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IMPULSIVE DIFFERENTIAL SYSTEM WITH VARIABLE TIMES

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Abstract

Results on mild solution of nonclassical ordinary differential equations (NODE) with variable times and impulsive conditions are studied. The moments of impulsive effect depend on the solution.

1. Introduction

Extensive study has been carried out on impulsive systems with fixed moments. See the references [3, 5, 6, 8]. For quantum stochastic differential equations (QSDEs), few results on analytical properties of solutions of QSDEs with fixed moments were established in [2, 3]. The applications and importance of systems with impulsive effects cannot be over emphasized especially when dealing with systems that exhibit abrupt changes due to small perturbations [2, 3]. In recent times, the study of impulsive differential systems with variable times has been of interest to some researchers. Notable amongst these are the works of [1, 7, 9, 11]. It is on this note we study the

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existence of solution of impulsive QSDE with variable times. This is to further enrich the qualitative theory of QSDE. We proceed as follows: Section 2 will show the preliminaries on which we intend to build the major result on and the major results will be discussed in Section 3.

2. Preambles

We study the following quantum stochastic differential equations also known as the nonclassical ordinary differential equation (NODE);

$$\langle \eta, x(t)\xi \rangle = P(t, x(t))(\eta, \xi), \; t \in I = [0, T], \; t \neq \tau_i(x(t)), \; i = 1, ..., n,$$

$$\langle \eta, x(t^+)\xi \rangle = I_i(x(t)(\eta, \xi)), \; t = \tau_i(x(t)), \; i = 1, ..., n,$$

$$x(t) = \psi(t), \quad t \in [-r, 0]. \tag{1}$$

Let the map $P(t, x)(\eta, \xi) \in A(\eta, \xi)$ be a stochastic process, where $A = \{\phi : [-r, 0] \to \tilde{\mathcal{A}} : \phi$ is continuous except for some t, where $\phi(t^-)$, $\phi(t^+)$ exist and $\phi(t^-) = \phi(t)\}$, $\phi \in A$, $0 < r < \infty$, $\tau_k \in C(\tilde{\mathcal{A}}, \mathbb{R})$, $k = 1, ..., m$ are given functions. Define $\| x(s) \|_{\eta\xi} = \| x(t + s) \|_{\eta\xi}$, $s \in [-r, 0]$ to be a function in A for $x \in [-r, T]$, $J_i : \tilde{\mathcal{A}} \to \tilde{\mathcal{A}}$, $\tau_i : \tilde{\mathcal{A}} \to \mathbb{R}$. The space $C(I, \tilde{\mathcal{A}})$ is a Banach space introduced in [2, 3, 11]. Its norm is given by

$$\| x \|_{\eta\xi} := \sup\{| x(t)(\eta, \xi) | : t \in I\}$$

and A is equipped with a similar norm for $t \in [-r, 0]$.

Define the space $\mathcal{A}$ as follows: Let $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T$, such that, $t_i = \tau_i(x(t_i))$, $x(t_i^-)$ and $x(t_i^+)$ with $x(t_i^-) = x(t_i)$, $i = 1, ..., n$. Then

$$\mathcal{A} = \{x : [-r, T] \to \tilde{\mathcal{A}}, \; x \in C(t_i, t_{i+1}], \tilde{\mathcal{A}})\}.$$

Definition 1. The map $(t, x) \to P(t, x)(\eta, \xi)$ satisfies Caratheodory conditions if:

(a) $P(\cdot, y)(\eta, \xi)$ is computable for each $y \in A$;

(b) $P(t, \cdot)(\eta, \xi)$ is continuous for almost all $t \in [0, T]$;

(c) there exist computable functions $h_{\eta\xi, q} : [0, T] \to \mathbb{R}_+$ so that

$$|P(t, y)(\eta, \xi)| \leq h_{\eta\xi, q}(t), \quad \forall \| y \|_{\eta\xi} \leq q \text{ and a.e. } t \in [0, T].$$

We use the Schaefer's fixed-point theorem. See [4, 10].

Definition 2. A *stochastic process* $x \in \mathcal{A}$ is a solution of (1) if it satisfies (1) a.e. on $I = [0, T]$, $t \neq \tau_i(x(t))$, $i = 1, ..., n$ and $x(t^+) = J_i(x(t))$, $t = \tau_i(x(t))$, $x(t) = \psi(t)$, $t \in [-r, 0]$.

The following conditions will be used to establish some major results:

(S_1) The function $\tau_i \in C^1(\tilde{\mathcal{A}}, \mathbb{R})$, $i = 1, ..., n$, $0 < \tau_1(x) < \cdots < t_n(x) < T$, $\forall x \in \tilde{\mathcal{A}}$.

(S_2) Let $\| J_i(x) \|_{\eta\xi} \leq l_i$, $i = 1, ..., n$, $x \in \tilde{\mathcal{A}}$, l_i are constants.

(S_3) There exist functions:

(i) $W : [0, \infty) \to (0, \infty)$ which is continuous and nondecreasing.

(ii) $M_{\eta\xi} : I \to \mathbb{R}_+$ which is measurable, such that $|P(t, y)(\eta, \xi)| \leq M_{\eta\xi}(t) W(\| y \|_{\eta\xi})$, for a.e. $t \in I$, $y \in A$ with $\int_1^\infty \frac{ds}{W(s)} = \infty$.

(S_4) $\left\langle \frac{d}{dt} \langle \eta, \tau_i(x)\xi \rangle, P(t, x)(\eta, \xi) \right\rangle \neq 1 \ \forall (t, x) \in [0, T] \times \tilde{\mathcal{A}}$.

(S_5) $\tau_i(J_i(x)) \leq \tau_i(x) \leq \tau_{i+1}(J_i(x))$, for $i = 1, ..., n$ and $\forall x \in \tilde{\mathcal{A}}$.

3. Major Results

Theorem. *Assume that hypothesis* (S_1)-(S_4) *hold*. *Then the problem* (1) *has at least a solution on* $[-r, T]$.

Proof. To apply the fixed point method, the problem (1) is transformed as follows: define the map $N : C([-r, T], \tilde{\mathcal{A}}) \to C([-r, T], \tilde{\mathcal{A}})$ by

$$N(x)(t)(\eta, \xi) = \| \psi(t) \|_{\eta\xi}, \quad t \in [-r, 0]$$

and

$$N(x)(t)(\eta, \xi) = \| \psi(0) \|_{\eta\xi} + \int_0^t P(s, x(s))(\eta, \xi) ds, \quad t \in [0, T].$$

We establish the result in stages:

Stage 1. N is continuous. Let the stochastic processes $\{x_n\}$ be a sequence, and let $x_n \to x$ in $C([-r, T], \tilde{\mathcal{A}})$. Then

$$| N(x_n)(t)(\eta, \xi) - N(x)(t)(\eta, \xi) | \leq \int_0^t | P(x, x_n(s))(\eta, \xi) - P(s, x(s))(\eta, \xi) | ds$$

$$\leq \int_0^T | P(s, x_n(s))(\eta, \xi) - P(s, x(s))(\eta, \xi) | ds.$$

By Lebesgue dominated convergence theorem,

$$\| N(x_n)(t) - N(x)(t) \|_{\eta\xi, \infty} \leq \| P(t, x_n(t)) - P(t, x(t)) \|_{\eta\xi} \to 0 \text{ as } n \to \infty.$$

Stage 2. N maps bounded sets into bounded sets in $C([-r, T], \tilde{\mathcal{A}}]$. For $q > 0,$ we have a constant $m \geq 0$ such that

$$x \in A_q = \{x \in C([-r, T], \tilde{\mathcal{A}}) : \| x(t) \|_{\eta\xi} \leq q\},$$

and $\| N(x) \|_{\eta\xi} \leq m.$ Therefore by (S_3),

$$N(x)(t)\ (\eta, \xi) \leq \| \psi(0) \|_{\eta\xi} + \int_0^s | P(s, x(s))(\eta, \xi) | ds \leq \| \phi \|_{\eta\xi} + \| h_{\eta\xi, q} \|.$$

Thus, $\| N(x) \|_{\eta\xi} \leq \| \phi \|_{\eta\xi} + | h_{\eta\xi, q} | := m.$

Stage 3. N maps bounded sets into equicontinuous sets of $C([-r, r], \tilde{\mathcal{A}}).$ Let $t_1, t_2 \in [0, T], t_1 < t_2,$ and A_q be as defined above. Then

$$\| N(x)(t_2) - N(x)(t_1) \|_{\eta\xi} \leq \int_{t_1}^{t_2} h_{\eta\xi, q}(s) ds \to 0 \text{ as } t_2 \to t_1.$$

The same holds for $t_1 < t_2 \leq 0$ and $t_1 \leq 0 \leq t_2.$ Hence, by Stages 1-3 and Arzela-Ascoli theorem, we deduce that the map N is completely continuous.

Stage 4. Define $\Gamma(N) := \{x \in C([-r, T], \tilde{\mathcal{A}}) : x = \rho N(x), 0 < \rho < 1\}$.

We claim that the set $\Gamma(N)$ is bounded: Let $x \in \Gamma(N)$. Then, $x = \rho N(x)$ for some $0 < \rho < 1$. For $t \in [0, T]$,

$$x(t)(\eta, \xi) = \rho\left(\phi_{\eta\xi}(0) + \int_0^t P(s, x(s))(\eta, \xi)ds\right).$$

Therefore by (S_2) and (S_3), we get

$$|x(t)(\eta, \xi)| \le \|\phi\|_{\eta\xi} + \int_0^t M_{\eta\xi, q}(s)W(\|x\|_{\eta\xi})ds, \quad t \in I. \tag{2}$$

Define a function g by $g(t) = \sup\{|x_{\eta\xi}(s)| : -r \le s \le t\}, 0 \le t \le T$.

Let $t^* \in [-r, t]$ be such that $g_{\eta\xi}(t) = |x_{\eta\xi}(t^*)|$. If $t^* \in [0, T]$, then by (2), we have

$$g_{\eta\xi}(t) \le \|\phi\|_{\eta\xi} + \int_0^t M_{\eta\xi}(s)W(g_{\eta\xi}(s))ds, \quad t \in [0, T]. \tag{3}$$

If $t^* \in [-r, 0]$, then $g_{\eta\xi}(t) = \|\phi\|_{\eta\xi}$ and (3) holds. Denote the R.H.S. of (3) by $v_{\eta\xi}(t)$. Then $v_{\eta\xi}(0) = \|\phi\|_{\eta\xi}$, $g_{\eta\xi}(t) \le v_{\eta\xi}(t)$, $t \in [0, 1]$ and

$$\frac{d}{dt}\langle \eta, v(t)\xi\rangle = M_{\eta\xi}(t)W(v_{\eta\xi}(t)), \text{ a.e. } t \in [0, T].$$

By W, $\frac{d}{dt}\langle \eta, v(t)\xi\rangle \le M_{\eta\xi}(t)W(v_{\eta\xi}(t))$. For $t \in [0, T]$, we get

$$\int_{v(0)}^{v(t)} \frac{ds}{W(s)} \le \int_0^T M_{\eta\xi}(s)ds < \int_{v(0)}^{\infty} \frac{ds}{W(s)}.$$

Let L be a constant so that $v_{\eta\xi}(t) \le L$, $t \in [0, T]$ and $g_{\eta\xi}(t) \le L$, $t \in [0, T]$. Let L^* be another constant depending on T and the functions $M_{\eta\xi}$, W. Since $\|x(t)\|_{\eta\xi} \le g_{\eta\xi}(t)$, $\|x(t)\|_{\eta\xi} \le L^* = \max\{\|\phi\|_{\eta\xi}, L\}$. Thus $\Gamma(N)$ is bounded, and hence a fixed point exists for N.

Next we show that the above stages are true for the case when the moments of impulsive effect depend on the solution. Denote the solution obtained by $\langle \eta, x_1 \xi \rangle$. Define

$$\gamma_{i,1}(t) = \langle \eta, \tau_i(x_1)\xi \rangle - t, \quad t \geq 0.$$

Remark. We remark that $\mathbb{C} \equiv \mathbb{R}^2$, hence $x_{\eta\xi}(t)$ is real valued.

By (S_1), $\gamma_{i,1}(0) \neq 0$ on $[0, T]$, that is $t \neq \langle \eta, \tau_i(x_1(t))\xi \rangle$ on $[0, T]$, $i = 1, ..., n$. Then $\langle \eta, x_1\xi \rangle$ is a solution of (1). Suppose $\gamma_{1,1}(t) = 0$, $t \in [0, T]$. Now if $\gamma_{1,1}(t) \neq 0$, due to its continuity, we find $t_1 > 0$ so that $\gamma_{1,1}(t_1) = 0$ and $\gamma_{1,1}(t) \neq 0$, $t \in [0, t_1)$. Thus, by (S_1), we get $\gamma_{i,1}(t) \neq 0$ for all $t \in [0, t_1)$, $i = 1, ..., n$.

Stage 5. Assume the following problem:

$$\langle \eta, x(\tau)\xi \rangle = \langle \eta, x_1(t)\xi \rangle, \quad t \in [t_1 - \gamma, t_1],$$

$$\frac{d}{dt}\langle \eta, x(t)\xi \rangle = P(t, x(t))(\eta, \xi) \text{ a.e. } t \in [t_1, T],$$

$$\langle \eta, x(t_1^+)\xi \rangle = \langle \eta, J_1(x_1(t_1))\xi \rangle. \tag{4}$$

By transforming (4) into a fixed point problem as follows, define $N_1 : C([t_1 - \gamma, T], \tilde{\mathcal{A}}) \to C([t_1 - \gamma, T], \tilde{\mathcal{A}})$ by

$$N_1(x(t)(\eta, \xi)) = \begin{cases} \langle \eta, x_1(t)\xi \rangle, & \text{if } t \in [t_1 - \gamma, t_1], \\ \langle \eta, J_1(x(t_1))\xi \rangle + \int_{t_1}^{t} P(s, x(s))(\eta\xi, q)ds, & \text{if } t \in [t_1, T]. \end{cases}$$

Again we show that N_1 is completely continuous as in previous section and that the set $\Upsilon(N_1) := \{x \in C([t_1 - \gamma, T], \tilde{\mathcal{A}}) : x = \rho N_1(x), 0 < \rho < 1\}$ is bounded.

Now define $B := C([t_1 - \gamma, T], \tilde{\mathcal{A}})$. By Schaefer's theorem, the desired result to problem (4) is obtained.

Again, let $x_{\eta\xi, 2}$ be a solution. Define $\gamma_{i,2}(t) = \tau_i(x_{\eta\xi, 2}(t)) - t$, for

$t \geq t_1$. If $\gamma_{i,2}(t) \neq 0$, on $(t_1, T]$ and for all $i = 1, \ldots, n$, then

$$x_{\eta\xi}(t) = \begin{cases} x_{\eta\xi,1}(t), & \text{if } t \in [0, t_1], \\ x_{\eta\xi,2}(t), & \text{if } t \in (t_1, T] \end{cases}$$

is a solution of equation (1). Now, when $\gamma_{2,2}(t) = 0$, for some $t \in (t_1, T]$, we obtain by (S_5),

$$\gamma_{2,2}(t_1^+) = \tau_2(x_{\eta\xi,2}(t_1^+)) = \tau_2(x_{\eta\xi,2}(t_1^+)) - t_1$$

$$= \tau_2(J_1(x_{\eta\xi,1}(t_1))) - t_1 > \tau_1(x_{\eta\xi,1}(t_1)) - t_1 = r_{1,1}(t_1) = 0.$$

Since $\gamma_{2,2}$ is continuous, we have $t_2 > t_1$, such that $\gamma_{2,2}(t_2) = 0$ and $\gamma_{2,2}(t) \neq 0$, for all $t \in (t_1, t_2)$. Now by (S_4), $\gamma_{i,2}(t) \neq 0$, for all $t \in (t_1, t_2)$. Let $\bar{s} \in (t_1, t_2]$ be such that $\gamma_{1,2}(t) = 0$. By (S_5), we get

$$\gamma_{1,2}(t_1^+) = \tau_1(x_{\eta\xi,2}(t_1^+)) - t_1 = \tau_1(J_1(x_{\eta\xi,1}(t_1))) - t_1$$

$$\leq \tau - 1(x_{\eta\xi,1}(t_1)) - t_1 = \gamma_{1,1}(t_1) = 0.$$

Thus $\gamma_{1,2}$ attains a non-negative maximum at some point $s_1 \in (t_1, T]$. Since $\frac{d}{dt}\langle \eta, x_2(t)\xi \rangle = P(t, x_2(t))(\eta, \xi)$, we have

$$\frac{d}{dt}\gamma_{1,2}(s_1) = \frac{d}{dt}\tau_1(x^2(s_1)) = \frac{d}{dt}\langle \eta, x_2(s_1)\xi \rangle - 1 = 0.$$

Therefore, we have the following, which contradicts (S_4):

$$\left\langle \frac{d}{ds_1}\tau_1(x_{\eta\xi,2}(s_1)), P(s-1, x_2(s_1))(\eta, \xi) \right\rangle = 1.$$

Stage 6. Continuing and letting $x_{n+1} = x|_{[t_n, T]}$ be a solution,

$$\langle \eta, x(t)\xi \rangle = \langle \eta, x_n(t)\xi \rangle, \quad t \in [t_n - r, t_n],$$

$$\frac{d}{dt}\langle \eta, x(t)\xi \rangle = P(t, x(t))(\eta, \xi), \text{ a.e. } t \in (t_n, T],$$

$$x(t_n^+)(\eta, \xi) = J_n(x_{n-1}(t_n))(\eta, \xi). \tag{5}$$

The solution $\langle \eta, x(t)\xi \rangle$ of (1) is then defined by

$$\langle \eta, x(t)\xi \rangle = \begin{cases} \langle \eta, x_1(t)\xi \rangle, & \text{if } t \in [-\gamma, t_1], \\ \langle \eta, x_2(t)\xi \rangle, & \text{if } t \in (t_1, t_2], \\ \vdots & \\ \langle \eta, x_{n+1}(t)\xi \rangle, & \text{if } t \in (t_n, T]. \end{cases}$$

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