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Some fixed point theorems in ordered partial metric spaces with applications

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Abstract: We defined the class of generalized weakly C-contractive mappings in partial metric spaces and proved some fixed-point results for such maps in ordered partial metric spaces without exploiting the continuity of any of the functions. We also establish fixed-point theorem for the integral type of these maps. Example is given to support the validity of our result. Our result generalizes the results of Chen and Zhu [3] and others in the literature

Subjects: Advanced Mathematics; Applied Mathematics; Foundations & Theorems; Computer Science (General)

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1. Introduction

Metric fixed-point theory has been a rigorous area of research in fixed-point theory and applications. A number of studies have been carried out concerning the generalization of metric spaces (see Eke 2016, Imaga, & Odetunmibi, 2017; Eke & Olaleru, 2013; Mustafa & Sims, 2006). Matthews (1992) introduced partial metric space to study the denotational semantics of dataflow networks. In the same reference, he proved the partial metric version of the Banach contraction principle. Alber and Guerre-Delabriere (1997), defined weakly contractive mappings on a Hilbert space and established a fixed-point theorem for such mappings. Subsequently, Rhoades (2001) use the

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PUBLIC INTEREST STATEMENT

Partial metric space was introduced in 1992 by Mathew to solve the denotational sematics problems in computer science. Fixed-point theorems in this space equally deal with functional equations in mathematics and applied mathematics such as computer sciences, engineering, economics, etc. Differential equations help to model real life problems into mathematical models and their solutions are accurately obtained through the application of fixed-point theorems. Thus fixed-point theorems in ordered partial metric spaces with applications provide unique solutions to differential and integral equations that model real life problems.

concept of weakly contractive mappings and obtained a fixed-point theorem in complete metric space. Choudhury (2009) introduced a class of weakly C-contractive mappings as follows:

A mapping $T: X \to X$, *where* (X, d) *is a complete metric space is said to be weakly* $C-$*contractive or weak* $C-$*contraction if for all* $x, y \in X$,

$$d(Tx, Ty) \leq \frac{1}{2}[d(x, Ty) + d(y, Tx)] - \psi(d(x, Ty), d(y, Tx)). \tag{1}$$

where $\psi: [0, \infty)^2 \to [0, \infty)$ *is a continuous mapping such that* $\psi(x, y) = 0$ *if and only if* $x = y = 0$.

Many authors had generalized the weak contractive mappings and proved fixed-point theorems for such mappings in various abstract spaces (see Aage & Salunke, 2012; Chi, Karapinar, & Thanh, 2013; Gairola & Krishan, 2015; Mishra, Tiwari, Mishra, & Mishra, 2015). Eke (2016) introduced a class of generalized weakly C-contractive maps by replacing C-contraction maps with Hardy–Rogers version of contractive maps. In the same reference, the fixed point of these maps in G-partial metric spaces is proved. For a decade, the existence of fixed points in ordered metric spaces was initiated by Ran and Reurings (2003). Olatinwo (2010) proved some fixed-point theorems using weak contraction of the integral type. Long, Son, and Hoa (2017) reestablished the uniqueness of two fuzzy weak solutions of fuzzy fractional partial differential equations via the unique fixed point of weakly contractive mappings in partially ordered metric spaces. Long and Dong (2018) established the integral solution of nonlocal problems of fuzzy implicit fractional differential system by employing Krasnoselskii's fixed-point theorem of generalized contractive mappings in generalized semilinear Banch space. Long, Son, and Rodriguez-Lopez (2018) prove that the fixed point of weakly contractive mappings in partially ordered metric spaces is unique. The authors further apply the result to obtain unique two types of fuzzy solution for fuzzy partial differential equations with local boundary conditions. In this work, we proved some fixed-point theorems for the generalized weakly C-contraction mappings in ordered partial metric spaces. Moreso, the application of these maps are established in the integral type.

2. Preliminaries

The following definitions and results are found in (Matthews, 1992).

Definition 2.1: Let X be a nonempty set, and let $p: X \times X \to R^+$ be a function satisfying the following:

$(p_1) p(x,y) = p(y,x)$
$(p_2) p(x,x) = p(x,y) = p(y,y)$ *iff* $x = y$,
$(p_3) p(x,x) \leq p(x,y)$,
$(p_4) p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$
for all $x, y, z \in X$ *and the pair* (X, p) *is called a partial metric space.*

Let (X,p) be a partial metric space, then a function $d^p: X \times X \to [0, \infty)$ defined as

$$d^p(x, y) = 2p(x, y) - p(y, y) - p(x, x)$$

is a metric on X.

Remark 2.2: In a partial metric space (X, p),

(1) $p(x, y) = 0 \Rightarrow x = y$ *but if* $x = y$ *then* $p(x, y)$ *may not be zero.*
(2) $p(x, y) > 0$ *for all* $x \neq y$, *for all* $x, y \in X$.

Example 2.3: Let $X = R^+$ *and define* $p(x, y) = max\{x, y\}$, *for all* $x, y \in X$. Then (X, p) is a complete partial metric space. Obviously, p is not a (usual) metric.

Definition 2.4: In a partial metric space (*X, p*),

(i) a sequence $\{x_n\}$ is said to converge to a point $x \in X$ if and only if $lim_{n\to\infty}p(x_n,\ x) = p(x,x)$.

(ii) a sequence $\{x_n\}$ is called Cauchy sequence if and only if $lim_{n,m\to\infty}p(x_n,\ x_m)$ is finite.

(iii) if every Cauchy sequence $\{x_n\}$ converges to a point $x \in X$ such that $lim_{n,m\to\infty}p(x_n,\ x_m) = p(x,x)$

then (*X, p*) is known as complete partial metric space.

Lemma 2.5 (Chi et al., 2013): In a partial metric space (*X, p*), if a sequence $\{x_n\}$ converges to a point $x \in X$, then $lim_{n\to\infty}\ p(x_n,x) \leq\ p(x,z)$ *for all* $z \in X$. *Also if* $p(x,\ x) = 0,$ then

$lim_{n\to\infty}\ p(x_n,z) \leq p(x,z)$ *for all* $z \in\ X$.

Lemma 2.6 (Long et al., 2018): In a partial metric space (*X, p*),

(i) a sequence $\{x_n\}$ is Cauchy if and only if, it is a Cauchy in $(X,\ d^p)$.

(ii) X is complete if and only if it is complete in $(X,\ d^p)$.

In addition, $lim_{n\to\infty}d^p(x_n,\ x) = 0$ if and only if

$lim_{n,m\to\infty}p(x_n,\ x_m) = lim_{n\to\infty}p(x_n,\ x) = p(x,\ x)$.

If $\{x_n\}$ is a Cauchy sequence in the metric space $(X,\ d^p)$, we have

$lim_{n,m\to\infty}d^p(x_n,\ x_m) = 0$

and therefore by definition of d^p, we have

$lim_{n,m\to\infty}p(x_n,\ x_m) = 0$

Definition 2.7 (Ran & Reurings, 2003): Let $(X,\prec)$be a partially ordered set. Then two elements $x,\ y \in X$ are said to be totally ordered or ordered if they are comparable. That is, $x \prec y$ *or* $y \prec x$.

Definition 2.8: Let X be a nonempty set. The triplet $(X,\prec,p)$ is called an ordered partial metric space if the following conditions hold:

(i) p is a partial metric on X;

(ii) $\prec$ is a partial order on X.

Definition 2.9 (Shatanawi, 2011): A self-mapping ψ on a positive real numbers is said to be an altering distance function, if holds for all $t \in [0,\infty)$ such that:

(i) ψ is continuous and non-decreasing,

(ii) $\psi(t) = 0$ *if and only if* $t = 0$.

Rhoades (2001) named the map introduced by Chatterjea after him as C-contraction map. The definition is as follows:

Definition 2.10 (Chatterjea, 1972) (C-contraction): Let $T:\ X \to X$where (*X, d*) is a metric space is called a C-contraction if there exists $0 < k < \frac{1}{2}$ such that for all $x,\ y\ \in X$ the following inequality holds:

$$d(Tx,\ Ty) \leq\ k[\ d(x,\ Ty) + d(y,\ Tx)]. \tag{2}$$

A more generalized C-contractive mapping is introduced by (Hardy and Rogers 1973) and defined as follow

Let *(X, d)* be a complete metric space and an operator $T:\ X \to X$ be a contractive mapping then there exist some numbers *a, b, c, e* and f, *a + b + c + e + f* < 1 such that for each $x,\ y\ \in X$,

$$d(Tx, Ty)\ \leq ad(x,y)\ + bd(x,Tx) + c\ d(y,Ty) + ed(x,Ty) + fd(y,Tx) \tag{3}$$

3. Main results

In this work, we introduced a class of generalized weak C-contractive mapping in partial metric spaces by replacing the C-contractive map by Hardy and Rogers contractive map.

Definition 3.1: Let *(X, p)* be a partial metric space and $T:\ X \to X$ be a mapping. Then *T* is said to be generalized weakly C-contractive if for all$x,\ y\ \in X$, the following inequality holds:

$$\begin{aligned} p(Tx,\ Ty\ ) \leq\ & a_1 p(x,y) + a_2 p(x,Tx) + a_3\ p(y,\ Ty) + a_4\ p(x,Ty) + a_5\ p(y,\ Tx) \\ & -\ \phi(\ p(x,y), p(x,Tx),\ p(y,\ Ty),\ p(x,Ty), p(y,\ Tx)) \end{aligned} \tag{4}$$

where $a_1, a_2, a_3, a_4, a_5 \in [0,\ 1),\ \sum_{i=1}^{5} a_i < 1$, and $\phi:\ [0,\infty)^5 \to\ [0,\ \infty)$ is a continuous function with $\phi(v,\ w,\ x,\ y,\ z)\ = 0$ if and only if *v* = *w* = *x* = *y* = *z* = 0.

Remark 3.2: If *v* = *w* = *x* = 0, $a_1 = a_2 = a_3 = 0, a_4 = a_5 = \frac{1}{2}$ and partial metric space is replace with metric space then (4) reduces to (1).

Example 3.3: Let $X = [0,\infty)$ be equipped with a partial metric which is defined by $p(x,y) = max\{x,y\}$. Define a mapping $T:\ X \to X$ by $Tx = \frac{x}{10}$. Define $\phi:\ [0,\infty)^5 \to\ [0,\ \infty)$ by $\phi(t) = \frac{x}{80}$ and let $a_1 = \frac{1}{4},\ a_2 = a_3 = a_4 = a_5 = \frac{1}{8}$. Then weakly C-contractive mapping is extended by Hardy and Rogers contractive mappings.

Theorem 3.4: Let $(X,\prec)$ be a partially ordered set and suppose that there exists a partial metric on X such that *(X, p)* is complete. Let $T:\ X \to X$ be a nondecreasing mapping such that for comparable $x,\ y\ \in X$,

$$\begin{aligned} \psi(p(Tx,\ Ty\ )\ \leq \varphi(\ & a_1\ p(x,y) + a_2 p(x,Tx) + a_3\ p(y,\ Ty) + a_4\ p(x,Ty) + a_5\ p(y,\ Tx)) \\ & - \phi(\ p(x,y),\ p(x,Tx),\ p(y,\ Ty),\ p(x,Ty),\ p(y,\ Tx)) \end{aligned} \tag{5}$$

where $a_1, a_2, a_3, a_4, a_5 \in [0,\ 1),\ \sum_{i=1}^{5} a_i <\ 1$, and ψ, φ are altering distance functions with

$$\psi(t) - \varphi(t)\ \geq 0 \tag{6}$$

for $t\ \geq 0$, and $\phi:\ [0,\infty)^5 \to\ [0,\ \infty)$ is a continuous function with $\phi(v,\ w,\ x,\ y,\ z)\ = 0$ if and only if *v* = *w* = *x* = *y* = *z* = 0. If there exists $x_0 \in X$such that $x_0 \prec Tx_0$, then *T* has a fixed point.

Proof: Observe that if *T* satisfies (5) then it satisfies

$$\begin{aligned} \psi(p(Tx,\ Ty\ )\ \leq \varphi(\ & a\ p(x,y)\ + bp(x,Tx)\ + bp(y,\ Ty)\ + cp(x,Ty)\ + cp(y,\ Tx)) \\ & - \phi(\ p(x,y),\ p(x,Tx),\ p(y,\ Ty),\ p(x,Ty),\ p(y,\ Tx)) \end{aligned} \tag{7}$$

where $a = a_1,\ 2b =\ a_2 + a_3, 2c = a_4 +\ a_5\ , a + 2b +\ 2c < 1$. We use (7) for our argument.

Let $x_0 \in X$ be arbitrary chosen. Suppose $x_0 = Tx_0$ then x_0 is the fixed point of T. Let $x_0 \prec Tx_0, x_1 \in X$ can be chosen such that $x_1 = Tx_0$. Since *T* is nondecreasing function, then

$$x_0 \prec x_1 = Tx_0 \prec x_2 = Tx_1 \prec x_3 = Tx_2.$$

Continuing the process, a sequence $\{x_n\}$ can be constructed such that $x_{n+1} = Tx_n$ with $x_0 \prec x_1 \prec x_2 \prec x_3 \prec \ldots \prec x_n \prec x_{n+1} \ldots$.

If $p(x_n, x_{n+1}) = 0$ for some $n \in \mathbb{N}$ then T has a fixed point. Letting $p(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$, we claim that

$$p(x_n, x_{n+1}) \le p(x_{n-1}, x_n), \; n \in \mathbb{N} \tag{8}$$

Suppose $x_n \neq x_{n+1}, \; p(x_n, x_{n+1}) > p(x_{n-1}, x_n)$ for some n_0 then

$$p(x_{n_0}, x_{n_{0+1}},) > p(x_{n_{0-1}}, x_{n_0},) \tag{9}$$

From (7) and (9) the proof of the claim is established as follows:

$$\psi\,(p(x_{n_0}, x_{n_{0+1}})) = \psi\,(p(Tx_{n_{0-1}}, Tx_{n_0}))$$

$$\le \varphi(\, a\, p(x_{n_{0-1}}, x_{n_0}) + b\, p(x_{n_{0-1}}, Tx_{n_{0-1}}) + b\, p(x_{n_0}, Tx_{n_0}) + c\, p(x_{n_{0-1}}, Tx_{n_0}) + c\, p(x_{n_0}, Tx_{n_{0-1}}s,))$$
$$-\ \phi(\, p(x_{n_{0-1}}, x_{n_0}),\ p(x_{n_{0-1}}, Tx_{n_{0-1}}),\ p(x_{n_0}, Tx_{n_0}),\ p(x_{n_{0-1}}, Tx_{n_0}),\ p(x_{n_0}, Tx_{n_{0-1}}))$$

$$= \varphi(\, a\, p(x_{n_{0-1}}, x_{n_0}) + b\, p(x_{n_{0-1}}, x_{n_0}) + b\, p(x_{n_0}, x_{n_{0+1}}) + c\, p(x_{n_{0-1}}, x_{n_{0+1}}) + c\, p(x_{n_0}, x_{n_0}))$$
$$-\ \phi(\, p(x_{n_{0-1}}, x_{n_0}),\ p(x_{n_{0-1}}, x_{n_0}),\ p(x_{n_0}, x_{n_{0+1}}), p(x_{n_{0-1}}, x_{n_{0+1}}),\ p(x_{n_0}, x_{n_0}))$$

$$\le \varphi\begin{pmatrix} a\, p(x_{n_{0-1}}, x_{n_0}) + b\, p(x_{n_{0-1}}, x_{n_0}) + b\, p(x_{n_0}, x_{n_{0+1}}) + c\, p(x_{n_{0-1}}, x_{n_0}) + cp(x_{n_0}, x_{n_{0+1}}) \\ -cp(x_{n_0}, x_{n_0}) + c\, p(x_{n_0}, x_{n_0}) \end{pmatrix}$$
$$-\ \phi(\, p(x_{n_{0-1}}, x_{n_0}),\ p(x_{n_{0-1}}, x_{n_0}),\ p(x_{n_0}, x_{n_{0+1}}),\ p(x_{n_0}, x_{n_0}))$$

$$\le \varphi(\, a\, p(x_{n_{0-1}}, x_{n_0}) + b\, p(x_{n_{0-1}}, x_{n_0}) + b\, p(x_{n_0}, x_{n_{0+1}}) + c\, p(x_{n_{0-1}}, x_{n_0}) + cp(x_{n_0}, x_{n_{0+1}}))$$
$$-\ \phi(\, p(x_{n_{0-1}}, x_{n_0}),\ p(x_{n_{0-1}}, x_{n_0}),\ p(x_{n_0}, x_{n_{0+1}}),\ p(x_{n_0}, x_{n_0}))$$

$$\le \varphi(\, (a + 2b + 2c)\max\{p(x_{n_{0-1}}, x_{n_0}), p(x_{n_0}, x_{n_0+1})\})$$
$$-\ \phi(\, p(x_{n_{0-1}}, x_{n_0}),\ p(x_{n_{0-1}}, x_{n_0}),\ p(x_{n_0}, x_{n_{0+1}}),\ p(x_{n_0}, x_{n_0}))$$

$$\le \varphi(p(x_{n_0}, x_{n_0+1})) - \phi(\, p(x_{n_{0-1}}, x_{n_0}),\ p(x_{n_{0-1}}, x_{n_0}),\ p(x_{n_0}, x_{n_{0+1}}),\ p(x_{n_0}, x_{n_0})) \tag{10}$$

Using (6), (10) becomes

$$\phi(\, p(x_{n_{0-1}}, x_{n_0}),\ p(x_{n_{0-1}}, x_{n_0}),\ p(x_{n_0}, x_{n_{0+1}}),\ p(x_{n_{0-1}}, x_{n_{0+1}}),\ p(x_{n_0}, x_{n_0})) = 0 \tag{11}$$

By property of ϕ, (11) yields

$$p(x_{n_{9-1}}, x_{n_0}) = 0,\ p(x_{n_{0-1}}, x_{n_0}) = 0,\ p(x_{n_0}, x_{n_{0+1}}) = 0,\ p(x_{n_{0-1}}, x_{n_{0+1}}) = 0,\ p(x_{n_0}, x_{n_0}) = 0 \tag{12}$$

Since

$$\psi(p(x_{n_{0+1}}, x_{n_{0+2}})) = \psi(p(Tx_{n_0}, Tx_{n_{0+1}}))$$

$$\le \varphi\ (\, a\, p(x_{n_0}, x_{n_{0+1}}) + b\, p(x_{n_0}, Tx_{n_0}) + b\, p(x_{n_{0+1}}, Tx_{n_{0+1}}) + c\, p(x_{n_0}, Tx_{n_{0+1}}) + c\, p(x_{n_{0+1}}, Tx_{n_0},))$$
$$-\ \phi(\, p(x_{n_0}, x_{n_{0+1}}),\ p(x_{n_0}, Tx_{n_0}),\ p(x_{n_{0+1}}, Tx_{n_{0+1}}),\ p(x_{n_0}, Tx_{n_{0+1}}),\ p(x_{n_{0+1}}, Tx_{n_0}))$$

$$= \varphi\ (\, a\, p(x_{n_0}, x_{n_{0+1}}) + b\, p(x_{n_0}, x_{n_{0+1}}) + b\, p(x_{n_{0+1}}, x_{n_{0+2}}) + c\, p(x_{n_0}, x_{n_{0+2}}) + c\, p(x_{n_{0+1}}, x_{n_{0+1}},))$$
$$-\ \phi(\, p(x_{n_0}, x_{n_{0+1}}),\ p(x_{n_0}, x_{n_{0+1}}),\ p(x_{n_{0+1}}, x_{n_{0+2}}),\ p(x_{n_0}, x_{n_{0+2}}),\ p(x_{n_{0+1}}, x_{n_{0+1}}))$$

$$\le \varphi\begin{pmatrix} a\, p(x_{n_0}, x_{n_{0+1}}) + b\, p(x_{n_0}, x_{n_{0+1}}) + b\, p(x_{n_{0+1}}, x_{n_{0+2}}) + c\, p(x_{n_0}, x_{n_{0+1}}) + c\, p(x_{n_{0+1}}, x_{n_{0+2}},) \\ -c\, p(x_{n_{0+1}}, x_{n_{0+1}},) + c\, p(x_{n_{0+1}}, x_{n_{0+1}},) \end{pmatrix}$$
$$-\ \phi(\, p(x_{n_0}, x_{n_{0+1}}),\ p(x_{n_0}, x_{n_{0+1}}),\ p(x_{n_{0+1}}, x_{n_{0+2}}), p(x_{n_{0+1}}, x_{n_{0+1}}))$$

$$\leq \varphi\ (\ a\ p(x_{n_0},\ x_{n_{0+1}})\ +\ b\ p(x_{n_0},x_{n_{0+1}})\ +\ b\ p(x_{n_{0+1}},\ x_{n_{0+2}})+\ c\ p(x_{n_0},x_{n_{0+1}})+c\ p(x_{n_{0+1}},\ x_{n_{0+2}},))$$
$$-\ \phi(\ p(x_{n_0},x_{n_{0+1}}),\ p(x_{n_0},x_{n_{0+1}}),\ p(x_{n_{0+1}},\ x_{n_{0+2}}),\ p(x_{n_{0+1}},\ x_{n_{0+1}}))$$

$$\leq \varphi\ ((a+2b+2c)\max\{p(x_{n_0},x_{n_{0+1}}),p(x_{n_{0+1}},x_{n_{0+2}})\})$$
$$-\ \phi(\ p(x_{n_0},x_{n_{0+1}}),\ p(x_{n_0},x_{n_{0+1}}),\ p(x_{n_{0+1}},\ x_{n_{0+2}}),\ p(x_{n_{0+1}},\ x_{n_{0+1}}))$$

$$\leq \varphi\ (\max\{p(x_{n_0},x_{n_{0+1}}),p(x_{n_{0+1}},x_{n_{0+2}})\})$$
$$-\ \phi(\ p(x_{n_0},x_{n_{0+1}}),\ p(x_{n_0},x_{n_{0+1}}),\ p(x_{n_{0+1}},\ x_{n_{0+2}}),\ p(x_{n_{0+1}},\ x_{n_{0+1}}))$$

$$\leq \varphi\ (p(x_{n_{0+1}},x_{n_{0+2}}))-\phi(\ p(x_{n_0},x_{n_{0+1}}),\ p(x_{n_0},x_{n_{0+1}}),\ p(x_{n_{0+1}},\ x_{n_{0+2}}),\ p(x_{n_{0+1}},\ x_{n_{0+1}})) \tag{13}$$

Using (6), (13) becomes

$$\phi(\ p(x_{n_0},x_{n_{0+1}}),\ p(x_{n_0},x_{n_{0+1}}),\ p(x_{n_{0+1}},\ x_{n_{0+2}}),\ p(x_{n_0},x_{n_{0+2}}),p(x_{n_{0+1}},\ x_{n_{0+1}}))=0 \tag{14}$$

By property of ϕ, (14) yields

$$p(x_{n_0},x_{n_{0+1}})=0,p(x_{n_{0+1}},\ x_{n_{0+2}})=0,p(x_{n_{0+1}},\ x_{n_{0+1}})=0,p(x_{n_0},x_{n_{0+2}})=0 \tag{15}$$

Thus, $\{p(x_n,\ x_{n+1})\}$is a decreasing sequence of nonnegative real numbers. Hence, there exists $k\geq 0$ such that

$lim_{n\to\infty}p(x_n,\ x_{n+1})=k.$

From (10) and the above facts, we have

$$\psi\ (p(x_n,\ x_{n+1}))\leq \varphi(\ p(x_n,\ x_{n+1})$$
$$-\ \phi(\ p(x_{n-1},x_n),\ p(x_{n-1},\ x_n),\ p(x_n,\ x_{n+1}),\ p(x_{n-1},x_{n+1}),p(x_n,\ x_n))$$

Taking the limit as $n\to\infty$ in the above inequality yields
$lim\ inf_{n\to\infty}\ (\phi(\ p(x_{n-1},x_n),\ p(x_{n-1},\ x_n),\ p(x_n,\ x_{n+1}),\ p(x_{n-1},x_{n+1}),p(x_n,\ x_n)))=0.$

By the continuity of ϕ we have

$\phi(\ lim\ inf_{n\to\infty}p(x_{n-1},\ x_n),\ lim\ inf_{n\to\infty}p(x_n,\ x_{n+1}),\ lim\ inf_{n\to\infty}p(x_{n-1},\ x_{n+1}),$
$lim\ inf_{n\to\infty}\ p(x_n,\ x_n))=0.$The property of ϕ gives that

$$lim\ inf_{n\to\infty}p(x_{n-1},\ x_n)=0,lim\ inf_{n\to\infty}p(x_n,\ x_{n+1})=0,lim\ inf_{n\to\infty}p(x_{n-1},\ x_{n+1})$$
$$=0,lim\ inf_{n\to\infty}\ p(x_n,\ x_n)=0. \tag{16}$$

Taking the inferior limit in (15) and using (16), $\psi(k)\ =0$, this implies that $k\ =\ 0$. Therefore $lim_{n\to\infty}\ p(x_n,\ x_{n+1})=0$.

Now we claim that $\{x_n\}$ is a Cauchy sequence. It is sufficient to show that $\{x_{2n}\}$ is a Cauchy sequence. On the contrary, suppose $\{x_{2n}\}$ is not a Cauchy sequence then there exists $\varepsilon>0$ and two subsequences $\{x_{2n_k}\}$and $\{x_{2m_k}\}$ of $\{x_{2n}\}$ such that $n(k) > m(k) > k$ and sequences in (7) tend to ε as $k\to\infty$. For two comparable elements $x=x_{2n_{k+1}}$ and $y\ =x_{2m_k}$ we can obtain from (7) that

$$\psi\ (p(x_{2n_{k+1}},\ x_{2m_k}))\ =\psi(p(Tx_{2n_k},\ Tx_{2m_{k-1}}))$$

$$\leq \varphi\ (\ a\ p(x_{2n_k},\ x_{2m_{k-1}})\ +\ b\ p(x_{2n_k},Tx_{2n_k})\ +\ b\ p(x_{2m_{k-1}},\ Tx_{2m_{k-1}})+\ c\ p(x_{2n_k},Tx_{2m_{k-1}})\ +\ c\ p(x_{2m_{k-1}},\ Tx_{2n_k}))$$
$$-\phi(\ p(x_{2n_k},\ x_{2m_{k-1}}),\ p(x_{2n_k},Tx_{2n_k}),\ p(x_{2m_{k-1}},\ Tx_{2m_{k-1}}),\ p(x_{2n_k},Tx_{2m_{k-1}}),\ p(x_{2m_{k-1}},\ Tx_{2n}))$$

$$\leq \varphi\ (\ a\ p(x_{2n_k},\ x_{2m_{k-1}})\ +\ b\ p(x_{2n_k},x_{2n_{k+1}})\ +\ b\ p(x_{2m_{k-1}},\ x_{2m_k})+\ c\ p(x_{2n_k},x_{2m_k})\ +\ c\ p(x_{2m_{k-1}},\ x_{2n_{k+1}}))$$
$$-\phi(\ p(x_{2n_k},\ x_{2m_{k-1}}),\ p(x_{2n_k},x_{2n_{k+1}}),\ p(x_{2m_{k-1}},\ x_{2m_k}),\ p(x_{2n_k},x_{2m_k}),\ p(x_{2m_{k-1}},\ x_{2n+1})) \tag{17}$$

As $k\to\infty$ in (17), we obtain

$$\psi(\varepsilon)\ \leq \varphi(\varepsilon)\ -\phi(\varepsilon,\varepsilon,\varepsilon,\varepsilon,\varepsilon),$$

this implies that $\phi(\varepsilon,\varepsilon,\varepsilon,\varepsilon,\varepsilon)=0$, hence$\varepsilon=0$, a contradiction. Thus $\{x_{2n}\}$ is a Cauchy sequence and so is $\{x_n\}$. Since (X,p) is complete so $(X,\ d^p)$is also complete (by Lemma 2.6). Therefore, the Cauchy sequence $\{x_n\}$converges in $(X,\ d^p)$, that is, $lim_{n\to\infty}p(x_n,\ z)=p(z,z)$then by Lemma 2.6, we have

$$lim_{n,m\to\infty}p(x_n,\ x_m)=lim_{n\to\infty}p(x_n,\ z)=p(z,z) \tag{18}$$

By Lemma 2.6, we obtain $lim_{n\to\infty}p(x_n,\ z)=0$,

so, by definition ofd^p, we obtain

$$d^p(x_n,\ x_m)=2p(x_n,\ x_m)-\ p(x_m,\ x_m)-\ p(x_n,\ x_n).$$

Using (16) and taking $n,\ m\to\infty$ in above inequality yields

$$lim_{n,m\to\infty}p(x_n,\ x_m)\ =0 \tag{19}$$

From (18) and (19), we obtain

$$lim_{n\to\infty}p(x_n,\ z)=p(z,z)=0 \tag{20}$$

By (P_4), we obtain

$$p(z,\ Tz)\le p(z,x_n)+p(x_n,\ Tz)-p(x_n,\ x_n)$$

Taking $n\to\infty$ and using Equation (16), (20) and Lemma 2.5 in the above inequality yields

$$p(z,Tz)\le\ p(Tz,Tz) \tag{21}$$

From (P_2), we have

$$p(Tz,Tz)\le\ p(z,Tz) \tag{22}$$

By (21) and (22), we obtain

$$\psi\ (p(z,Tz))=\psi(p(Tz,Tz))$$

$$\le\varphi(ap(z,\ z)+\ bp(z,\ Tz)+bp(z,\ Tz)+cp(z,\ Tz)+cp(z,\ Tz))$$
$$-\phi(p(z,\ z),\ p(z,\ Tz),\ p(z,\ Tz),\ p(z,\ Tz),\ p(z,\ Tz))$$

$$\le\varphi(ap(z,\ z)+\ bp(z,\ Tz)+bp(z,\ Tz)+cp(z,\ Tz)+cp(z,\ Tz))-\phi(p(z,\ z),\ p(z,\ Tz))$$

$$\le\varphi((a\ +2b+\ 2c)max\{p(z,\ z),\ p(z,\ Tz)\})-\phi(p(z,z),\ p(z,\ Tz))$$

$$\le\varphi(max\{p(z,\ z),\ p(z,\ Tz)\})\ -\phi(p(z,\ z),\ p(z,\ Tz)).$$

Using (20) and (6) in above inequality, we obtain

$$\psi(p(z,Tz))\ -\varphi(p(z,\ Tz))\le-\ \phi(0,\ p(z,\ Tz)),$$

this gives $\phi(0,\ p(z,\ Tz))\le\ 0$ this implies that $\phi(0,\ p(z,\ Tz))=0$. Hence p(z, Tz) = 0. Thus, Tz = z.

Corollary 3.5 (Chen & Zhu, 2013): Let $(X,\ \prec)$ be a partially ordered set and suppose that there exists a partial metric in *X* such that (*X, p*) is complete. Let $T:\ X\to\ X$be continuous nondecreasing mapping. Suppose that for comparable$x,\ y\in X$, we have

$$\psi(p(Tx,\ Ty\ ))\le\varphi\left(\frac{p(x,Ty)+p(y,\ Tx)}{2}\right)\ -\phi(\ p(x,Ty),\ p(y,\ Tx)) \tag{23}$$

$$\text{where } \psi(t)-\varphi(t)\ge0 \tag{24}$$

for all $t \geq 0$, and $\phi : [0, \infty)^2 \to [0, \infty)$ is a continuous function with $\phi(y, z) = 0$ if and only if $y = z = 0$. If there exists $x_0 \in X$ such that $x_0 \prec Tx_0$ then T has a fixed point.

Corollary 3.6: Let $(X, \prec)$ be a partially ordered set and suppose that there exists a partial metric in X such that (X, p) is complete. Let $T : X \to X$ be continuous nondecreasing mapping. Suppose that for comparable$x, y \in X$, we have

$$\psi\,(p(Tx, Ty)) \leq \varphi\,(p(x,y)) - \phi(p(x,y)), \tag{25}$$

$$\text{where } \psi(t) - \varphi(t) \geq 0 \tag{26}$$

for all $t \geq 0$, and $\phi : [0, \infty) \to [0, \infty)$ is a continuous function with $\phi(x) = 0$ if and only if $x = 0$. If there exists $x_0 \in X$ such that $x_0 \prec Tx_0$ then T has a fixed point.

The proof of the corollary follows from Theorem 3.3.

Remarks 3.7: If we replace ordered partial metric space with G-metric space and $\psi(k) = k$, $\varphi(t) = t$ in (25) then corollary 3.5 gives Theorem 2.1 of Chi et al. (2013).

Example 3.8 (Ran & Reurings, 2003): Let X = [0,1] with usual order $\prec$ be a partially ordered set and endowed with a partial metric$p : X \times X \to R^+$. This partial metric is defined by p(x, y) = max{x, y}. Then the partial metric space is complete. Also, we define the mapping $T : X \to X$ by $Tx = \frac{x}{3}$. Let us take $\psi, \varphi : [0, +\infty) \to [0, +\infty)$such that $\psi(t) = t^2$ and $\varphi(t) = \frac{t^2}{3}$, respectively, and take $\phi : [0, +\infty)^5 \to [0, +\infty)$ such that $\phi\,(u, v, x, y, z) = \frac{(u+v+x+y+z)^2}{9}$.

By simple calculation we have,

$$p(Tx, Ty) \leq \frac{1}{3} p(x, y) \tag{27}$$

$$p(Tx, Ty) \leq \frac{1}{3}\,[\,p(x, Tx,\,) + p(y, Ty)] \tag{28}$$

$$p(Tx, Ty) \leq \frac{1}{3}\,[p(x, Ty) + p(y, Tx)] \tag{29}$$

If $x \geq y$ then

$$p(Tx, Ty) = max\left\{\frac{x}{3}, \frac{y}{3}\right\} = \frac{x}{3}.$$

Also,

$$p(x, y) + p(x, Tx) + p(y, Ty) + p(x, Ty) + p(y, Tx)$$

$$= p(x, y) + p\left(x, \frac{x}{3}\right) + p\left(y, \frac{y}{3}\right) + p\left(x, \frac{y}{3}\right) + p\left(y, \frac{x}{3}\right)$$

$$= max\,\{x, y\} + max\left\{x, \frac{x}{3}\right\} + p\left(y, \frac{y}{3}\right) + max\left\{x, \frac{y}{3}\right\} + p\left(y, \frac{x}{3}\right)$$

$$= 3x + p\left(y, \frac{y}{3}\right) + p\left(y, \frac{x}{3}\right).$$

Hence,

$$\psi(p(Tx, Ty)) = \frac{x^2}{9} \leq \frac{\left(3x + p\left(y, \frac{y}{3}\right) + p\left(y, \frac{x}{3}\right)\right)^2}{9}$$

$$\leq \frac{\left(3x+\ p\left(y,\frac{y}{3}\right)+p\left(y,\frac{x}{3}\right)\right)^2}{3}-\frac{\left(3x+\ p\left(y,\frac{y}{3}\right)+p\left(y,\frac{x}{3}\right)\right)^2}{9}$$

$$=\varphi\ (\ a_1p(x,y)+a_2p(x,Tx)+a_3p(y,\ Ty)+a_4p(x,Ty)+\ a_5p(y,\ Tx))$$
$$-\phi(\ p(x,y),\ p(x,Tx),\ p(y,\ Ty),p(x,Ty),\ p(y,\ Tx)).$$

If $y\geq\ x$ then we have

$$p(Tx,\ Ty\ )=max\ \left\{\frac{x}{3},\frac{y}{3}\right\}=\frac{y}{3}$$

Also,

$$p(x,\ y\ )+p(x,\ Tx\ )+p(y,\ Ty\ )+p(x,\ Ty\ )+p(y,\ Tx\ )$$

$$=\ p(x,\ y\ )+p\left(x,\frac{x}{3}\ \right)+p\left(y,\ \frac{y}{3}\right)+p\left(x,\ \frac{y}{3}\right)+p\left(y,\ \frac{x}{3}\right)$$

$$=\ max\ \{x,\ y\}+max\ \left\{x,\frac{x}{3}\ \right\}+\ p\left(y,\ \frac{y}{3}\right)+max\left\{x,\ \frac{y}{3}\right\}+p\left(y,\frac{x}{3}\right)$$

$$=\ 3y+p\left(x,\frac{x}{3}\right)+p\left(x,\ \frac{y}{3}\right).$$

Therefore,

$$\psi(p(Tx,\ Ty\ ))=\frac{y^2}{9}\ \leq\frac{\left(3y+p\left(x,\frac{x}{3}\right)+p\left(x,\frac{y}{3}\right)\right)^2}{9}$$

$$\leq \frac{\left(3y+\ p\left(x,\frac{x}{3}\right)+p\left(x,\frac{y}{3}\right)\right)^2}{3}-\frac{\left(3y+\ p\left(x,\frac{x}{3}\right)+p\left(x,\frac{y}{3}\right)\right)^2}{9}$$

$$=\varphi\ (\ a_1p(x,y)+a_2p(x,Tx)+a_3p(y,\ Ty)+a_4p(x,Ty)+\ a_5p(y,\ Tx))$$
$$-\phi(\ p(x,y),\ p(x,Tx),\ p(y,\ Ty),p(x,Ty),\ p(y,\ Tx)).$$

For a comparable $x,\ y\ \in X$ and with the above argument, we conclude that (5) holds. Therefore all the conditions of Theorem 3.4 are satisfied. The fixed point of T is 0.

4. Application to integral type

Theorem 4.1: Let $(X,\ \prec)$ be a partially ordered set and suppose that there exists a partial metric in X such that (X, p) is complete. Let $T:\ X\rightarrow\ X$ be continuous nondecreasing mapping. Suppose that for comparable$x,\ y\in X$, we have

$$\int_0^{p(Tx,\ Ty\ )}\alpha(s)ds\ \leq\int_0^{(\ a_1\ +\ a_2\ +\ a_3\ +\ a_4\ +\ a_5)\ max\ \{p(x,y),\ p(x,Tx),p(y,\ Ty),p(x,Ty)\ ,\ p(y,\ Tx)\}}\beta(s)ds$$
$$-\ \phi\int_0^{(\ p(x,y),\ p(x,Tx),\ p(y,\ Ty),\ p(x,Ty),\ p(y,\ Tx))}\gamma(s)ds \tag{30}$$

where $a_1,\ a_2,a_3,a_4,a_5\in\ [0,\ 1),\ \sum_{i=1}^{5}a_i<1,$ and $\alpha,\beta,\gamma:\ [0,\ \infty)\ \rightarrow[0,\ \infty)$ is a Lebesgue–Stieltjes integrable mapping which are summable and nonnegative . Suppose $\phi:\ [0,\ \infty)^5\ \rightarrow[0,\ \infty)$ is a continuous function with $\phi(v,\ w,\ x,\ y,\ z)\ =0$ if and only if $v = w = x = y = z = 0$. If there exists $x_0\in\ X$ such that $x_0\prec Tx_0$, then T has a fixed point.

Proof: We consider the functions $\psi,\varphi:\ [0,\ \infty)\ \rightarrow[0,\infty)$ defined by

$$\psi(t) = \int_0^t \alpha(s)ds, \varphi(t) = \int_0^t \beta(s)ds,$$

and ψ and φ altering distance functions satisfying

$$\psi(t) - \varphi(t) \geq 0 \tag{31}$$

for all $t \geq 0$. Since ψ and φ satisfied the above condition then the result follows immediately from Theorem 3.3. This completes the proof.

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