Mathematical Methods and their Applications

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Acknowledgments

But there is a Spirit in man and the Inspiration of the Almighty gives them understanding (Job 32,8). To God be the glory for His divine Inspiration to accomplish this book project.

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Preface

Here, we put together basic mathematics concepts and applied research contents, integrating them for proper understanding both to the undergraduate and the postgraduate students. In this book, there are three originalities, and they are related. First, the foundation theoretical concepts are presented in a format that makes application very easy to follow. Thereafter, the life application problems were presented to enable the readers see immediately, the application of the theories learned. Finally, we explain basic data structures that find application in our life application problems. No book has presented data structures this way.

We present the content of this book in three parts, namely, Part I: The Fundamentals, Part II: Expansion and Advanced Integration Methods, and Part III: Differential Equations and Numerical Analysis. Part I is all about the fundamental topics of mathematics. It provides the building bricks or languages with which all complex forms of mathematics can be built. For example, Gamma functions and related integrals of chapter 10 can not be taught without a minimum knowledge of Real Numbers and Functions of Real Variables, Limits, Continuity and Differentiability, Indefinite, Definite, and Improper Integrals and Infinite Series discussed in chapters 2, 4, 5, and 6 respectively. Or can Differentiation and Integration of Integrals in chapter 11 be taught without a minimum knowledge of Real Numbers and Functions of Real Variables, Limits, Continuity and Differentiability, and Indefinite, Definite and
Improper Integrals? Basically all topics discussed in part II and III required the understanding of all the topics discussed in part I.

In part II, we discussed techniques useful for differentiation and integration of (complex) integrals as introduced in Limits, Continuity and Differentiability and Indefinite, Definite and Improper Integrals of chapters 4 and 5 respectively. We also discussed here (that is part II) how expansions of functions that can not be done using Taylor and Maclaurin expansion techniques can be carried out. Lastly part III is all about differential equations, which basically consist of ordinary differential equations (ODEs) and the partial differential equations (PDEs). Since not all equations (algebraic or differential) can be solved analytically, we presented in this part, an introduction to numerical analysis, an alternate method for solving equations.

The unique additional context in this book to teaching mathematical methods to Computer Science students was discovered during the time Mr. Fatumo and myself were teaching the (2003/2004) 200 level class of Computer Science and Management Information Science (MIS) students. We realized that as we changed our approach from teaching only basic mathematics to a mixture of basic mathematics and their application, the students enjoyed our classes and found their final examination easier.

The book is basically useful for teaching mathematical methods and analysis of algorithms to computer science students, but note that it can also be generally used as a mathe-
mathematic methods textbook for the science and technology students, both at the undergraduate and the post-graduate levels.

The German word 'Rigorism' employed in this book to describe Abstract questions after each chapter comes from the English word rigorous. Furthermore, C programming language is the pseudo-code language employed in this book. This language, just like English, has become a universal programming language for scientists across the globe.

The book can be read as follows:

1. An advanced Computer Science (CS) person familiar with mathematical methods may go straight to study the life application questions and get back to the basic mathematical methods discussed in this book, if the need arises.

2. The chapters has been arranged to allow systematic understanding of all the topics considered. Therefore, the book content may be divided into three. Each of these is ideal for usage as course content per semester.

3. Those who are interested in knowing further details about the methods discussed in this book should consider the publications listed under the bibliography section.

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14.6 a) Trapezoidal rule: Interval with \( h = 3/2\), b) Trapezoidal rule: Interval with \( h = 3/4\), c) Trapezoidal rule: Interval with \( h = 3/8\), and d) Trapezoidal rule: Interval with \( h = 1/2\). \hspace{1cm} 568

14.7 Partition of interval \([a, b]\) into an even number \( N \) of equal subintervals. \hspace{1cm} 568