Coupled Method for Solving Time-Fractional Navier-Stokes Equation

S. O. Edeki, and G. O. Akinlabi

Abstract— This paper witnesses the coupling of two basic transforms: the He-Laplace transform (HLT) which is a blend of Laplace transformation and Homotopy perturbation methods and the fractional complex transform (FCT). This coupling technique is applied for the solutions of the time-fractional Navier-Stokes model equation. Two examples are considered in demonstrating the effectiveness of the coupled technique. The exact solutions of the solved problems are obtained with less computational work, while still maintaining high level of accuracy with little knowledge of fractional calculus being required. Thus, the proposed method is recommended for handling linear and nonlinear fractional models arising in pure and applied sciences.

Keywords— Fractional complex transform; Analytical solutions; Laplace transform; HPM; Navier-Stokes model.

I. Introduction

In applied sciences, Navier-Stokes equations (NSEs) act as vital models used in describing the physics of many phenomena of scientific and engineering interest. They have wider applications in modelling of weather, ocean currents, water flow in a pipe and air flow around a wing. These NSEs establish the connection between pressure and external forces acting on fluid to the response of the fluid flow [1]. In general, we consider the time-fractional NSE of the form:

$$\begin{cases} \dfrac{\partial^\alpha w}{\partial t^\alpha} + \left(\underline{w}\bullet\nabla\right)\underline{w} = -\rho^{-1}\nabla P + v\nabla^2 \underline{w} \\ \nabla\bullet\underline{w} = 0 \end{cases} \tag{1.1}$$

where w is the flow velocity, $\underline{w}$ is the velocity, v is the kinematics viscosity, P is the pressure, t is the time, ρ is the density, and ∇ is a del operator. For a one dimensional motion of a viscous fluid in a tube; the equations of motion governing the flow field in the tube are Navier-Stokes equations in cylindrical coordinates [1, 2]. These are denoted by:

$$\frac{\partial^\alpha w}{\partial t^\alpha} - P = v\left(\frac{\partial^2 w}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial w}{\partial \eta}\right),\ \ w(\eta,0) = g(\eta) \tag{1.2}$$

where $P = -\dfrac{\partial P}{\rho \partial z}$.

In relation to stochastic dynamics, it appears a tradition to represent the solutions of partial differential equations associated with the Navier-Stokes models as the expected functionals of stochastic processes [3, 4]. In that regard, a coupled forward-backward stochastic differential system (FBSDS) is formulated in spaces of fields for the incompressible Navier-Stokes equation in the whole space [3].

Providing solutions (numerical or exact) to linear and nonlinear differential equations has led to the development and adoption of direct and semi-analytical methods [5-7]. A lot of semi-analytical, analytical, and approximate methods have been proposed in literature [10-30].

Fractional Complex Transform (FCT) transforms fractional order differential equations to integer differential equations with the help of Riemann-Liouville derivatives [31-33]. FCT as a solution method for fractional differential equations (FDEs) was first proposed by [34]. The notion of Jumarie's fractional derivative is introduced as follows before the overview of FCT.

In this work, our aim is to provide analytical solutions to the NSEs using the He-Laplace method which combines the basic features of the Laplace transform and those of He's polynomials method.

II. The Overview of the He-LapLace Method [35, 36]

Let Ξ be an integral or a differential operator on the equation of the form:

$$\Xi(\Im) = 0. \tag{2.1}$$

Let $H(\Im, p)$ be a convex homotopy defined by:

$$H(\Im, p) = p\Xi(\Im) + (1-p)G(\Im) \tag{2.2}$$

where $G(\Im)$ is a functional operator with $\Im_0$ is a known solution. Thus, we have:

$$H(\Im,0) = G(\Im) \text{ and } H(\Im,1) = \Xi(\Im) \tag{2.3}$$

whenever $H(\Im, p) = 0$ is satisfied, and $p \in (0,1]$ is an embedded parameter. In HPM, p is used as an expanding parameter to obtain:

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$$\Im = \sum_{j=0}^{\infty} p^j \Im_j = \Im_0 + p\Im_1 + p^2\Im_2 + \cdots . \quad (2.4)$$

From (2.4) the solution is obtained as $p \to 1$. The convergence of (2.4) as $p \to 1$ has been considered in [25].

The method considers $N(\Im)$ (the nonlinear term) as:

$$N(\Im) = \sum_{j=0}^{\infty} p^j H_j \quad (2.5)$$

where H_k's are the so-called He's polynomials, which can be computed using:

$$H_i(\Im_0, \Im_1, \Im_2, \Im_3, \cdots, \Im_i) = \frac{1}{i!}\frac{\partial^i}{\partial p^i}\left(N\left(\sum_{j=0}^{i} p^j \Im_j\right)\right)_{p=0}, \; n \ge 0 . \quad (2.6)$$

A. The He-Laplace Method

Let $\hbar(y', y, x) = f(x)$ expressed as:

$$y' + p_1 y + p_2 g(y) = g(x), \; y(0) = \beta \quad (2.7)$$

be a first order initial value problem (IVP), where $p_1(x)$ and $p_2(x)$ are coefficient of y and $g(y)$ respectively, $g(y)$ a nonlinear function and $g(x)$ a source term. Suppose we define the Laplace transform (resp. inverse Laplace transform) as $\tilde{L}\{(\cdot)\}$ $\left(resp.\ \tilde{L}^{-1}\{(\cdot)\}\right)$. So the Laplace transform of (2.7) is as follows:

$$\tilde{L}\{y'\} + \tilde{L}\{p_1 y\} + \tilde{L}\{p_2 g(y)\} = \tilde{L}\{g(x)\}. \quad (2.8)$$

Applying linearity property of Laplace transform on (12) yields:

$$\tilde{L}\{y'\} + p_1\tilde{L}\{y\} + p_2\tilde{L}\{g(y)\} = \tilde{L}\{g(x)\}. \quad (2.9)$$

Therefore, by differential property of Laplace transform, (2.9) is expressed as follows:

$$s\tilde{L}\{y\} - y(0) = \tilde{L}\{g(x)\} - \left(p_1\tilde{L}\{y\} + p_2\tilde{L}\{g(y)\}\right) \quad (2.10)$$

$$\Rightarrow \tilde{L}\{y\} = \frac{y(0)}{(s+p_1)} + \frac{1}{(s+p_1)}\left[\tilde{L}\{g(x)\} - p_2\tilde{L}\{g(y)\}\right]. \quad (2.11)$$

Thus, by inverse Laplace transform, (2.11) becomes:

$$y(x) = H(x) + \tilde{L}^{-1}\left(\frac{1}{(s+p_1)}\begin{bmatrix}\tilde{L}\{g(x)\} \\ -p_2\tilde{L}\{g(y)\}\end{bmatrix}\right) \quad (2.12)$$

$$\tilde{L}^{-1}\left(\frac{\beta}{(s+p_1)}\right) = H(x). \quad (2.13)$$

Suppose we the solution $y(x)$ assumes an infinite series, then by convex homotopy, (2.12) can be expressed as:

$$\sum_{i=0}^{\infty} p^i y_i = z(x) + \tilde{L}^{-1}\left(\frac{1}{(s+p_1)}\begin{bmatrix}\tilde{L}\{g(x)\} \\ -p_2 p\tilde{L}\left\{\sum_{i=0}^{\infty} p^i H_i(y)\right\}\end{bmatrix}\right) \quad (2.14)$$

where $g(y) = \sum_{i=0}^{\infty} p^i H_i(y)$ for some He's polynomials H_i, and p an expanding parameter as defined earlier.

B. Jumarie's Fractional Derivative (JFD)

It is noted here that JFD is a modified form of the Riemann-Liouville derivatives [22]. Hence, the definition of JFD and its basic properties as follows:

Let $h(v)$ be a continuous real function of v (not necessarily differentiable), and $D_v^\alpha h = \frac{\partial^\alpha h}{\partial v^\alpha}$ denoting JFD of h, of order α w.r.t. v. Then,

$$D_v^\alpha h = \begin{cases} \dfrac{1}{\Gamma(-\alpha)}\dfrac{d}{dv}\displaystyle\int_0^v (v-\lambda)^{-\alpha-1}\left(h(\lambda)-h(0)\right)d\lambda, \; \alpha \in (-\infty, 0) \\ \dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dv}\displaystyle\int_0^v (v-\lambda)^{-\alpha}\left(h(\lambda)-h(0)\right)d\lambda, \; \alpha \in (0,1) \\ \left(h^{(\alpha-\eta)}(v)\right)^{(\eta)}, \; \alpha \in [\eta, \eta+1), \; \eta \ge 1 \end{cases} \quad (2.15)$$

where $\Gamma(\cdot)$ denotes a gamma function. As summarized in [20], the basic properties of JFD are stated as *P1-P5*:

P1: $D_v^\alpha k = 0, \; \alpha > 0$,

P2: $D_v^\alpha (kh(v)) = kD_v^\alpha h(v), \; \alpha > 0$,

P3: $D_v^\alpha v^\beta = \dfrac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} v^{\beta-\alpha}, \; \beta \ge \alpha > 0$,

P4: $D_v^\alpha (h_1(v)h_2(v)) = D_v^\alpha h_1(v)(h_2(v)) + h_1(v)D_v^\alpha h_2(v)$,

P5: $D_v^\alpha (h(v(g))) = D_v^1 h \cdot D_g^\alpha v$,

where k is a constant.

Note: *P1, P2, P3, P4,* and *P5* are referred to as fractional derivative of: constant function, constant multiple function, power function, product function, and function of function respectively. P5 can be linked to Jumarie's chain rule of fractional derivative.

III. The Fractional Complex Transform and DTM

Here, we briefly introduce the concept of the FCT and the RDTM.

A. The Fractional Complex Transform

Let us consider a general fractional differential equation of the form:

$$f\left(\varpi, D_t^{\alpha}\varpi, D_x^{\beta}\varpi, D_y^{\lambda}\varpi, D_z^{\gamma}\varpi\right)=0,\ \ \varpi=\varpi\left(t,x,y,z\right). \quad (3.1)$$

Then, the Fractional Complex Transform [24] is defined as follows:

$$\begin{cases} T=\dfrac{at^{\alpha}}{\Gamma\left(1+\alpha\right)},\ \alpha\in\left(0,1\right], \\ X=\dfrac{bx^{\beta}}{\Gamma\left(1+\beta\right)},\ \beta\in\left(0,1\right], \\ Y=\dfrac{cy^{\lambda}}{\Gamma\left(1+\lambda\right)},\ \lambda\in\left(0,1\right], \\ Z=\dfrac{dz^{\gamma}}{\Gamma\left(1+\gamma\right)},\ \gamma\in\left(0,1\right], \end{cases} \quad (3.2)$$

where $a,\ b,\ c,$ and d are unknown constants.

From *P3*,

$$D_v^{\alpha}v^{\beta}=\frac{\Gamma\left(1+\beta\right)}{\Gamma\left(1+\beta-\alpha\right)}v^{\beta-\alpha},\ \beta\geq\alpha>0,$$

$$\therefore\ \ D_t^{\alpha}T=D_t^{\alpha}\left[\frac{at^{\alpha}}{\Gamma\left(1+\alpha\right)}\right]=\frac{a}{\Gamma\left(1+\alpha\right)}D_t^{\alpha}t^{\alpha}$$

$$=\frac{a}{\Gamma\left(1+\alpha\right)}\left[\frac{\Gamma\left(1+\alpha\right)}{\Gamma\left(1+\alpha-\alpha\right)}\right]t^{\alpha-\alpha}=a. \quad (3.3)$$

Obviously in a similar manner, using properties *P1-P5*, and the FCT in (2.3), the following are easily obtained:

$$\begin{cases} D_t^{\alpha}T=\dfrac{\partial^{\alpha}T}{\partial t^{\alpha}}=a, \\ D_x^{\beta}X=\dfrac{\partial^{\beta}X}{\partial x^{\beta}}=b, \\ D_y^{\lambda}Y=\dfrac{\partial^{\lambda}Y}{\partial y^{\lambda}}=c, \\ D_z^{\gamma}Z=\dfrac{\partial^{\gamma}Z}{\partial z^{\gamma}}=d. \end{cases} \quad (3.4)$$

Hence,

$$\begin{cases} D_t^{\alpha}\varpi\left(t,x,y,z\right)=D_t^{\alpha}\varpi\left(T\left(t\right)\right)=D_T^{1}\varpi\cdot D_t^{\alpha}T=a\dfrac{\partial\varpi}{\partial T}, \\ D_x^{\beta}\varpi\left(t,x,y,z\right)=D_x^{\beta}\varpi\left(X\left(x\right)\right)=D_X^{1}\varpi\cdot D_x^{\beta}X=b\dfrac{\partial\varpi}{\partial X}, \\ D_y^{\lambda}\varpi\left(t,x,y,z\right)=D_y^{\lambda}\varpi\left(Y\left(y\right)\right)=D_Y^{1}\varpi\cdot D_y^{\lambda}Y=c\dfrac{\partial\varpi}{\partial Y}, \\ D_z^{\gamma}\varpi\left(t,x,y,z\right)=D_z^{\gamma}\varpi\left(Z\left(z\right)\right)=D_Z^{1}\varpi\cdot D_z^{\gamma}Z=d\dfrac{\partial\varpi}{\partial Z}. \end{cases} \quad (3.5)$$

$$D_t^{\alpha}\varpi\left(t,x,y,z\right)=D_t^{\alpha}\varpi\left(T\left(t\right)\right)=D_T^{1}\varpi\cdot D_t^{\alpha}T=a\frac{\partial\varpi}{\partial T}, \quad (3.6)$$

$$D_x^{\beta}\varpi\left(t,x,y,z\right)=D_x^{\beta}\varpi\left(X\left(x\right)\right)=D_X^{1}\varpi\cdot D_x^{\beta}X=b\frac{\partial\varpi}{\partial X}, \quad (3.7)$$

$$D_y^{\lambda}\varpi\left(t,x,y,z\right)=D_y^{\lambda}\varpi\left(Y\left(y\right)\right)=D_Y^{1}\varpi\cdot D_y^{\lambda}Y=c\frac{\partial\varpi}{\partial Y}, \quad (3.8)$$

$$D_z^{\gamma}\varpi\left(t,x,y,z\right)=D_z^{\gamma}\varpi\left(Z\left(z\right)\right)=D_Z^{1}\varpi\cdot D_z^{\gamma}Z=d\frac{\partial\varpi}{\partial Z}. \quad (3.9)$$

IV. Applications:

In this section, the proposed method is applied to time-fractional Navier-Stokes models as follows:

A. *Problem 1:*

Consider the following time-fractional Navier-Stokes model:

$$\begin{cases} \dfrac{\partial^{\alpha}\psi}{\partial t^{\alpha}}=\dfrac{\partial^{2}\psi}{\partial\xi^{2}}+\dfrac{1}{\xi}\dfrac{\partial\psi}{\partial\xi}, \\ \psi\left(\xi,0\right)=\xi. \end{cases} \quad (4.1)$$

Procedure w.r.t Problem 1:

Solution procedure:

By FCT,

$$T=\frac{at^{\alpha}}{\Gamma\left(1+\alpha\right)},$$

which according to section 3 gives $D_t^{\alpha}u=\dfrac{\partial u}{\partial T}$ for $a=1.$

Hence, (4.1) becomes:

$$\begin{cases} \dfrac{\partial w}{\partial T} = \dfrac{\partial^2 w}{\partial \eta^2} + \dfrac{1}{\eta}\dfrac{\partial w}{\partial \eta}, \\ w(\eta, 0) = \eta. \end{cases} \tag{4.2}$$

We take the Laplace transform (LT) of (3.1) as follows:

$$\tilde{L}\left\{\frac{\partial \psi}{\partial T} = \frac{\partial^2 \psi}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial \psi}{\partial \xi}\right\} \tag{4.3}$$

$$\Rightarrow \quad \tilde{L}\{\psi\} = \frac{\psi(0)}{s} + \frac{1}{s}\tilde{L}\left\{\frac{\partial^2 \psi}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial \psi}{\partial \xi}\right\}. \tag{4.4}$$

By applying the inverse Laplace transform, $\tilde{L}^{-1}\{(\cdot)\}$ of $\tilde{L}\{(\cdot)\}$ on both sides of (4.4), we have:

$$w = \begin{cases} \tilde{L}^{-1}\left\{\dfrac{1}{s}\tilde{L}\left\{\dfrac{\partial^2 \psi}{\partial \xi^2} + \dfrac{1}{\xi}\dfrac{\partial \psi}{\partial \xi}\right\}\right\} \\ +\tilde{L}^{-1}\left\{\dfrac{\xi}{s}\right\}, \psi = \psi(\xi, t) \end{cases}$$

$$= \xi + \tilde{L}^{-1}\left\{\frac{1}{s}\tilde{L}\left\{\frac{\partial^2 \psi}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial \psi}{\partial \xi}\right\}\right\}. \tag{4.5}$$

By Convex Homotopy approach (4.5) becomes:

$$\sum_{i=0}^{\infty} p^i \psi_i = \xi + \tilde{L}^{-1}\left\{\frac{1}{s}\tilde{L}\left\{\sum_{i}^{\infty} p^{i+1}\left(\frac{\partial^2 \psi_i}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial \psi_i}{\partial \xi}\right)\right\}\right\}. \tag{4.6}$$

Thus, comparing the coefficients of the p powers in (4.6) gives:

$$p^{(0)} : w_0 = \eta$$

$$p^{(1)} : \psi_1 = \tilde{L}^{-1}\left\{\frac{1}{s}\tilde{L}\left\{\frac{\partial^2 \psi_0}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial \psi_0}{\partial \xi}\right\}\right\},$$

$$p^{(2)} : \psi_2 = \tilde{L}^{-1}\left\{\frac{1}{s}\tilde{L}\left\{\frac{\partial^2 \psi_1}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial \psi_1}{\partial \xi}\right\}\right\},$$

$$p^{(3)} : \psi_3 = \tilde{L}^{-1}\left\{\frac{1}{s}\tilde{L}\left\{\frac{\partial^2 \psi_2}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial \psi_2}{\partial \xi}\right\}\right\},$$

$$p^{(4)} : \psi_4 = \tilde{L}^{-1}\left\{\frac{1}{s}\tilde{L}\left\{\frac{\partial^2 \psi_3}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial \psi_3}{\partial \xi}\right\}\right\},$$

$$p^{(5)} : \psi_5 = \tilde{L}^{-1}\left\{\frac{1}{s}\tilde{L}\left\{\frac{\partial^2 \psi_4}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial \psi_4}{\partial \xi}\right\}\right\},$$

$$\vdots$$

$$p^{(j)} : \psi_j = \tilde{L}^{-1}\left\{\frac{1}{s}\tilde{L}\left\{\frac{\partial^2 \psi_{j-1}}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial \psi_{j-1}}{\partial \xi}\right\}\right\}, j \geq 1.$$

So, for $\psi_0 = \xi$, we have the following:

$$\begin{cases} \psi_0 = \xi, \ \psi_2 = \dfrac{1}{2}\dfrac{T^2}{\xi^3}, \\ \psi_4 = \dfrac{75}{8}\dfrac{T^4}{\xi^7}, \ \psi_6 = \dfrac{19845}{16}\dfrac{T^6}{\xi^{11}},, \cdots \end{cases}, \tag{4.7}$$

$$\begin{cases} \psi_1 = \dfrac{T}{\xi}, \ \psi_3 = \dfrac{3}{2}\dfrac{T^3}{\xi^5}, \\ \psi_5 = \dfrac{735}{8}\dfrac{T^5}{\xi^9}, \ \psi_7 = \dfrac{343035}{16}\dfrac{T^7}{\xi^{13}}, \cdots \ . \end{cases} \tag{4.8}$$

(3.8)

Thus, the solution of (4.2) is as follows:

$$w(\xi, T) = \begin{cases} \xi + \dfrac{T}{\xi} + \dfrac{1}{2}\dfrac{T^2}{\xi^3} + \dfrac{3}{2}\dfrac{T^3}{\xi^5} + \dfrac{75}{8}\dfrac{T^4}{\xi^7} \\ + \dfrac{735}{8}\dfrac{T^5}{\xi^9} + \dfrac{19845}{16}\dfrac{T^6}{\xi^{11}} + \dfrac{343035}{16}\dfrac{T^7}{\xi^{13}} + \cdots \end{cases}$$

$$= \xi + \sum_{\iota=1}^{\infty} \frac{1^1 \times 3^2 \times 5^2 \times \cdots \times (2\iota - 3)^2}{\xi^{2\iota - 1}} \frac{1}{\iota!} T^{\iota} \ . \tag{4.9}$$

Hence, the exact solution of (4.1) is:

$$u(\xi, t) = \xi + \sum_{\iota=1}^{\infty} \frac{1^1 \times 3^2 \times 5^2 \times \cdots \times (2\iota - 3)^2}{\xi^{2\iota - 1}} \frac{1}{\iota!}\left(\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^{\iota} \tag{4.10}$$

Our solution (4.10) is very much in line with those obtained in [2, 23].

B. *Problem 2:*

Consider the following time-fractional Navier-Stokes model:

$$\frac{\partial^{\alpha} w}{\partial t^{\alpha}} = p + \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} \tag{4.11}$$

subject to:

$$w(r, 0) = 1 - r^2 . \tag{4.12}$$

Solution procedure:

By FCT,

$$T = \frac{a t^{\alpha}}{\Gamma(1+\alpha)},$$

which according to section 3 gives $D_t^{\alpha} u = \dfrac{\partial u}{\partial T}$ for $a = 1.$

Hence, (4.5) and (4.6) become:

$$\begin{cases} \dfrac{\partial w}{\partial T} = p + \dfrac{\partial^2 w}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial w}{\partial r}, \\ w(r, 0) = 1 - r^2. \end{cases} \tag{4.13}$$

We take the Laplace transform (LT) of (4.13) as follows:

$$\tilde{L}\left\{\frac{\partial w}{\partial T}=\rho+\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right\} \tag{4.14}$$

$$\Rightarrow\ \tilde{L}\{w\}=\frac{w(0)}{s}+\frac{1}{s}\tilde{L}\left\{\rho+\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right\} \tag{4.15}$$

By applying the inverse Laplace transform, $\tilde{L}^{-1}\{(\cdot)\}$ of $\tilde{L}\{(\cdot)\}$ on both sides of (4.15), we have:

$$w=\begin{cases}\tilde{L}^{-1}\left\{\frac{1}{s}\tilde{L}\left\{\rho+\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right\}\right\}\\ +\tilde{L}^{-1}\left\{\frac{1-r^2}{s}\right\}, \psi=\psi(r,t)\end{cases}$$

$$=\left(1-r^2\right)+\tilde{L}^{-1}\left\{\frac{1}{s}\tilde{L}\left\{\rho+\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right\}\right\} \tag{4.16}$$

By Convex Homotopy Approach (4.16) becomes:

$$\sum_{i=0}^{\infty}p^i w_i=\begin{pmatrix}\left(1-r^2\right)\\ +\tilde{L}^{-1}\left\{\frac{1}{s}\tilde{L}\left\{\sum_{i}^{\infty}p^{i+1}\left(\rho+\frac{\partial^2 w_i}{\partial r^2}+\frac{1}{r}\frac{\partial w_i}{\partial r}\right)\right\}\right\}\end{pmatrix}. \tag{4.17}$$

Thus, comparing the coefficients of the p powers in (4.17) gives:

$$p^{(0)}: w_0=\left(1-r^2\right)$$

$$p^{(1)}: w_1=\tilde{L}^{-1}\left\{\frac{1}{s}\tilde{L}\left\{\left(\rho+\frac{\partial^2 w_0}{\partial r^2}+\frac{1}{r}\frac{\partial w_0}{\partial r}\right)\right\}\right\},$$

$$p^{(2)}: w_2=\tilde{L}^{-1}\left\{\frac{1}{s}\tilde{L}\left\{\left(\rho+\frac{\partial^2 w_1}{\partial r^2}+\frac{1}{r}\frac{\partial w_1}{\partial r}\right)\right\}\right\},$$

$$p^{(3)}: w_3=\tilde{L}^{-1}\left\{\frac{1}{s}\tilde{L}\left\{\left(\rho+\frac{\partial^2 w_2}{\partial r^2}+\frac{1}{r}\frac{\partial w_2}{\partial r}\right)\right\}\right\},$$

$$p^{(4)}: w_4=\tilde{L}^{-1}\left\{\frac{1}{s}\tilde{L}\left\{\left(\rho+\frac{\partial^2 w_3}{\partial r^2}+\frac{1}{r}\frac{\partial w_3}{\partial r}\right)\right\}\right\},$$

$$p^{(5)}: w_5=\tilde{L}^{-1}\left\{\frac{1}{s}\tilde{L}\left\{\left(\rho+\frac{\partial^2 w_4}{\partial r^2}+\frac{1}{r}\frac{\partial w_4}{\partial r}\right)\right\}\right\},$$

$$\vdots$$

$$p^{(j)}: w_j=\tilde{L}^{-1}\left\{\frac{1}{s}\tilde{L}\left\{\left(\rho+\frac{\partial^2 w_{j-1}}{\partial r^2}+\frac{1}{r}\frac{\partial w_{j-1}}{\partial r}\right)\right\}\right\}, j\geq 1.$$

Thus, applying the He-Laplace Method in sections 2.1 and 2.2 to (4.7) gives the solution of (4.7) as:

So, simplifying the process for $w_0=\left(1-r^2\right)$, we have the following:

$$w(r,T)=\left(1-r^2\right)+(p-4)T. \tag{4.18}$$

Hence, the exact solution of (3.1) is:

$$w(r,t)=\left(1-r^2\right)+\frac{(p-4)t^\alpha}{\Gamma(1+\alpha)}. \tag{4.20}$$

Our solution (4.20) is very much in line with those obtained in [2, 23].

Remark: for $p=1$, the solution is:

$$w(r,t)=\left(1-r^2\right)-\frac{3t^\alpha}{\Gamma(1+\alpha)}. \tag{4.21}$$

For $\alpha=1$, and $p=1$ we have $w(r,t)=\left(1-r^2\right)-3t$ as the corresponding exact solution.

Here, we present in Fig. 1 through Fig. 6, the relationship between the exact solutions of the integer cases $\alpha=1$, and the fractional cases for $\alpha\in\mathbb{Q}$ as regards example II.

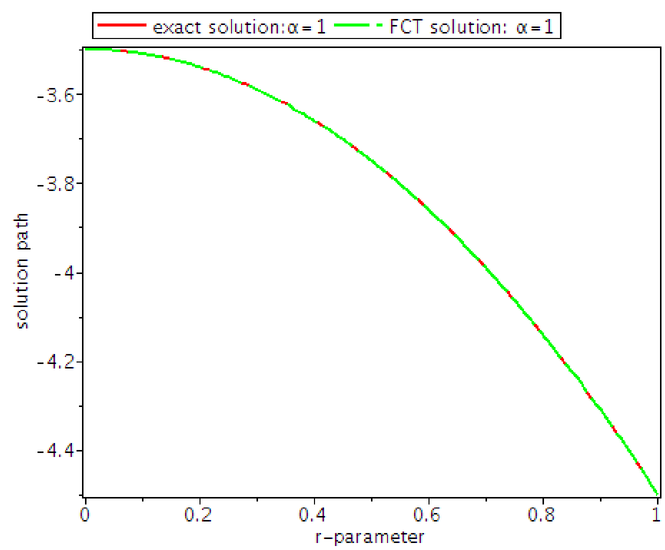

Fig. 1: The solution graph for $t=1.5$ & $r\in[0,1]$

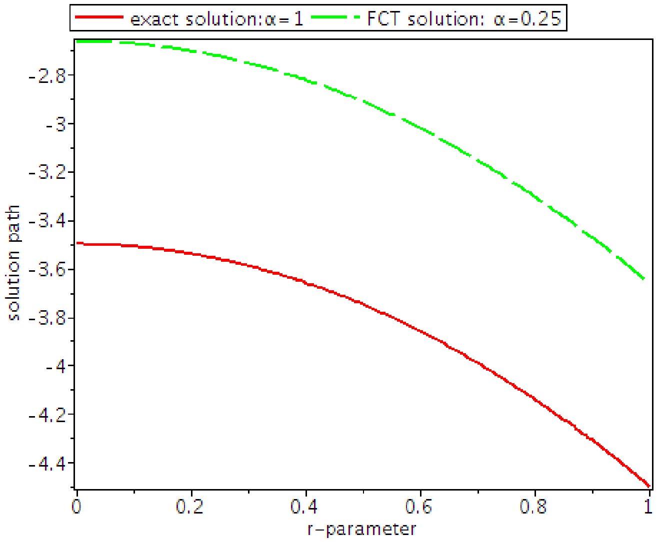

Fig. 2: The solution graph for $t = 1.5$ & $r \in [0,1]$

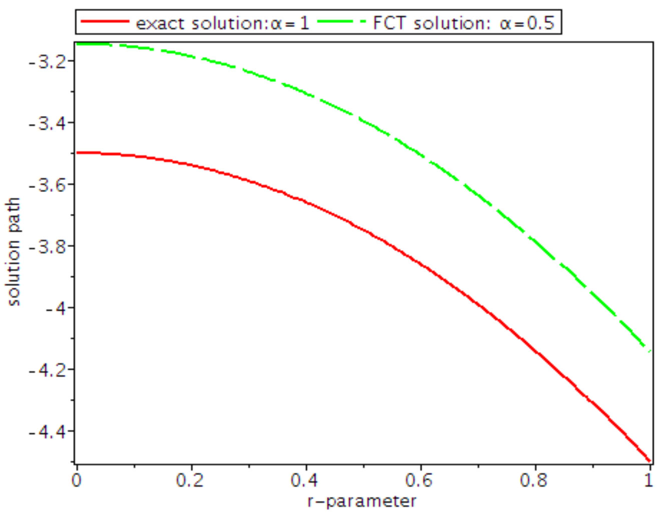

Fig. 3: The solution graph for $t = 1.5$ & $r \in [0,1]$

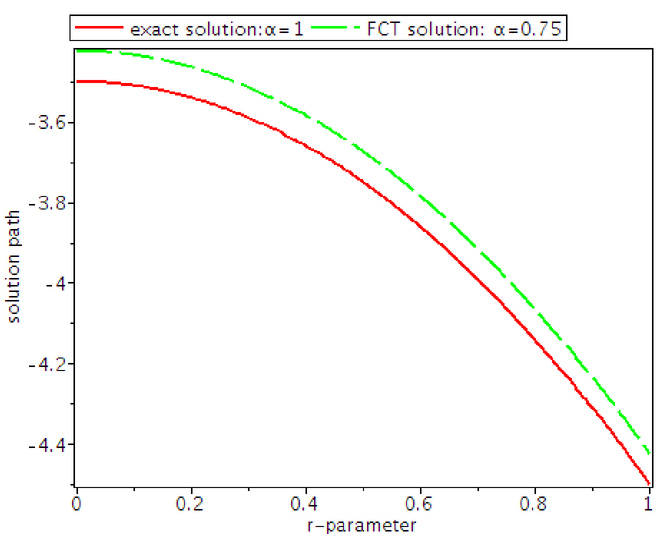

Fig. 4: The solution graph for $t = 1.5$ & $r \in [0,1]$

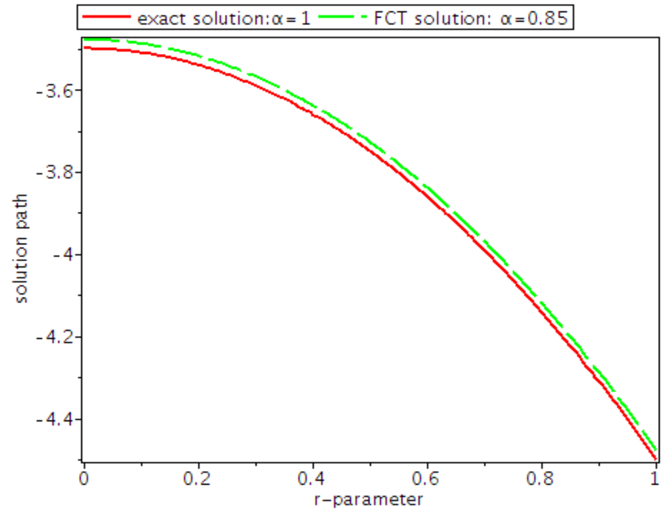

Fig. 5: The solution graph for $t = 1.5$ & $r \in [0,1]$

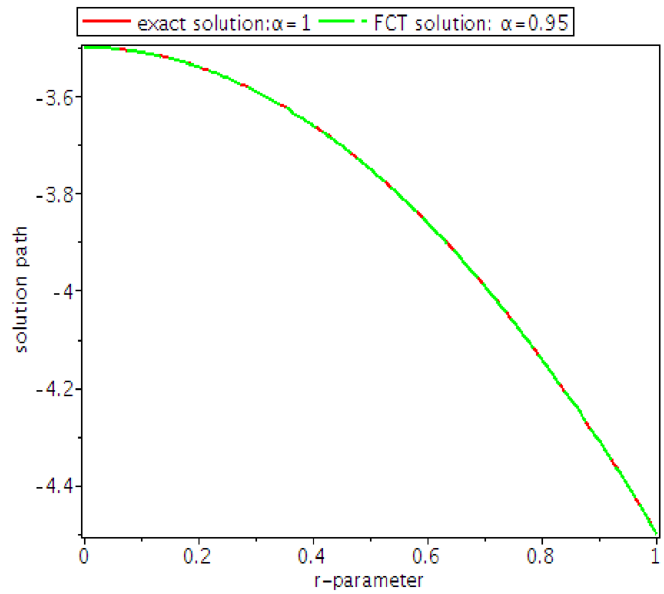

Fig. 6: The solution graph for $t = 1.5$ & $r \in [0,1]$

Remark: In Fig. 7 and Fig. 8, the exact and approximate solutions of problem 1 are displayed respectively. These are considered at $\alpha = 1$.

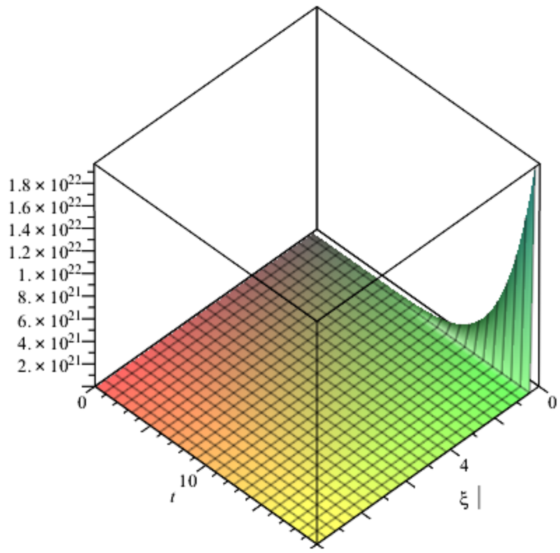

Fig. 7: The exact solution graph for problem 1.

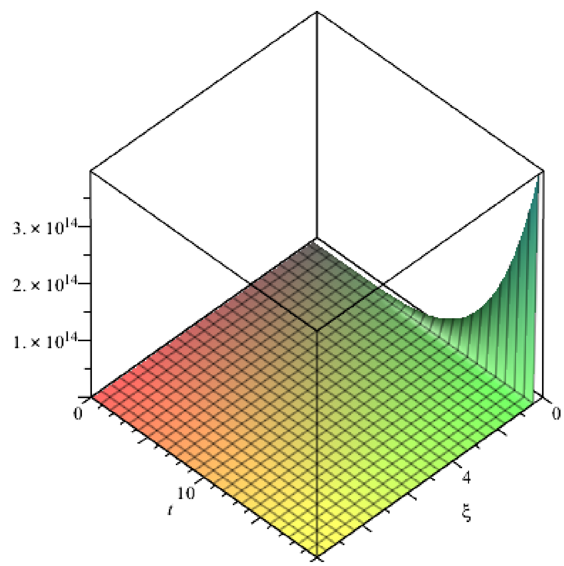

Fig. 8: The approximate solution graph for problem 1

V. Concluding Remarks

In this paper, exact solutions of time-fractional Navier-Stokes model equation were provided in series form with easily computable components. The method of solutions involved the coupling of two basic transforms: the He-Laplace transform (HLT) which is a blend of Laplace transformation and Homotopy perturbation methods and the fractional complex transform (FCT) with the requirement of little knowledge of fractional calculus while still maintaining high

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