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ITERATIVE METHOD FOR CONSTRUCTING ANALYTICAL SOLUTIONS TO THE HARRY-DYM INITIAL VALUE PROBLEM

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Abstract: In this paper, an analytical technique, namely the new iterative method (NIM), is applied to obtain an approximate analytical solution of the nonlinear Harry-Dym equation which is often used in the theory of solitons. The rapid convergence of the method results in qualitatively accurate solutions in relatively few iterations; this is obvious upon comparing the obtained analytical solutions with the exact solutions. Our results indicate that NIM is highly accurate and efficient, therefore can be considered a very useful and valuable method.

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1. Introduction

Nonlinear differential equations play an important role in modelling numerous

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problems in physics, chemistry, biology, economy and engineering science. Many problems can be modelled as systems of differential equations, integral equations, integro-differential equations, partial differential equations, fractional order differential equations. Many of these equations are nonlinear. Recently, Daftardar-Gejji and Jafari [1] proposed a new technique for solving nonlinear functional equations namely: New Iterative Method (NIM). The NIM has been extensively used by many researchers for the treatment of linear and nonlinear ordinary, and partial differential equations of integer and fractional orders, see [2, 3, 4]. The method converges to the exact solution, if it exists through successive approximations. However, for concrete problems, a few number of approximations can be used for numerical purposes with high degree of accuracy. The NIM possesses a great potential in solving different kinds of functional equations. Both linear and nonlinear equations, and systems of such types are all amenable to the method. In the nonlinear case for differential equations and partial differential equations, the method has the advantage of dealing directly with the problem. These equations are solved without transforming them to more simple ones. The method avoids linearization, perturbation, discretization, or any unrealistic assumptions.

Considering the Harry-Dym equation being it nonlinear, most of the semi-analytical methods such as Adomian Decomposition Method (ADM), Differential Transform Method (DTM), Homotopy Perturbation Method (HPM), etc., and their modified versions will require the involvement of Adomian polynomials but this is completely avoided via the NIM, yet high rate of accuracy and convergence is not neglected [5, 6].

In the present work, we will utilize the NIM to solve the Harry-Dym equation [15]. This equation is a nonlinear partial differential equation which is of great importance in terms of applications; for example, in the analysis of the Saffman-Taylor problem with surface tension [16, 17]. We will design an algorithm to solve the Harry-Dym equation subject to some initial conditions. Finally, our proposed solution method is illustrated for effectiveness and reliability by considering the Harry-Dym equation for two different initial conditions based on their usual nature in the existing literature.

Our work is divided in several sections. In the "Basic idea of new iterative method (NIM)" section, we present, in a brief and self-contained manner, the NIM. Some references are given to delve deeper into the subject and to study its mathematical foundation that is beyond the scope of the present work. In "The Harry-Dym equation" section, we give a brief introduction to the model described by the Harry-Dym equation. In the "General solution of the Harry-Dym equation through NIM" section, we establish that NIM can be used to solve this equation as a problem of initial value. After, in "Numerical examples" section, we show by means of two examples, the quality and precision of our method, comparing the obtained results with the only exact solutions available in the literature [19]. Finally, in the "Conclusions" section, we summarize our findings and present our final conclusions.

2. Basic idea of new iterative method

To describe the idea of the NIM, consider the following general functional equation [1, 14]:

$$u = N(u) + f, \tag{1}$$

where N is a nonlinear mapping between Banach spaces such that $N : B \to B$ and f is a known function. We are looking for a solution u of Eq. (1) having the series form

$$u = \sum_{i=0}^{\infty} u_i. \tag{2}$$

The nonlinear operator N can be decomposed as

$$N\Big(\sum_{i=0}^{\infty} u_i\Big) = N(u_0) + \sum_{i=1}^{\infty} \Big[N\Big(\sum_{j=0}^{i} u_j\Big) - N\Big(\sum_{j=0}^{i-1} u_j\Big)\Big]. \tag{3}$$

From (2) and (3), (1) is equivalent to

$$\sum_{i=0}^{\infty} u_i = f + N(u_0) + \sum_{i=1}^{\infty} \Big[N\Big(\sum_{j=0}^{i} u_j\Big) - N\Big(\sum_{j=0}^{i-1} u_j\Big)\Big]. \tag{4}$$

We define the recurrence relation as

$$\begin{cases} u_0 = f, \\ u_1 = N(u_0), \\ u_{m+1} = N\big(\sum_{i=0}^{m} u_i\big) - N\big(\sum_{i=0}^{m-1} u_i\big), \ m = 1, 2, \ldots. \end{cases} \tag{5}$$

Then

$$\sum_{i=0}^{m+1} u_i = N\Big(\sum_{i=0}^{m} u_i\Big) \tag{6}$$

and

$$u = f + \sum_{i=0}^{\infty} u_i. \tag{7}$$

The $m-$term approximate solution of (2) is given by $u = \sum_{i=0}^{m-1} u_i$.

It may be observed here that in this decomposition method, computation of complicated quantities is not required and therefore the computing time is minimal.

If the operator N is a contraction, i.e.,

$$\|N(x) - N(y)\| \leq k\|x - y\|, \quad 0 < k < 1,$$

then:

$$\begin{aligned}\|u_{m+1}\| &= \|N(u_0 + \cdots + u_m) - N(u_0 + \cdots + u_{m-1})\| \\ &\leq k\|u_m\| \leq \cdots \leq k^m\|u_0\|, \quad m = 0, 1, 2, \ldots,\end{aligned}$$

and the series $\sum_{i=0}^{\infty} u_i$ absolutely and uniformly converges to a solution of (1), which is unique, in view of the Banach fixed point theorem [13].

This method decreases considerably the volume of calculations. The decomposition procedure given by NIM will be easily set, without linearising the problem. In this approach, the solution is found in the form of a convergent series with easily computed components; in many cases, the convergence of this series is very fast and only a few terms are needed in order to have an idea of how the solutions behave. For more details about convergence, we refer the reader to [14].

3. The Harry-Dym equation

In physics-mathematics, and in particular in the theory of solitons, the Harry-Dym equation is the third-order nonlinear partial differential equation given as:

$$\frac{\partial u}{\partial t} = u^3 \frac{\partial^3 u}{\partial x^3}. \tag{8}$$

The Harry-Dym (HD) equation has nonlinearity and dispersion coupled together. It was discovered by H. Dym in 1973-1974 while its first appearance in the literature occurred in a 1975 paper of Kruskal [15], where it was named after its discoverer. The Harry-Dym equation has strong links to the Korteweg-de Vries equation and the solitons theory, [18].

To make the description of the problem complete, we will consider some initial conditions for (8):

$$u(x, 0) = f(x). \tag{9}$$

In the next section, we will develop an algorithm using the method described in Section 2 in order to solve the nonlinear Harry-Dym equation (8) without resort to any truncation or linearization.

4. General solution of the Harry-Dym equation through NIM

Integrating (8) and considering the initial condition (9), we obtain

$$u(x,t) = \int_0^t u^3(x,s)u_{xxx}(x,s)ds + f(x). \tag{10}$$

Comparing (10) with (1), we have that the nonlinear term is given by

$$N(u) = \int_0^t u^3(x,s)u_{xxx}(x,s)ds. \tag{11}$$

By using (5) through the NIM, we obtain the following, recursively:

$$\begin{cases} u_0(x,t) = f(x), \\ u_1(x,t) = \int_0^t u_0^3(x,s)u_{0,xxx}(x,s)ds, \\ u_{m+1}(x,t) = \sum_{i=0}^{m} \big(\int_0^t u_i^3(x,s)u_{i,xxx}(x,s)ds \big) \\ -\sum_{i=0}^{m-1} \big(\int_0^t u_i^3(x,s)u_{i,xxx}(x,s)ds \big), \ m = 1,2,\dots. \end{cases} \tag{12}$$

Finally, the exact solution of the Harry-Dym equation (8) is given by

$$u(x,t) = f(x) + \sum_{i=0}^{\infty} u_i(x,t). \tag{13}$$

The $m-$term approximate solution of (8) is given by $u = \sum_{i=0}^{m-1} u_i$.

Using the expressions obtained above for (8), we will illustrate, with two examples, the effectiveness of NIM to solve the nonlinear Harry-Dym equation.

5. Numerical examples

To validate the present iterative approach, results are compared with the exact solution of the Harry-Dym equation provided by Mokhtari in [19] which can be expressed as

$$u(x,t) = \Big(a - \frac{3\sqrt{b}}{2}(x+ct) \Big)^{2/3}, \tag{14}$$

where a, b and c are constants.

In all examples the package of *Mathematica Version 11.0* has been used to solve the test problems.

Example 1

Using the NIM, we solve the Harry-Dym equation subject to the initial condition $f(x) = (1-3x)^{\frac{2}{3}}$.

Following the recurrent formula (12) we have:
$u_0(x,t) = (1-3x)^{\frac{2}{3}}$,
$u_1(x,t) = \int_0^t u_0^3(x,s) u_{0,xxx}(x,s) ds = -\frac{8t}{\sqrt[3]{1-3x}}$,
$u_2(x,t) = \frac{16t^2\left(7168t^3+3040t^2(3x-1)+400t(1-3x)^2+5(3x-1)^3\right)}{5(1-3x)^{13/3}}$, and

$$u_3(x,t) = K \left(\begin{array}{c} 1233637173587190579126599968t^{18} \\ +193719587414863520628473856t^{17}(3x-1) \\ +136875836595891063698227200t^{16}(1-3x)^2 \\ +56790926743757769911828480t^{15}(3x-1)^3 \\ +15013293579608519564328960t^{14}(1-3x)^4 \\ +2491321806795244547604480t^{13}(3x-1)^5 \\ +203312399629040505323520t^{12}(1-3x)^6 \\ -1229024414934478356480 0t^{11}(3x-1)^7 \\ -59436035529770532864 00t^{10}(1-3x)^8 \\ -77152195104952647680 0t^9(3x-1)^9 \\ -3454490952563589120 0t^8(1-3x)^{10} \\ +398055730474321920 0t^7(3x-1)^{11} \\ +79277044861414400 0t^6(1-3x)^{12} \\ +5355122803430400 0t^5(3x-1)^{13} \\ -2672902003200 0t^4(1-3x)^{14} \\ -292556876612000t^3(3x-1)^{15} \\ -23265676434000t^2(1-3x)^{16} \\ -737782670625t(3x-1)^{17} \\ -3233230000(1-3x)^{18} \end{array} \right)$$

where $K = \frac{256t^3}{606230625(1-3x)^{61/3}}$. Then, the approximate solution for this example is given by $u_{NIM} = u_0 + u_1 + u_2 + u_3$.

In Figure 1 we plot both the approximate solution and the exact solution for Harry-Dym equation. The approximate solution appears under the exact solution but as it can be observed, the approximate solution, obtained using NIM, converges to the exact solution in such a way that it becomes difficult to distinguish them. All the numerical work and the graphics was accomplished with the *Mathematica* software package.

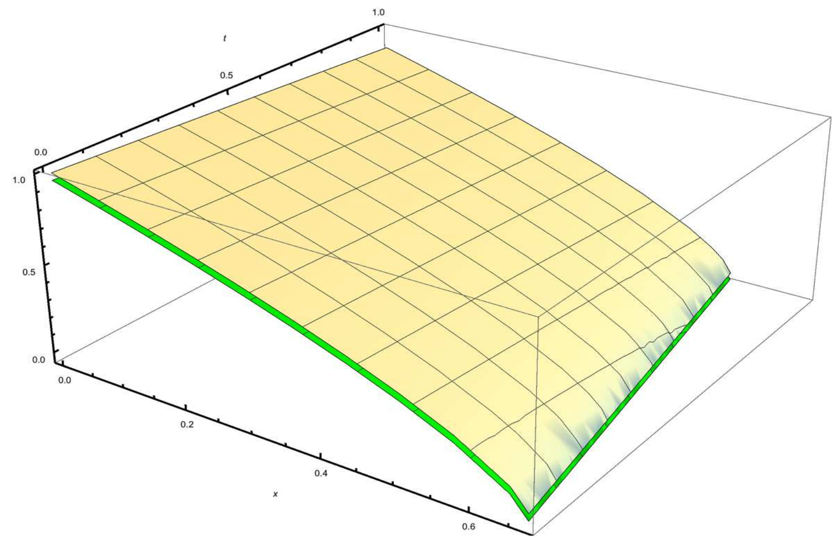

Figure 1: Plot of $u_{NIM}(x,t)$ (below) and $u_{ex}(x,t)$ (above) for nonlinear Harry-Dym equation with the initial condition $f(x) = (1-3x)^{\frac{2}{3}}$ corresponding to Example 1.

	$t = 0.01$			$t = 0.02$		
x	u_{NIM}	u_{ex} [19]	Error	u_{NIM}	u_{ex} [19]	Error
0.00	0.9985997498	0.9989997498	0.00000000	0.9975989986	0.9975989986	0.00000000
0.05	0.9479266016	0.9479266234	2.101×10^{-8}	0.9468994402	0.9468995371	9.691×10^{-8}
0.10	0.8958611352	0.8958612311	9.590×10^{-8}	0.8948045348	0.8948053108	7.701×10^{-7}
0.15	0.8422363927	0.8422364367	4.400×10^{-8}	0.8411466591	0.8411472554	5.963×10^{-7}
0.20	0.7868468658	0.7868479856	1.119×10^{-6}	0.7857194085	0.7857196522	2.437×10^{-7}
0.25	0.7294343591	0.7294356781	1.319×10^{-6}	0.7282633449	0.7282644128	1.067×10^{-6}
0.30	0.6696662665	0.6696674318	1.165×10^{-6}	0.6684440750	0.6684452319	1.156×10^{-6}
0.35	0.6071005221	0.6071017108	1.188×10^{-6}	0.6058168437	0.6058271731	1.032×10^{-5}
0.40	0.5411254648	0.5411266729	1.208×10^{-6}	0.5397657013	0.5397768201	1.111×10^{-5}
0.45	0.4708480035	0.4708493592	1.355×10^{-6}	0.4693901583	0.4694231902	3.303×10^{-5}
0.50	0.3948612702	0.3948624809	1.210×10^{-6}	0.3932690771	0.3933698128	1.007×10^{-4}

Table 1: For $t = 0.01$ and $t = 0.02$, related to Example 1

Example 2

Using the NIM, we solve this Harry-Dym equation subject to the initial condition $f(x) = (2 - \frac{9}{2}x)^{\frac{2}{3}}$.

Again, using the recursive formula (12) we have

$$u_0(x,t) = \left(2 - \frac{9x}{2}\right)^{2/3},$$

$$u_1(x,t) = \int_0^t u_0^3(x,s)u_{0,xxx}(x,s)ds = -\frac{27t}{\sqrt[3]{2 - \frac{9x}{2}}},$$

$$\begin{aligned}
u_2(x,t) = {} & \frac{803538792\sqrt[3]{2}t^5}{5(4-9x)^{13/3}} - \frac{40389516\sqrt[3]{2}t^4}{(4-9x)^{13/3}} + \frac{90876411\sqrt[3]{2}t^4x}{(4-9x)^{13/3}} \\
& + \frac{15943230\sqrt[3]{2}t^3x^2}{(4-9x)^{13/3}} + \frac{3149280\sqrt[3]{2}t^3}{(4-9x)^{13/3}} - \frac{14171760\sqrt[3]{2}t^3x}{(4-9x)^{13/3}} \\
& + \frac{531441t^2x^3}{2^{2/3}(4-9x)^{13/3}} - \frac{354294\sqrt[3]{2}t^2x^2}{(4-9x)^{13/3}} - \frac{23328\sqrt[3]{2}t^2}{(4-9x)^{13/3}} \\
& + \frac{157464\sqrt[3]{2}t^2x}{(4-9x)^{13/3}} - \frac{177147\sqrt[3]{2}tx^4}{(4-9x)^{13/3}} + \frac{314928\sqrt[3]{2}tx^3}{(4-9x)^{13/3}} \\
& - \frac{209952\sqrt[3]{2}tx^2}{(4-9x)^{13/3}} + \frac{27t}{\sqrt[3]{2 - \frac{9x}{2}}} - \frac{6912\sqrt[3]{2}t}{(4-9x)^{13/3}} + \frac{62208\sqrt[3]{2}tx}{(4-9x)^{13/3}},
\end{aligned}$$

$$
\begin{aligned}
u_3(x,t) = H\Big[& 41755659739294094842089537332425843723468 8t^{18} \\
& + 97139902865718901195138854210851789217792 \\
& \times t^{17}(9x-4) \\
& + 10168271724530912624176064009104155340800 \\
& \times t^{16}(4-9x)^2 \\
& + 6250223386879307853593198018881259174 40 \\
& \times t^{15}(9x-4)^3 \\
& + 2447872470342202312748498215022942976 0 \\
& \times t^{14}(4-9x)^4 \\
& + 601781548616684359018704784199659635t^{13}(9x-4)^5 \\
& + 72756053146549639761388415025904 80t^{12}(4-9x)^6 \\
& - 65157138446771921494923637272900t^{11}(9x-4)^7 \\
& - 4668179566361326613796010289400t^{10}(4-9x)^8 \\
& - 89772275536125677895558883725t^{9}(9x-4)^9 \\
& - 5954896719423477989742033 00t^{8}(4-9x)^{10} \\
& + 101655359747060009890749 75t^{7}(9x-4)^{11} \\
& + 29993702623714945390275 0t^{6}(4-9x)^{12} \\
& + 300156871406258166300 0t^{5}(9x-4)^{13} \\
& - 221951517731658000t^{4}(4-9x)^{14} \\
& - 3598998126471247 50t^{3}(9x-4)^{15} \\
& - 4240169530096500t^{2}(4-9x)^{16} \\
& - 19920132106875t(9x-4)^{17} \\
& - 12932920000(4-9x)^{18}\Big],
\end{aligned}
$$

where $H = \frac{6561t^3}{202076875\ 2^{2/3}(4-9x)^{61/3}}$. Then, the approximate solution for this example it is given by $u_{NIM} = u_0 + u_1 + u_2 + u_3$.

In Figure 2 we plot both the approximate solution and the exact solution for Harry-Dym equation. The approximate solution appears under the exact solution but as it can be observed, the approximate solution, obtained using NIM, converges to the exact solution in such a way that it becomes difficult to distinguish them.

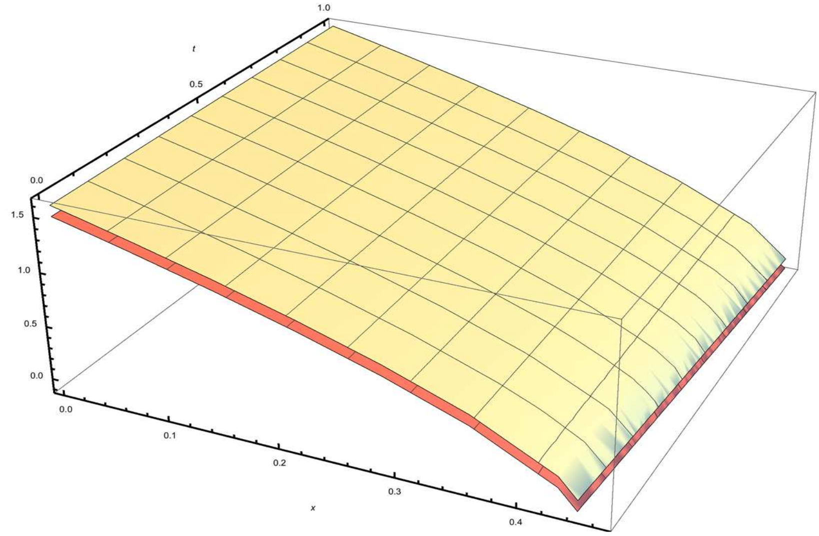

Figure 2: Plot of $u_{NIM}(x,t)$ (below) and $u_{ex}(x,t)$ (above) for nonlinear Harry-Dym equation with the initial condition $f(x) = (2 - \frac{9}{2}x)^{\frac{2}{3}}$ corresponding to Example 2.

From Tables 1 and 2, we can conclude that the difference between the exact and the obtained NIM approximate solution is very small. This fact tells us about the effectiveness and accuracy of the NIM method.

	$t = 0.01$			$t = 0.03$		
x	u_{NIM}	u_{ex} [19]	Error	u_{NIM}	u_{ex} [19]	Error
0.00	1.5872772408	1.5872772408	0.00000000	1.5872296181	1.5872296181	0.00000000
0.05	1.4658693758	1.4658693180	5.780×10^{-8}	1.4658198203	1.4658198762	5.590×10^{-8}
0.10	1.3392047001	1.3392042319	4.682×10^{-7}	1.3391528541	1.3391539312	1.077×10^{-6}
0.15	1.2062335848	1.2062337188	3.341×10^{-7}	1.2061789559	1.2061897250	1.076×10^{-5}
0.20	1.0654731746	1.0654739281	7.535×10^{-7}	1.0654150493	1.0654421775	2.712×10^{-5}
0.25	0.9146950617	0.9146976701	2.608×10^{-6}	0.9146323286	0.9146783221	4.599×10^{-5}
0.30	0.7502355547	0.7502464282	1.087×10^{-5}	0.7501662866	0.7502991445	1.328×10^{-4}
0.35	0.5651344572	0.5651469427	1.248×10^{-5}	0.5650546481	0.5657982318	7.435×10^{-4}
0.40	0.3418438881	0.3418669251	2.303×10^{-5}	0.3417412741	0.3429892145	1.247×10^{-3}

Table 2: For $t = 0.01$ and $t = 0.03$, related to Example 2

6. Conclusions

Very few exact solutions of the nonlinear Harry-Dym equation were known in the literature. In this work, we have obtained accurate approximate solutions for the Harry-Dym nonlinear partial differential equation using the new iterative method (NIM); thereby illustrating in this way, the use of NIM in the solution of nonlinear partial differential equations. We have chosen the Harry-Dym equation due to its importance in the theory of solitons. In order to show the accuracy and efficiency of our method, we have solved two examples, comparing our results with the exact solution of the equation that was obtained in [19]. The obtained results demonstrate the reliability of the algorithm and its wider applicability to nonlinear partial differential equations.

We therefore, conclude that the NIM is a notable non-sophisticated powerful tool that produces high quality approximate solutions for nonlinear partial differential equations using simple calculations and that attains converge with only few terms.

References

[1] V. Daftardar-Gejji, H. Jafari, An iterative method for solving nonlinear functional equations, *J. Math. Anal. Appl.*, **316** (2006), 753-763.

[2] S. Bhalekar, V. Daftardar-Gejji, Solving a system of nonlinear functional equations using revised new iterative method, *World Acad. Sci. Eng. Tech.*, **6** (2012), 968-972.

[3] S. Bhalekar, V. Daftardar-Gejji, New iterative method: application to partial differential equations, *Applied Math. and Comput.*, **203** (2008), 778-783.

[4] V. Daftardar-Gejji, S. Bhalekar, Solving fractional boundary value problems with Dirichlet boundary conditions using a new iterative method, *Comput. & Math. with Applications*, **59** (2010), 1801-1809.

[5] O. González-Gaxiola, J. Ruiz de Chávez, J.A. Santiago, A nonlinear option pricing model through the Adomian decomposition method, *Int. J. Appl. Comput. Math.*, **2** (2016), 453-467.

[6] S.O. Edeki, O.O. Ugbebor, O. González-Gaxiola, Analytical solutions of the Ivancevic option pricing model with a nonzero adaptive market potential, *Int. J. of Pure and Applied Math.*, **115** (2017), 187-198.

[7] S. Abbasbandy, Improving Newton-Raphson method for nonlinear equations by modified Adomian decomposition method, Applied Math. and Comput., **145** (2003), 887-893.

[8] C. Chun, Iterative methods improving Newton's method by the decomposition method, *Comput. & Math. with Applications*, **50** (2005), 1559-1568.

[9] J.H. He, A new iteration method for solving algebraic equations, *Applied Math. and Comput.*, **135** (2003), 81-84.

[10] J.H. He, Newton-like iteration method for solving algebraic equations, *Commun. Nonlinear Sci. Numer. Simul.*, **3** (1998), 106-109.

[11] E. Babolian, J. Biazar, Solution of nonlinear equations by modified adomian decomposition method, *Applied Math. and Comput.*, **132** (2002), 167-172.

[12] R. Luck, J.W. Stevens, Explicit solutions for transcendental equations, *SIAM Review*, **44** (2002), 227-233.

[13] E. Kreyszig, *Introductory Fuctional Analysis with Applications*, John Wiley & Sons, New York (1978).

[14] S. Bhalekar, V. Daftardar-Gejji, Convergence of the new iterative method, *Int. J. of Differential Equations*, **2011** (2011), Article ID 989065.

[15] M.D. Kruskal, Nonlinear wave equations, In: *Dynamical Systems, Theory and Applications*, Springer, Heidelberg (1975), 310-354.

[16] L.P. Kadanoff, Exact solutions for the Saffman-Taylor problem with surface tension, *Phys. Rev. Lett.*, **65** (1990), 2986-2988.

[17] G.L. Vasconcelos, L.P. Kadanoff, Stationary solutions for the Saffman-Taylor problem with surface tension, *Phys. Rev. A*, **44** (1991), 6490-6495.

[18] W. Hereman, P.P. Banerjee, M.R. Chatterjee, Derivation and implicit solution of the Harry-Dym equation and its connections with the Korteweg-de Vries equation, *J. Phys. A: Math. Gen.*, **22** (1989), 241-255.

[19] R. Mokhtari, Exact solutions of the Harry-Dym equation, *Commun. Theor. Phys.*, **55** (2011), 204-208.