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On some higher order boundary value problems at resonance with integral boundary conditions

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Abstract. This paper investigates the existence of solutions for higher-order multipoint boundary value problems at resonance. We obtain existence results by using coincidence degree arguments.

Keywords: Multipoint boundary value problems; Resonance; Integral boundary conditions; Coincidence degree; Higher-order

Mathematics Subject Classification: 34B15; 34B40

1. Introduction

In this article, we consider the following higher-order boundary value problems:

$$x^{(n)}(t) = f\left(t, x(t), x'(t), \ldots, x^{(n-1)}(t)\right) + e(t) \tag{1.1}$$

$$x^{(n-1)}(0) = \alpha x(\xi), \quad x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = 0, \quad x(1) = \int_0^1 x(s)dg(s) \tag{1.2}$$

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where $\alpha \geq 0$, $0 < \xi < 1$, $f : [0,1] \times \Re^n \to \Re$ is a continuous function, $e : [0,1] \to \Re$ is a function in $L^1[0,1]$ and $g : [0,1] \to [0,\infty)$ is a nondecreasing function with $g(0) = 0$ and $g(1) = 1$. The integral in (1.2) is a Riemann–Stieltjes integral.

Multipoint boundary value problems of ordinary differential equations arise in many areas of Physics, Engineering and Applied Mathematics. In particular, integral boundary conditions are encountered in various applications such as population dynamics, blood flow models and cellular systems. In recent years, higher-order boundary value problems have appeared in many papers, for example, see [1–7] and the references therein. To the best of our knowledge, the corresponding problem for higher-order ordinary differential equations with integral boundary conditions at resonance has received little attention.

The boundary value problem (1.1)–(1.2) is called a problem at resonance if $Lx = x^{(n)}(t) = 0$ has non-trivial solutions under boundary condition (1.2), that is, when $dimKerL \geq 1$. When $KerL = 0$, the differential operator L is invertible. In this case, the problem is at non-resonance. The remainder of this paper is organized as follows. In Section 2 we provide some results and lemmas which are important in stating and proving the main existence theorems. In Section 3, the statement and proof of the main existence results are provided.

2. Preliminaries

In this section we present some preliminaries that will be used in the subsequent sections. Let X and Z be real Banach spaces and let $L : domL \subset X \to Z$ be a linear Fredholm operator of index zero. Let $P : X \to X$ and $Q : Z \to Z$ be continuous projections such that $ImP = KerL$, $KerQ = ImL$ and $X = KerL \oplus KerP$, $Z = ImL \oplus ImQ$. It follows that $L|_{domL \cap KerP} : domL \cap KerP \to ImL$ is invertible. We denote this inverse by K_p.

If Ω is an open bounded subset of X such that $domL \cap \Omega \neq \emptyset$, then the map $N : X \to Z$ is called $L-$ compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N : \overline{\Omega} \to X$ is compact, with P and Q as above.

In what follows we shall use the classical spaces $C^{n-1}[0,1]$, $L^1[0,1]$. For $x \in C^{n-1}[0,1]$, we use the norm $\|x\|_\infty = \max_{t\in[0,1]}|x(t)|$ and $\|x\| = max\{\|x\|_\infty, \|x'\|_\infty, \ldots, \|x^{n-1}\|_\infty\}$. We denote the norm in $L^1[0,1]$ by $\|\cdot\|_1$. We use the Sobolev space $W^{n,1}(0,1)$ defined by $W^{n,1}(0,1) = \{x : [0,1] \to \Re | x, x', \ldots, x^{(n-1)}$ are absolutely continuous on [0,1] with $x^{(n)} \in L^1[0,1]\}$. Let $X = C^{n-1}[0,1]$, $Z = L^1[0,1]$. L is the linear operator from $domL \subset X \to Z$ with $domL = \{x \in W^{n,1}(0,1) : x^{(n-1)}(0) = \alpha x(\xi), x'(0) = x''(0) = \cdots = x^{n-2}(0) = 0, x(1) = \int_0^1 x(s)dg(s)\}$. We define $L : domL \subset X \to Z$ by $Lx = x^{(n)}(t)$ and $N : X \to Z$ by $Nx = f\left(t, x(t), x'(t), \ldots, x^{n-1}(t)\right) + e(t)$. Then, the boundary value problem (1.1)–(1.2) becomes

$$Lx = Nx$$

We shall discuss existence results for (1.1)–(1.2) in the following cases.
Case 1: $\alpha = \frac{(n-1)!}{\xi^{n-1}}$, $\int_0^1 s^{n-1}dg(s) = 1$, $\int_0^1 s^n dg(s) \neq 1$, $g(1) = 1$, $g(0) = 0$
Case 2: $\alpha = 0$, $\int_0^1 s^n dg(s) \neq 1$, $g(1) = 1$, $g(0) = 0$

3. Existence results

We shall use the following fixed point theorem of Mawhin [8] to obtain our existence results.

Theorem 3.1. *Let L be a Fredholm operator of index zero and let N be $L-$ compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*
(i). $Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(domL \setminus KerL) \cap \partial\Omega] \times (0, 1)$
(ii). $Nx \notin ImL$ for every $x \in KerL \cap \partial\Omega$
(iii). $deg(JQN|_{\partial\Omega\cap KerL}, \Omega \cap KerL, 0) \neq 0$
Then, the equation $Lx = Nx$ has at least one solution in $domL \cap \overline{\Omega}$.

We shall first consider Case 1.

Lemma 3.1. *If $\alpha = \frac{(n-1)!}{\xi^{n-1}}$, $\int_0^1 s^{n-1}dg(s) = 1$, $\int_0^1 s^n dg(s) \neq 1$, $g(1) = 1$, $g(0) = 0$, then,*
(i). $KerL = \{x \in domL : x = ct^{n-1},\ c \in \Re,\ t \in [0, 1]\}$
(ii). $ImL = \{y \in Z : \int_0^1 \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)d\tau_1 \cdots d\tau_n - \int_0^1 \int_0^s \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)d\tau_1 \cdots d\tau_n dg(s) = 0\}$

Proof. (i). For $x \in KerL$, we have $x^{(n)} = 0$. Hence, $x(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1}t^{n-1}$, $a_i \in \Re$. In view of $x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = 0$ we obtain $x(t) = a_0 + a_{n-1}t^{n-1}$ $x(\xi) = a_0 + a_{n-1}\xi^{n-1}$. From $\alpha = \frac{(n-1)!}{\xi^{n-1}}$, and noting that $x^{(n-1)}(0) = (n-1)!a_{n-1}$ we get

$$(n-1)!a_{n-1} = x^{(n-1)}(0) = \alpha x(\xi) = \frac{(n-1)!}{\xi^{n-1}}a_0 + \frac{(n-1)!}{\xi^{n-1}}a_{n-1}\xi^{n-1}$$

which implies that $a_0 = 0$.

From $x(1) = \int_0^1 x(s)dg(s)$, we derive $a_{n-1} = a_{n-1}\int_0^1 s^{n-1}dg(s)$ or $a_{n-1}\left(1 - \int_0^1 s^{n-1} dg(s)\right) = 0$

In view of $\int_0^1 s^{n-1}dg(s) = 1$, we have $KerL = \{x \in domL : x = ct^{n-1},\quad c \in \Re\}$.

(ii) We next show that

$$ImL = \{y \in Z : \int_0^1 \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)d\tau_1 \cdots d\tau_n - \int_0^1 \int_0^s \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)d\tau_1 \cdots d\tau_n dg(s) = 0\} \tag{3.1}$$

To do this, we consider the problem

$$x^{(n)}(t) = y(t) \tag{3.2}$$

We show that Problem (3.2) has a solution $x(t)$ satisfying

$$x^{(n-1)}(0) = \frac{(n-1)!}{\xi^{n-1}}x(\xi), \quad x'(0) = x''(0) = x^{(n-1)}(0) = 0, \quad x(1) = \int_0^1 x(s)dg(s) \tag{3.3}$$

if and only if

$$\int_0^1 \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)d\tau_1 \cdots d\tau_n - \int_0^1 \int_0^s \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)d\tau_1 \cdots d\tau_n dg(s) = 0 \tag{3.4}$$

Suppose (3.2) has a solution $x(t)$ satisfying (3.3) then from (3.2) we have

$$\begin{aligned} x(t) &= x(0) + \frac{x'(0)t}{1!} + \cdots + \frac{1}{(n-1)!}x^{(n-1)}(0)t^{n-1} \\ &\quad + \int_0^t \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)d\tau_1 \cdots d\tau_n \\ &= x(0) + \frac{1}{(n-1)!}t^{n-1} \int_0^t \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)d\tau_1 \cdots d\tau_n \end{aligned}$$

$$x(1) = x(0) + \frac{x^{(n-1)}(0)}{(n-1)!} + \int_0^1 \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)d\tau_1 \cdots d\tau_n \tag{3.5}$$

$$\begin{aligned} &= \int_0^1 x(s)dg(s) \\ &= \int_0^1 \left[x(0) + \frac{1}{(n-1)!}x^{(n-1)}(0)s^{n-1} + \int_0^t \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)d\tau_1 \cdots d\tau_n\right] dg(s) \\ &= x(0)g(1) + \frac{1}{(n-1)!}x^{(n-1)}(0) \int_0^1 s^{n-1}dg(s) \\ &\quad + \int_0^1 \int_0^s \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)d\tau_1 \cdots d\tau_n dg(s) \end{aligned} \tag{3.6}$$

Since $\int_0^1 s^{n-1}dg(s) = 1$ we obtain from (3.5) and (3.6) that

$$\int_0^1 \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)d\tau_1 \cdots d\tau_n - \int_0^1 \int_0^s \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)d\tau_1 \cdots d\tau_n dg(s) = 0$$

If however (3.4) holds, then setting

$$x(t) = dt^{n-1} + \int_0^t \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)d\tau_1 \cdots d\tau_n$$

where d is an arbitrary constant, then $x(t)$ is a solution of (3.2) satisfying (3.3). $\square$

Lemma 3.2. *If the conditions of Lemma 3.1 holds then*
(i) $L : domL \subset X \to Z$ is a Fredholm operator of index zero and furthermore the linear continuous projection $Q : Z \to Z$ can be written as

$$\begin{aligned} Qy = \frac{n!}{1 - \int_0^1 s^n dg(s)} \Bigg[&\int_0^1 \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)d\tau_1 \cdots d\tau_n \\ &- \int_0^1 \int_0^s \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)d\tau_1 \cdots d\tau_n dg(s)\Bigg] \end{aligned}$$

(ii) Let $P : X \to X$ be defined as

$$Px = x^{(n-1)}(0)t^{n-1} \tag{3.7}$$

Then, the generalized inverse $K_p : ImL \to domL \cap Ker P$ can be defined as

$$K_p y = \int_0^t \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)d\tau_1 \cdots d\tau_n - \int_0^{\xi} \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)d\tau_1 \cdots d\tau_n dg(s)$$

(iii) $\|K_p y\| \le 2\|y\|_1$, for all $y \in ImL$.

Proof. (i). For $y \in Z$, we define the projection Q as

$$Qy = \frac{n!}{1-\int_0^1 s^n dg(s)}\left[\int_0^1\int_0^{\tau_n}\cdots\int_0^{\tau_2} y(\tau_1)d\tau_1\cdots d\tau_n - \int_0^1\int_0^s\cdots\int_0^{\tau_2} y(\tau_1)d\tau_1\cdots d\tau_n dg(s)\right]$$

Let $y_1 = y - Q_y$, then

$$\begin{aligned}&\int_0^1\int_0^{\tau_n}\cdots\int_0^{\tau_2} y_1(\tau_1)d\tau_1\cdots d\tau_n - \int_0^1\int_0^s\int_0^{\tau_n}\cdots\int_0^{\tau_2} y_1(\tau_1)d\tau_1\cdots d\tau_n dg(s)\\ &= \int_0^1\int_0^{\tau_n}\cdots\int_0^{\tau_2} y(\tau_1)d\tau_1\cdots d\tau_n - \int_0^1\int_0^s\int_0^{\tau_n}\cdots\int_0^{\tau_2} y(\tau_1)d\tau_1\cdots d\tau_n dg(s)\\ &- Qy\frac{\left(1-\int_0^1 s^n dg(s)\right)}{n!} = 0\end{aligned}$$

Thus, $y_1 \in ImL$ and hence $Z = ImL + \Re$. Since $ImL \cap \Re = \{0\}$ we conclude that $Z = ImL \oplus \Re$ and therefore

$$dimKerL = dim\Re = CodimImL = 1$$

Therefore, L is a Fredholm operator of index zero.

To prove (ii), we define the generalized inverse $K_p : ImL \to domL \cap KerP$ as

$$K_p y = \int_0^t\int_0^s\cdots\int_0^{\tau_2} y(\tau_1)d\tau_1\cdots d\tau_n - \int_0^\xi\int_0^s\cdots\int_0^{\tau_2} y(\tau_1)d\tau_1\cdots d\tau_n$$

Now for $y \in ImL$, we have

$$\left(LK_p\right) y(t) = \left[\left(K_p y\right)(t)\right]^{(n)} = y(t)$$

and for $x \in domL \cap KerP$, we get

$$\begin{aligned}\left(K_p L\right) x(t) &= \int_0^t\int_0^s\cdots\int_0^{\tau_2} x^{(n)}(\tau_1)d\tau_1\cdots d\tau_n - \int_0^\xi\int_0^s\cdots\int_0^{\tau_2} x^{(n)}(\tau_1)d\tau_1\cdots d\tau_n\\ &= x(t) - \frac{x^{(n-1)}(0)}{(n-1)!}t^{n-1} - x(\xi) + \frac{x^{(n-1)}(0)}{(n-1)!}\xi^{n-1}\end{aligned}$$

In view of $x \in domL \cap KerP$, $Px = x^{(n-1)}(0)t^{n-1}$ and $x^{(n-1)}(0) = \alpha x(\xi) = \frac{(n-1)!}{\xi^{n-1}}x(\xi)$ we obtain

$$\left(K_p L\right) x(t) = x(t)$$

We conclude that $K_p = (L|_{domL\cap KerP})^{-1}$.

To prove (ii) we note that from the definition of K_p, we derive that

$$\begin{aligned}\|K_p y\|_\infty \le& \int_0^1\int_0^1\cdots\int_0^1 |y(\tau_1)|d\tau_1\cdots d\tau_n\\ &+ \int_0^1\int_0^1\cdots\int_0^1 |y(\tau_1)|d\tau_1\cdots d\tau_n = 2\|y\|_1\end{aligned}$$

$$\left(K_p y\right)'(t) = \int_0^t\int_0^{\tau_{n-1}}\cdots\int_0^{\tau_2} y(\tau_1)d\tau_1\cdots d\tau_n$$

$$\|\big(K_p y\big)'\|_\infty \le \|y\|_1$$

$$\vdots$$

$$\|\big(K_p y\big)^{(n-1)}\|_\infty \le \|y\|_1$$

Thus,

$$\|K_p y\| \le 2\|y\|_1. \quad \square$$

Theorem 3.2. *Let $f : [0, 1] \times \Re^n \to \Re$ be a continuous function and assume that*
(A1) There exist functions $a_1(t), \ldots, a_n(t)$, $r(t) \in L^1[0, 1]$ such that for all $(x_1, x_2, \ldots, x_n) \in \Re^n$, $t \in [0, 1]$

$$|f(x_1, x_2, \ldots, x_n)| \le \sum_{i-1}^{n} a_i(t)|x_i| + r(t)$$

(A2) There exists a constant $M_1 > 0$ such that for $x \in domL$, if $x^{(n-1)}(t) > M_1$ for all $t \in [0, 1]$ then

$$\int_0^1 \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)d\tau_1 \cdots d\tau_n - \int_0^1 \int_0^s \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)d\tau_1 \cdots d\tau_n dg(s) \neq 0$$

(A3) There exists a constant $M_2 > 0$ such that for all $d \in \Re$, $|d| > M_2$ then either

$$d \cdot QN(d) \ge 0 \quad \text{or} \quad d \cdot QN(d) \le 0 \tag{3.8}$$

Then, for $e(t) \in L^1[0, 1]$, the boundary value problem (1.1)– (1.2) with $\alpha = \frac{(n-1)!}{\xi^{n-1}}$, $\int_0^1 s^{n-1}dg(s) = 1$, $\int_0^1 s^n dg(s) \neq 1$ and $g(1) = 1$, $g(0) = 0$ has at least one solution in $C^{n-1}[0, 1]$ provided $\sum_{i=1}^n \|a_i\|_1 < \frac{1}{3}$

To prove Theorem 3.2, we shall first establish the following lemmas.

Lemma 3.3. *Let $\Omega_1 = \{x \in domL \setminus KerL : Lx = \lambda Nx, \quad \lambda \in (0, 1]\}$. Then Ω_1 is a bounded set in X.*

Proof. Let $x \in \Omega_1$. We assume that $Lx = \lambda Nx$, $0 < \lambda \le 1$. Then, $Nx \in ImL = KerQ$ and hence from (A2) there exist $t_0 \in [0, 1]$ such that $|x^{(n-1)}(t_0)| \le M_1$. Then

$$x^{(n-1)}(0) = x^{(n-1)}(0) - \int_0^{t_0} x^{(n)}(s)ds$$

$$\|Px\| = |x^{(n-1)}(0)| \le M_1 + \|Nx\|_1 \tag{3.9}$$

For $x \in \Omega_1$, $x \in domL \setminus KerL$, then $(I - P)x \in domL \cap KerP$

$$\|(I - P)x\| = \|K_p L(I - P)x\| \le \|K_p Lx\| \le 2\|Lx\|_1 \le 2\|Nx\|_1 \tag{3.10}$$

where I is the identity operator on X.

Using (3.9) and (3.10) we obtain

$$\|x\| = \|Px + (I-P)x\| \le \|Px\| + \|(I-P)x\| \le M_1 + 3\|Nx\|_1 \tag{3.11}$$

By (A1) and the definition of N, we obtain

$$\begin{aligned} \|Nx\|_1 &\le \int_0^1 |f\left(s, x(s), x'(s), \ldots, x^{(n-1)}(s)\right) + e(s)| ds \\ &\le \sum_{i=1}^{n} \|a_i\|_1 \|x^{(i-1)}\|_\infty + |r\ \|_1 + |e\ \|_1 \le \sum_{i=1}^{n} \|a_i\|_1 |x\ \| + |r\|_1 + |e\|_1 \end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12), we obtain

$$\|x\| \le \frac{3|r\|_1 + 3|e\|_1 + M_1}{1 - 3\sum_{i=1}^{n} \|a_i\|_1} = M_3 \tag{3.13}$$

From (A1) and (3.13), we get

$$\|x^{(n)}\|_1 \le 3M_3 \sum_{i=1}^{n} \|a_i\|_1 + |r\|_1 + |e\|_1$$

Therefore, Ω_1 is bounded in X. □

Lemma 3.4. *The set $\Omega_2 = \{x \in KerL : Nx \in ImL\}$ is a bounded set in X.*

Proof. Let $x \in \Omega_2$, $x \in dt^{n-1}$, $d \in \Re$, $t \in [0, 1]$ and $QNx = 0$. Therefore,

$$\begin{aligned} &\int_0^1 \int_0^{\tau_n} \cdots \int_0^{\tau_2} \left[f\left(\tau_1, d\tau_1^{n-1}, d(n-1)\tau^{n-2}, \ldots, d(n-1)!\right) + e(\tau_1)\right] d\tau_1 \cdots d\tau_n \\ &- \int_0^1 \int_0^s \int_0^{\tau_n} \cdots \int_0^{\tau_2} \left[f\left(\tau_1, d\tau_1^{n-1}, d(n-1)\tau^{n-2}, \ldots, d(n-1)!\right)\right. \\ &\left. + e(\tau_1)\right] d\tau_1 \cdots d\tau_n dg(s) = 0 \end{aligned}$$

From (A2) there exists $t_0 \in [0, 1]$ such that $|x^{(n-1)}(0)| \le M_1$. That is $|(n-1)d \le M_1$. Hence,

$$\|x\| = Max\{\|x\|_\infty, \ldots, \|x^{(n-1)}\|_\infty\} = \{\|x^{(n-1)}\|_\infty \le M_1\}$$

Therefore, Ω_2 is bounded in X. □

Lemma 3.5. *Let*

$$\Omega_3^+ = \{x \in KerL : \lambda x + (1-\lambda)QNx = 0, \quad \lambda \in [0, 1]\} \tag{3.14}$$

and

$$\Omega_3^- = \{x \in KerL : -\lambda x + (1-\lambda)QNx = 0, \quad \lambda \in [0, 1]\} \tag{3.15}$$

Then Ω_3^+ and Ω_3^- are bounded in X provided (3.14) and (3.15) *are satisfied, respectively.*

Proof. Let $x \in \Omega_3^+$. Then, there exists $d \in \Re$ such that $x(t) = dt^{n-1}$. From the first part of (3.8) we have for $|d| > M_2$, $d \cdot QN(d) \ge 0$ and from (3.14) we obtain

$$(1-\lambda)\, QNx = -\lambda x \tag{3.16}$$

If $\lambda = 0$, it follows that $QNx = 0$ and therefore $Nx \in KerQ = ImL$, that is $Nx \in \Omega_2$ and by Lemma 3.4 we can deduce that $\|x\| \leq M_1$. However, if $\lambda \in (0, 1]$ and $|d| > M_2$ then, by assumption (A3) we obtain

$$0 \leq (1 - \lambda)dQN(d) = -\lambda|d|^2 < 0$$

which is a contradiction. Thus, $\|x\| = |d| \leq M_2$. Therefore, Ω_3^+ is bounded. By a similar argument we can prove that Ω_3^- is bounded in X. □

Theorem 3.3. *Let the assumptions (A1)–(A3) hold. Then, problem (1.1)–(1.2) has at least one solution in X.*

Proof. As a consequence of Theorem 3.2, we only show that all the conditions of Theorem 3.1 are fulfilled. Let Ω be a bounded subset of X such that $\bigcup_{i=1}^{3} \Omega_i \subset \Omega$ where $\Omega_3 = \Omega_3^+$ if (3.14) holds or $\Omega_3 = \Omega_3^-$ if (3.15) holds. It is easily seen that conditions (i) and (ii) of Theorem 3.1 are satisfied if we use Lemmas 3.3 and 3.4. To verify the third condition we apply the invariance under a homotopy property of the degree. Let

$$H(x, \lambda) = \pm\lambda x + (1 - \lambda)QNx$$

and let $I : ImQ \to KerL$ be the identity operator. By Lemma 3.5 we know that $H(x, \lambda) \neq 0$ for $(x, \lambda) \in KerL \cap \partial\Omega \times [0, 1]$. Therefore,

$$\begin{aligned} deg\,(QN|_{KerL}, \Omega \cap KerL, 0) &= deg\,(H(\cdot, 0), \Omega \cap KerL, 0) \\ &= deg\,(H(\cdot, 1), \Omega \cap KerL, 0) \\ &= deg\,(\pm I, \Omega \cap KerL, 0) = \pm 1 \end{aligned}$$

This proves Theorem 3.3. □

Next, we consider Case 2. By using the same procedure as in the proof of Lemma 3.1 and 3.2, we can prove the following lemma.

Lemma 3.6. *If $\alpha = 0$, $\int_0^1 s^n dg(s) \neq 0$, $g(1) = 1$, $g(0) = 0$, then*
(i) $KerL = \{x \in domL : x = d,\ \ d \in \Re,\ \ t \in [0, 1]\}$
(ii) $ImL = \{y \in Z : \int_0^1 \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)d\tau_1 \cdots d\tau_n - \int_0^1 \int_0^s \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)d\tau_1 \cdots d\tau_n\, dg(s)\}$
(iii) $L : domL \subset X \to Z$ is a Fredholm operator of index zero and furthermore, the linear continuous projection $Q : Z \to Z$ can be defined as

$$\begin{aligned} Qy = \frac{n!}{1 - \int_0^1 s^n dg(s)} \Bigg[&\int_0^1 \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)d\tau_1 \cdots d\tau_n \\ &- \int_0^1 \int_0^s \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)d\tau_1 \cdots d\tau_n dg(s) \Bigg] \end{aligned}$$

(iv) The linear operator $K_p : ImL \to domL \cap KerP$ can be written as

$$K_p y = \int_0^t \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)d\tau_1 \cdots d\tau_n$$

and

$$\|K_p y\|_\infty \leq \|y\|_1, \quad \textit{for all} \quad y \in ImL$$

Theorem 3.4. *Let $f : [0,1] \times \Re^n \to \Re$ be a continuous function. Assume that the condition (A1) in Theorem 3.2 holds as well as the following two additional conditions*
(A4) There exists a constant $M_1 > 0$ such that for any $x \in domL$, if $x(t) > M_1$ for all $t \in [0,1]$, then,

$$\int_0^1 \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)d\tau_1 \cdots d\tau_n - \int_0^1 \int_0^s \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)d\tau_1 \cdots d\tau_n dg(s) \neq 0.$$

(A5) There exists a constant $M_2 > 0$ such that for $d \in \Re$, if $|d| \geq M_2$ then either

$$d \cdot QN(d) \geq 0 \;\; or \;\; d \cdot QN(d) \leq 0.$$

Then for every $e(t) \in L^1[0,1]$ the boundary value problem (1.1)– (1.2) has at least one solution in $C^{n-1}[0,1]$ provided $\sum_{i=1}^n \|a_i\|_1 < \frac{1}{2}$.

Proof. Let Ω_1 be defined as in Lemma 3.3. We prove that Ω_1 is bounded in X. If $x \in \Omega_1$, then following the procedure in the proof of Lemma 3.3, there exist $t_0 \in [0,1]$ such that $|x(t_0)| \leq M_1$. Then, from $x(0) = x(t_0) - \int_0^{t_0} x'(s)ds$ we get

$$|x(0)| \leq M + \|x'\|_\infty$$

Since $x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = x^{(n-1)}(0) = 0$ we derive

$$\|x'\|_\infty \leq \|x''\|_1 \leq \|x''\|_\infty$$

$$\|x''\|_\infty \leq \|x'\|_1 \leq \|x'\|_\infty$$

$$\vdots$$

$$\|x^{(n-1)}\|_\infty \leq \|x^{(n)}\|_1 \leq \|x^{(n)}\|_\infty$$

$$\begin{aligned} \|Px\| = |x(0)| \leq M + \|x'\|_\infty \leq M + \|x''\|_\infty \leq \cdots \leq M + \|x^{(n)}\|_1 \\ \leq M + \|Lx\|_1 \leq M + \|Nx\|_1 \end{aligned} \tag{3.17}$$

where $P : X \to X$ is defined as $Px = x(0)$

$$\|(I-P)x\| = \|K_pL(I-P)x\| \leq \|K_pLx\| \leq \|Lx\| \leq \|Nx\| \tag{3.18}$$

and from (3.17) and (3.18) we obtain

$$\|x\| \leq \|Px\| + \|(I-P)x\| \leq M + 2\|Nx\|_1 \tag{3.19}$$

By (A1) and the definition of N we obtain

$$\begin{aligned} \|Nx\|_1 &\leq \int_0^1 |f\left(s, x(s), x'(s), \ldots, x^{(n-1)}(s)\right) + e(s)|ds \\ &\leq \sum_{i=1}^n \|a_i\|_1 \|x^{(i-1)}\|_\infty + \|r\|_1 + \|e\|_1 \\ &\leq \sum_{i=1}^n \|a_i\|_1 \|x\| + \|r\|_1 + \|e\|_1 \end{aligned} \tag{3.20}$$

From (3.19) and (3.20) we obtain

$$\|x\| \leq \frac{2\|r\|_1 + 2\|e\|_1 + M}{1 - 2\sum_{i=1}^{n} \|a_i\|_1}$$

The remainder of the proof of Theorem 3.4 is similar to the proof of Theorem 3.2. □

Theorem 3.5. *Let the assumptions A1, A4 and A5 hold. Then problem* (1.1)– (1.2) *has at least one solution in* X.

Proof. The proof follows the same steps as in Theorem 3.3. □

References

[1] R.P. Agarwal, Boundary Value Problems for Higher Order Differential Equations, World Scientific, Singapore, 1986.

[2] A. Cabada, E. Liz, Boundary value problems for higher order ordinary differential equations with impulses, Nonlinear Anal. Method. Theory Appl. 32 (1998) 775–786.

[3] Z. Du, X. Lin, W. Ge, Some higher order multi-point boundary value problems at resonance, J. Comput. Appl. Math. 177 (1) (2005) 55–65.

[4] A. Frioui, A. Guezane-Lakoud, R. Khaldi, Higher order boundary value problems at resonance on an unbounded interval, Electron. J. Differ. Equ. 2016 (2016) 1–10.

[5] X. Lin, Existence of solutions to a nonlocal boundary value problem with nonlinear growth, Bound. Value Probl. 2011 (2011) 1–15.

[6] X. Lin, Z. Du, W. Ge, Solvability of multipoint boundary value problems at resonance for higher order ordinary differential equations, Comput. Math. Appl. 49 (2005) 1–11.

[7] Y. Liu, W. Ge, Solutions of a multipoint boundary value problem for higher order differential equatios at resonanc, Tamakang J. Math. 36 (2) (2005) 119–130.

[8] J. Mawhin, Topological degree methods in nonlinear boundary value problems, in: NS-FCBMS Regional Conference in Maths, American Math Society, Providence RI, 1979.