11)

to arrive at $a_k, k=0,1,2,3,\ldots,8$ and substituting the values of $a_k{}'s$ into (6) to get the continuous block-Milne's device as

$$y[u\_]=\left(1+\frac{11}{6}\frac{(u-u[n])}{h}+\frac{(u-u[n])^2}{h^2}+\frac{(u-u[n])^3}{6h^2}\right)y[n]+\left(\frac{-3(u-u[n])}{h}-\frac{5}{2}\frac{(u-u[n])^2}{h^2}-\frac{(u-u[n])^3}{2h^2}\right)y[n-1]+\left(\frac{3(u-u[n])}{2h}+\frac{2(u-u[n])^2}{h^2}+\frac{(u-u[n])^3}{2h^2}\right)y[n-2]+\left(\frac{-(u-u[n])}{3h}-\frac{1}{2}\frac{(u-u[n])^2}{h^2}-\frac{(u-u[n])^3}{6h^2}\right)y[n-3]+\left(\frac{193}{20160}\left(\frac{u-u[n]}{h}\right)+\frac{4463}{120960}\left(\frac{u-u[n]}{h}\right)^2+\frac{53}{960}\left(\frac{u-u[n]}{h}\right)^3+\frac{1}{24}\left(\frac{u-u[n]}{h}\right)^4+\frac{5}{288}\left(\frac{u-u[n]}{h}\right)^5+\frac{7}{1728}\left(\frac{u-u[n]}{h}\right)^6+\frac{1}{2016}\left(\frac{u-u[n]}{h}\right)^7+\frac{1}{40320}\left(\frac{u-u[n]}{h}\right)^8\right)f[n]h^4+\left(\frac{181}{1008}\left(\frac{u-u[n]}{h}\right)+\frac{10531}{30240}\left(\frac{u-u[n]}{h}\right)^2+\frac{23}{120}\left(\frac{u-u[n]}{h}\right)^3-\frac{1}{30}\left(\frac{u-u[n]}{h}\right)^5-\frac{13}{1080}\left(\frac{u-u[n]}{h}\right)^6-\frac{1}{560}\left(\frac{u-u[n]}{h}\right)^7-\frac{1}{10080}\left(\frac{u-u[n]}{h}\right)^8\right)f[n-1]h^4+\left(\frac{209}{3360}\left(\frac{u-u[n]}{h}\right)+\frac{1327}{20160}\left(\frac{u-u[n]}{h}\right)^2-\frac{1}{96}\left(\frac{u-u[n]}{h}\right)^3+\frac{1}{40}\left(\frac{u-u[n]}{h}\right)^5+\frac{19}{1440}\left(\frac{u-u[n]}{h}\right)^6+\frac{1}{420}\left(\frac{u-u[n]}{h}\right)^7+\frac{1}{6720}\left(\frac{u-u[n]}{h}\right)^8\right)f[n-2]h^4+\left(\frac{-1}{720}\left(\frac{u-u[n]}{h}\right)+\frac{283}{30240}\left(\frac{u-u[n]}{h}\right)^2+\frac{1}{60}\left(\frac{u-u[n]}{h}\right)^3-\frac{1}{90}\left(\frac{u-u[n]}{h}\right)^5-\frac{7}{1080}\left(\frac{u-u[n]}{h}\right)^6-\frac{1}{720}\left(\frac{u-u[n]}{h}\right)^7-\frac{1}{10080}\left(\frac{u-u[n]}{h}\right)^8\right)f[n-3]h^4+\left(\frac{1}{20160}\left(\frac{u-u[n]}{h}\right)-\frac{241}{120960}\left(\frac{u-u[n]}{h}\right)^2-\frac{1}{320}\left(\frac{u-u[n]}{h}\right)^3+\frac{1}{480}\left(\frac{u-u[n]}{h}\right)^5+\frac{11}{8640}\left(\frac{u-u[n]}{h}\right)^6+\frac{1}{3360}\left(\frac{u-u[n]}{h}\right)^7+\frac{1}{40320}\left(\frac{u-u[n]}{h}\right)^8\right)f[n-4]h^4, \quad (12)$$

$$y[u\_]=\left(1+\frac{11}{6}\frac{(u-u[n])}{h}+\frac{(u-u[n])^2}{h^2}+\frac{(u-u[n])^3}{6h^2}\right)y[n]+\left(\frac{-3(u-u[n])}{h}-\frac{5}{2}\frac{(u-u[n])^2}{h^2}-\frac{(u-u[n])^3}{2h^2}\right)y[n-1]+\left(\frac{3(u-u[n])}{2h}+\frac{2(u-u[n])^2}{h^2}+\frac{(u-u[n])^3}{2h^2}\right)y[n-2]+\left(\frac{-(u-u[n])}{3h}-\frac{1}{2}\frac{(u-u[n])^2}{h^2}-\frac{(u-u[n])^3}{6h^2}\right)y[n-3]+\left(\frac{10511}{1440}\left(\frac{u-u[n]}{h}\right)+\frac{109609}{241920}\left(\frac{u-u[n]}{h}\right)^2+\frac{409}{1920}\left(\frac{u-u[n]}{h}\right)^3+\frac{1}{64}\left(\frac{u-u[n]}{h}\right)^4-\frac{3}{640}\left(\frac{u-u[n]}{h}\right)^5+\frac{7}{17280}\left(\frac{u-u[n]}{h}\right)^6+\frac{1}{13440}\left(\frac{u-u[n]}{h}\right)^7+\frac{1}{80640}\left(\frac{u-u[n]}{h}\right)^8\right)f[n-1]h^4+\left(\frac{2833}{201600}\left(\frac{u-u[n]}{h}\right)+\frac{23783}{1209600}\left(\frac{u-u[n]}{h}\right)^2+\frac{43}{960}\left(\frac{u-u[n]}{h}\right)^3-\frac{1}{960}\left(\frac{u-u[n]}{h}\right)^4+\frac{1}{5760}\left(\frac{u-u[n]}{h}\right)^5+\frac{1}{17280}\left(\frac{u-u[n]}{h}\right)^6-\frac{1}{40320}\left(\frac{u-u[n]}{h}\right)^7+\frac{1}{403200}\left(\frac{u-u[n]}{h}\right)^8\right)f[n-3]h^4+\left(\frac{-179}{2688}\left(\frac{u-u[n]}{h}\right)-\frac{4553}{80640}\left(\frac{u-u[n]}{h}\right)^2+\frac{97}{1920}\left(\frac{u-u[n]}{h}\right)^3+\frac{3}{64}\left(\frac{u-u[n]}{h}\right)^5+\frac{3}{640}\left(\frac{u-u[n]}{h}\right)^5-\frac{11}{5760}\left(\frac{u-u[n]}{h}\right)^6-\frac{1}{13440}\left(\frac{u-u[n]}{h}\right)^7+\frac{1}{26880}\left(\frac{u-u[n]}{h}\right)^8\right)f[n+1]h^4+\left(\frac{101}{1800}\left(\frac{u-u[n]}{h}\right)+\frac{4357}{75600}\left(\frac{u-u[n]}{h}\right)^2-\frac{13}{600}\left(\frac{u-u[n]}{h}\right)^3-\frac{1}{40}\left(\frac{u-u[n]}{h}\right)^4+\frac{1}{540}\left(\frac{u-u[n]}{h}\right)^6+\frac{1}{25200}\left(\frac{u-u[n]}{h}\right)^8\right)f[n+2]h^4+\left(\frac{-115}{8064}\left(\frac{u-u[n]}{h}\right)-\frac{3769}{241920}\left(\frac{u-u[n]}{h}\right)^2+\frac{7}{1920}\left(\frac{u-u[n]}{h}\right)^3+\frac{1}{192}\left(\frac{u-u[n]}{h}\right)^4-\frac{1}{5760}\left(\frac{u-u[n]}{h}\right)^5-\frac{7}{17280}\left(\frac{u-u[n]}{h}\right)^6+\frac{1}{40320}\left(\frac{u-u[n]}{h}\right)^7+\frac{1}{80640}\left(\frac{u-u[n]}{h}\right)^8\right)f[n+3]h^4 \quad (13)$$

Evaluating par (12) and (13) at some favored points of $u_{n+j},\ j=1,2,3$ will formulate the block-Milne's device as

$$y[u\_]=\alpha_0 y[n]+\alpha_1 y[n-1]+\alpha_2 y[n-2]+\alpha_3 y[n-3]+h^4(\beta_0 f[n]+\beta_1 f[n-1]+\beta_2 f[n-2]+\beta_3 f[n-3]+\beta_4 f[n-4]), \quad (14)$$

$$y[u\_]=\alpha_0 y[n]+\alpha_1 y[n-1]+\alpha_2 y[n-2]+\alpha_3 y[n-3]+h^4(\beta_0 f[n-1]+\beta_1 f[n-3]+\beta_2 f[n+1]+\beta_3 f[n+2]+\beta_4 f[n+3]), \quad (15)$$

where $\alpha_j, j=0,\ldots 4$ and $\beta_j, j=0,1\ldots 4$ are known values of the block-Milne's device. Consider Abell and Braselton (2009), Akinfenwa et al. (2013), and Oghonyon et al. (2016) for more details.

To set up the programming codes of block-Milne's device, $q-step$ predictor method and $q-1-step$ corrector technique are used as predictor-corrector pair having the same order. A combination of Ascher and Petzold (1998), Dormand (1996), Faires and Burden (2012), Lambert (1973,1991), and Oghonyon et al. (2015,2016) has been established from existing literatures such that it is more proficient to estimate the principal local truncation error of the parallel processing predictor-corrector pair without finding the estimate of higher derivatives of $y(x)$. Suppose that $\tilde{p}=\bar{p}$, where $\bar{p}$ and $\tilde{p}$ proves the order of the predictor and corrector method. Now, for a method of order $\tilde{p}$, the $q-step$ block predictor method is examined to produce the principal local truncation errors as

$$\begin{aligned}\tilde{C}^{[1]}_{p+9}h^{p+9}y^{(p+9)}(\tilde{x}_n)&=y(x_{n+1})-y^{[r_1]}_{n+1}+O(h^{p+10}),\\ \tilde{C}^{[2]}_{p+9}h^{p+9}y^{(p+9)}(\tilde{x}_n)&=y(x_{n+2})-y^{[r_2]}_{n+2}+O(h^{p+10}),\\ \tilde{C}^{[3]}_{p+9}h^{p+9}y^{(p+9)}(\tilde{x}_n)&=y(x_{n+3})-y^{[r_3]}_{n+1}+O(h^{p+10}).\end{aligned} \quad (16)$$

Similar analysis of the $q-1-step$ block corrector method will bring forth the principal local truncation errors as

$$\begin{aligned}\bar{C}^{[1]}_{p+9}h^{p+9}y^{(p+9)}(\bar{x}_n)&=y(x_{n+1})-y^{[l_1]}_{n+1}+O(h^{p+10}),\\ \bar{C}^{[2]}_{p+9}h^{p+9}y^{(p+9)}(\bar{x}_n)&=y(x_{n+2})-y^{[l_2]}_{n+2}+O(h^{p+10})\\ \bar{C}^{[3]}_{p+9}h^{p+9}y^{(p+9)}(\bar{x}_n)&=y(x_{n+3})-y^{[l_3]}_{n+3}+O(h^{p+10}),\end{aligned} \quad (17)$$

where $\tilde{C}^{[1]}_{p+9}, \tilde{C}^{[2]}_{p+9}, \tilde{C}^{[1]}_{p+9}$, $\bar{C}^{[1]}_{p+9}, \bar{C}^{[2]}_{p+9}$ and $\bar{C}^{[3]}_{p+9}$ exist as different component irrespective of the step-size h and $y(x)$ is seen as the solution of the differential coefficient gratifying the initial condition $y(x_n)\approx y_n$.

To advance further, the assumption for small values of h is accepted as

$$y^{(9)}(\tilde{x}_n) \approx y^{(9)}(\bar{x}_n),$$

and thus, the derivation of the convergence limit and execution of the programming codes depends on this precondition.

Simplifying the expression of (16) and (17) above, in continuous manner, removing terms of degree $O(h^{p+10})$, it turns out the computed principal local truncation errors of block-Milne's device can be realized as

$$\bar{C}^{[1]}_{p+9} h^{p+9} y^{(p+9)}(\bar{x}_n) \approx \frac{\bar{C}^{[1]}_{p+9}}{\bar{C}^{[1]}_{p+9} - \bar{C}^{[1]}_{p+9}} \left[y^{[r_1]}_{n+1} - y^{[l_1]}_{n+1} \right] < \delta_1,$$
$$\bar{C}^{[2]}_{p+9} h^{p+9} y^{(p+9)}(\bar{x}_n) \approx \frac{\bar{C}^{[2]}_{p+9}}{\bar{C}^{[2]}_{p+9} - \bar{C}^{[2]}_{p+9}} \left[y^{[r_2]}_{n+2} - y^{[l_2]}_{n+2} \right] < \delta_2, \quad (18)$$
$$\bar{C}^{[3]}_{p+9} h^{p+9} y^{(p+9)}(\bar{x}_n) \approx \frac{\bar{C}^{[3]}_{p+9}}{\bar{C}^{[3]}_{p+9} - \bar{C}^{[3]}_{p+9}} \left[y^{[r_3]}_{n+3} - y^{[l_3]}_{n+3} \right] < \delta_3.$$

Noting that $y^{[r_1]}_{n+1} \neq y^{[l_1]}_{n+1}$, $y^{[r_2]}_{n+2} \neq y^{[l_2]}_{n+2}$ and $y^{[r_3]}_{n+3} \neq y^{[l_3]}_{n+3}$ are given values of the block-predictor-corrector brought forth y the block Mine's device of order p, although $\bar{C}^{[1]}_{p+9} h^{p+9} y^{(p+9)}(\bar{x}_n)$, $\bar{C}^{[2]}_{p+9} h^{p+9} y^{(p+9)}(\bar{x}_n)$ and $\bar{C}^{[3]}_{p+9} h^{p+9} y^{(p+9)}(\bar{x}_n)$ represents the different the principal local truncation errors. δ_1, δ_2 and δ_3 are the bounds of the convergence criteria of block-Milne's device.

Still, the estimates of the principal local truncation error (18) are employed to determine whether to accept the results of the current step or to repeat the step with a smaller variable step-size. This process is veritably satisfactory on try-out expressed in (18) as quoted in (18) (Ascher and Petzold, 1998; Dormand, 1996; Faires and Burden, 2012; Lambert, 1973; Lambert, 1991; Oghonyon et al., 2015; Oghonyon et al., 2016). The principal local truncation errors par (18) is called the convergence criteria of block-Milne, differently referred to as block-Milne's device (estimate) for adjusting to convergence.

3. Results and discussion

In this section, the programming codes show the effectiveness of the block-Milne's device for solving fourth order ODEs. The completed result provided was obtained with the assistance of Mathematica 9 Kernel on Microsoft windows (64-bit). See appendix for P-CB-MD1 and P-CB-MD2. The terminology used is named below:

Problem examined: Two problems were examined and worked out enforcing P-CB-MD employing several distinguishable convergence limit 10^{-8}, 10^{-9}, 10^{-10}, 10^{-11}, 10^{-12} and 10^{-15}.

Problem 1:
$y''''(x) = x, \quad y(0) = 0,\ y'(0) = 1, y''(0) = 0, y'''(0) = 0.$

Exact solution:

$$y(x) = \frac{x^2}{120} + x.$$

Problem 2:

$$y''''(x) = \frac{-(8+25x+30x^2+12x^3+x^4)}{(1+x^2)}$$
$, y(0) = 0,\ \ y'(0) = 1,\ y''(0) = 0,$
$y'''^{(0)} = -3.$

Exact Solution:

$$y(x) = x(1-x)e^x.$$

Table 1 and Table 2 presents the completed results of the examined problem 1 and 2 employing P-CB-MD compared with existing methods. The terminology defined on Table 1 and Table 2 is presented below:

- P-CB-MD: error in P-CB-MD (Programming Codes of Block-Milne's Device for Solving Fourth Order Ordinary Differential Equations).
- C_{cr}: convergence criteria.
- Mth: method used.
- Max_{error}: the magnitude of the maximum errors of P-CB-MD.
- AAFO-SPIBM: error in AAFO-SPIBM (An Accurate Five Off-Step Points Implicit Block Method for Direct Solution of Fourth Order Differential Equations) for examine problem 2 as sited in (Duromola, 2016).
- AS-SCMM: error in AS-SCMM (A Six-Step Continuous Multistep Method for the Solution of General Fourth Order Initial Value Problems of ODEs) for examines problems 1 and 2 as discoursed in (Awoyemi et al., 2015).
- DBP-CMS: error in DBP-CM (Direct Block-Predictor-Corrector Method for the Solution of General Fourth Order ODEs) Initial Value Problems of Fourth Order ODEs) for examine problem 1 as seen in (Olabode and Alabi, 2013).
- DSIVP: error in DSIVP (Direct Solution of Initial Value Problems of Fourth Order Ordinary Differential Equations Using Modified Implicit Hybrid Block Method) for examined problem 2 (Kayode et al., 2014).
- NS-SNM: error in NS-SNM (New Seven-Step Numerical Method for Direct Solution of Fourth Order Ordinary Differential Equations) for examined problem 1 and 2 as cited (Omar and Kuboye, 2016).
- SSBM: error in SSBM (A Six Step Block Method for Solution of Fourth Order Ordinary Differential Equations) for examined problem 2 (Mohammed and Tech, 2010).
- S-SS: error in S-SS (A Six-Step Scheme for the Solution of Fourth Order Ordinary Differential Equations) for examined problem 2 (Olabode, 2009).

Table 1: The completed results of the examined problem 1

Method	$\text{Max}_{\text{error}}$	C_{cr}
AS-SCMM	1.2977E − 09	10^{-9}
DBP-CMS	2.020367E − 09	
P-CB-MD	6.94412E − 10	
P-CB-MD	5.60978E − 10	10^{-9}
P-CB-MD	4.48954E − 10	
NS-SNM	4.06899E − 11	10^{-11}
P-CB-MD	4.18463E − 12	
P-CB-MD	4.06899E − 12	10^{-11}
P-CB-MD	4.30495E − 12	

Table 2: The completed results of the examined problem 2

Method	$\text{Max}_{\text{error}}$	C_{cr}
SSBM	2.600E − 08	10^{-8}
P-CB-MD	4.45746E − 09	
P-CB-MD	4.84151E − 09	10^{-8}
P-CB-MD	5.25158E − 09	
S-SS	2.42999998E − 10	10^{-9}
P-CB-MD	4.25111E − 10	
P-CB-MD	4.30802E − 10	10^{-9}
P-CB-MD	4.36554E − 10	
AAFO-SPIBM	2.024E − 10	10^{-10}
DSIVP	2.865E − 10	
P-CB-MD	4.5646E − 11	
P-CB-MD	4.65995E − 11	10^{-10}
P-CB-MD	4.75799E − 11	
NS-SNM	4.75997E − 12	10^{-12}
P-CB-MD	5.75134E − 12	
P-CB-MD	5.76773E − 12	10^{-12}
P-CB-MD	5.79042E − 12	
AS-SCMM	1.55431E − 15	10^{-15}
P-CB-MD	5.1955E − 16	
P-CB-MD	6.96491E − 16	10^{-15}
P-CB-MD	8.56086E − 16	

The defining steps for evaluating the maximum errors and determining the convergence limit are set below:

$$C_{P+4}h^{P+4}y^{(P+4)}(x_n) \cong \frac{C_{P+4}}{C^*_{P+4}-C_{P+4}}\left|\bar{P}_{n+k}-\bar{C}_{n+k}\right| < \varepsilon$$

Noting that $C^*_{P+4} \neq C_{P+4}$ and $\bar{P}_{n+k} \neq \bar{C}_{n+k}$. C^*_{P+4} and C_{P+4} are autonomous of h.

Where C^*_{P+4} and C_{P+4} are the computes of the principal local truncation error of the predictor and corrector method. $\bar{P}_{n+j}$ and $\bar{C}_{n+j}$ are called the predicted and corrected approximations estimations permitted by the method of order p.

Table 1 and Table 2 displays the computational results for calculating the examined problems in the previous section applying P-CB-MD.

Step by step algorithm: A framework of step by step algorithm to design afresh h and valuing maximum errors of P-CB-MD, if the mode is run times, if the mode is run m times.

- Item #1: Prime h.
- Item #2: Order of the multiprocessing predictor-corrector approach must be equal.
- Item #3: Step number of multiprocessing predictor approach must be unit step more eminent compare to multiprocessing corrector approach.
- Item #4: Posit main local truncation errors of both predictor-corrector approach.
- Item #5: Set convergence limit.
- Item #6: Insert multiprocessing predictor-corrector approach in whatever mathematical programming-language.
- Item #7: Employ Taylor's series method to bring forth starting-out economic measures if called for, else disregard item #7 and advance to item #8.
- Item #8: Execute P-CB-MD together with the main local truncation errors.
- Item #9: If item 8# diverges, reiterate procedure once more and split up h into 2 parts from item #1 or else, continue item #10.
- Item 10#: Valuing maximum calculated errors only when convergence has been attained.
- Item #11: Write maximum calculated errors.
- Item #12: Utilize formula expressed infra to formulate a new step size since converge is reached.

$$wh = \left|\frac{\varepsilon}{2(C^*_{P+4}-C_{P+4})}\right|^{\frac{100}{4000}}$$

4. Conclusion

The completed results displayed on Table 1 and Table 2 declared that the P-CB-MD is attained with the support of the convergence limit and a suited/changing step size. Nevertheless, these factors introduce an alternative to decide either to accept or reject result. This terminal result also demonstrate the functioning of the P-CB-MD were discovered to obtain an improve maximum errors than AAFO-SPIBM, AS-SCMM, DBP-CMS, DSIVP, NS-SNM, SSBM, S-SS at all convergence criteria as cited in Awoyemi et al. (2015), Duromola (2016), Kayode et al. (2014), Mohammed (2010), Olabode (2009), Olabode and Alabi (2013), Omar and Kuboye (2016). Thusly, it will be concluded that the P-CB-MD formulated is worthy for solving fourth order ODEs.

Acknowledgement

The generators would ilk to express gratitude to Covenant University for furnishing financial backing through grants during the study period.

References

Abell ML and Braselton JP (2009). Mathematica by example. 4th Edition. Elsevier, USA.

Adesanya AO, Momoh AA, Alkali AM, and Tahir A (2012). Five steps block method for the solution of fourth order ordinary differential equations. International Journal of Engineering Research and Applications, 2(4): 991-998.

Akinfenwa AO, Jator SN, and Yao NM (2013). Continuous block backward differentiation formula for solving stiff ordinary differential equations. Computers and Mathematics with Applications, 65(7): 996-1005.

Anake TA and Adoghe LO (2013). A four point integration method for the solutions of IVP in ODE. Australian Journal of Basic and Applied Sciences, 7(10): 467-473.

Anake TA, Adesanya AO, Oghonyon GJ, and Agarana MC (2013). Block algorithm for general third order ODEs. Icastor Journal of Mathematical Sciences, 7(2): 127-136.

Ascher UM and Petzold LR (1998). Computer methods for ordinary differential equations and differential algebraic equations. SIAM, Philadelphia, USA.

Awoyemi DO, Kayode SJ, and Adoghe LO (2014). A five-step p-stable method for the numerical integration of third order ODEs. American Journal of Computational Mathematics, 4(3): 119-126.

Awoyemi DO, Kayode SJ, and Adoghe LO (2015). A six-step continuous multistep method for the solution of general fourth order IVP of ODEs. Journal of Natural Sciences Research, 5(5): 131-138.

Dormand JR (1996). Numerical methods for differential equations: A computational approach. Vol. 3, CRC Press, USA.

Duromola MK (2016). An accurate five off-step points implicit block method for direct solution of fourth order differential equations. Open Access Library Journal, 3(6): 1-14.

Faires JD and Burden RL (2012). Initial-value problems for ODEs, variable step-size multistep methods. Dublin City University, Dublin, Republic of Ireland.

Gear CW (1971). Numerical initial value problems in ODEs (Automatic Computation). Prentice-Hall, New Jersey, USA.

Ibrahim ZB, Othman KI, and Suleiman M (2007). Implicit r-point block backward differentiation formula for solving first-order stiff ODEs. Applied Mathematics and Computation, 186(1): 558-565.

Jain MK, Iyengar SRK, and Jain RK (2007). Numerical methods for scientific and engineering computation. 5th Edition, New Age International, New Delhi, India.

Kayode SJ (2008). An order six zero-stable method for direct solution of fourth order ODEs. American Journal of Applied Sciences, 5(11): 1461-1466.

Kayode SJ, Duromola MK, and Bolaji B (2014). Direct solution of initial value problems of fourth order ODEs using modified implicit hybrid block method. Journal of Scientific Research and Reports, 3(21): 2792-2800.

Lambert JD (1973). Computational methods in ordinary differential equations. John Wiley and Sons, New York, USA.

Lambert JD (1991). Numerical methods for ordinary differential systems, the initial value problem. John Wiley and Sons, New York, USA.

Majid ZA and Suleiman M (2008). Parallel direct integration variable step block method for solving large system of higher order ordinary differential equations. World Academy of Science, Engineering and Technology, 2(4): 269-273.

Majid ZA and Suleiman MB (2007). Implementation of four-point fully implicit block method for solving ordinary differential equations. Applied Mathematics and Computation, 184(2): 514-522.

Mehrkanoon S, Majid ZA, and Suleiman M (2010). A variable step implicit block multistep method for solving first-order ODEs. Journal of Computational and Applied Mathematics, 233(9): 2387-2394.

Mohammed U and Tech M (2010). A six step block method for solution of fourth order ordinary differential equations. The Pacific Journal of Science and Technology, 11(1): 259-265.

Oghonyon JG, Okunuga SA, and Iyase SA (2016). Milne's implementation on block predictor-corrector methods. Journal of Applied Sciences, 16(5): 236-241.

Oghonyon JG, Okunuga SA, Omoregbe NA, and Olasunmbo OO (2015). Adopting a variable step size approach in implementing implicit block multi-step method for non-stiff ODEs. Journal of Engineering and Applied Sciences, 10(7): 174-180.

Olabode BT (2009). A six-step scheme for the solution of fourth order ordinary differential equations. The Pacific Journal of Science and Technology, 10(1): 143-148.

Olabode BT and Alabi TJ (2013). Direct block predictor-corrector method for the solution of general fourth order ODEs. Journal of Mathematical Research, 5(1): 26-33.

Omar Z and Kuboye JO (2016). New seven-step numerical method for direct solution of fourth order ordinary differential equations. Journal of Mathematical and Fundamental Sciences, 48(2): 94-105.

Zarina BI, Khairil IO, and Suleimann M (2007). Variable step block backward differentiation formula for solving first-order stiff ODEs. In the World Congress on Engineering, London, UK, 2: 2-6.