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Mathematical Modelling and Analysis of Human Arm as a Triple Pendulum System using Euler – Lagragian Model

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Abstract. This study attempts to model the human arm as a dynamical triple pendulum system. The equation of motion of te human arm was obtained using Euler-Lagrange equation. The resulted second order differential equation was solved analytically. Simulated results were presented with the aid of a computer software – Maple. It was observed that the angular displacement values of the three segments are directly proportional to their respective angular acceleration, which is in the modelling and analysis of human arm motion as a multiple pendulum system. Generally, the longer the segments of the human arm the longer it takes to swing back and forth, and the fewer back-and-forth swings there are in a second.

1. Introduction

Human parts move when involved in activities such as walking, running dancing, jumping and so on. The proper movement of these parts of the body results in good body balance. Proper functioning of human arms results in a good human activity. Using mechanics and other mathematical concepts for the human body part motion modelling and analysis is constantly expanding and becoming very important in body mechanics via the application of Newtonian mechanics to the human skeletal system [1]. Agarana et al [1] considered the movement of human arm during dance and pointed out the importance of balanced arm movement in a good dance. In dancing, the arm locomotion is one of the most complicated motions of a human body [1-3]. Human body or part of it always strives to maintain balance. So, during any activity the balancing of human arm ensures good and sustained position, at least for a considerable long period of time.

Body mechanics involves the coordinated effort of muscles, bones, the nervous system to maintain balance, posture, and alignment during moving, transferring, and positioning a body. Proper body mechanics allows individual to carry out human activities without excessive use of energy. It also helps in preventing injuries [3, 4]. A balance of the body, during human activities, is maintained via body mechanics. When a vertical line falls from the centre of gravity through the base of support, body balance is achieved, otherwise the body will lose its balance [4]. Balance in this sense means ability to maintain the line of gravity of a body within the base of support with minimal postural sway [4]. In this study, the three human arm parts represent the three segments in the triple pendulum system.

The external forces responsible for the motion of the triple pendulum, can be accounted for by splitting it into a sum of kinetic and potential forces [8, 9]. The Lagrangian [11] was used in this paper to model the equation of motion of the human as a triple pendulum system. The analysis of the dynamics of different segments of the human arm during activities were also carried out. The relevance of this study to body mechanics cannot be overemphasized.

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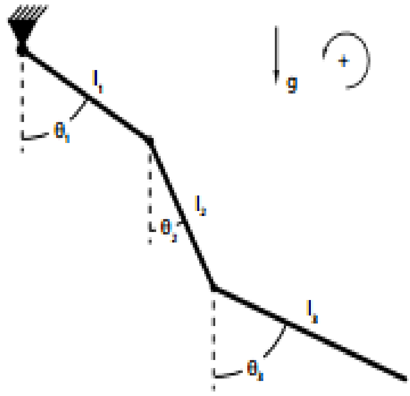

Fig.1. Schematic of Human arm showing the three segments [1]

2. Modelling the Human Arm

The human arm can be simulated as three links, where the upper arm is the first link that is jointed with elbow. The lower part of the arm is connected to the wrist. The third part of the arm is the palm. For simplification sake, the rotation angle from a vertical position of the three human arm parts are denoted as $\theta_1, \theta_2, \theta_3$ respectively. Also the corresponding length of the links are represented by l_1, l_2, l_3 respectively, as shown in schematic of triple pendulum in Agarana's work [1].

2.1. Mathematical Model Formulation

A triple pendulum consists of three pendulum such that one pendulum is attached to another, then to another. Such dynamic system is capable of exhibiting chaotic behaviour. Considering human arm modelled as a triple bob pendulum with masses m_1, m_2, m_3 attached massless wire of lengths l_1, l_2, l_3. The angles the wire make with the vertical, as stated above, are represented as $\theta_1, \theta_2, \theta_3$ respectively. Following the work of Agarana [1], the acceleration due to gravity is g and the positions of the bobs are given respectively as:

(x_1,y_1), (x_2,y_2) and (x_3,y_3)
where

$$x_1 = l_1 sin\theta_1 \quad (1)$$

$$x_2 = l_1 sin\theta_1 + l_2 sin\theta_2 \quad (2)$$

$$x_3 = l_1 sin\theta_1 + l_2 sin\theta_2 + l_3 sin\theta_3 \quad (3)$$

$$y_1 = -l_1 \cos\theta_1 \quad (4)$$

$$y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2 \quad (5)$$

$$y_3 = -l_1 \cos\theta_1 - l_2 \cos\theta_2 - l_3 \cos\theta_3 \quad (6)$$

It was assumed in this study that the values of θ_1, θ_2 *and* θ_3 ranges from 0 to 90 degrees. This implies that none of the segments of the human arm during dance should make more than 90 degrees with the vertical.

Going by Agarana [1], the potential energy of the system is given as:

$$V = m_1 g y_1 + m_2 g y_2 + m_3 g y_3 \tag{7}$$

While the kinetic energy of the system is given as:

$$T = \frac{1}{2}\mathrm{m}_1\mathrm{v}_1^2 + \frac{1}{2}\mathrm{m}_2\mathrm{v}_2^2 + \frac{1}{2}\mathrm{m}_3\mathrm{v}_3^2 \tag{8}$$

V and T can be written respectively as:

$$V = -(m_1 + m_2 + m_3) g l_1 \cos\theta_1 - (m_2 + m_3) g l_2 \cos\theta_2 - m_3 g l_3 \cos\theta_3 \tag{9}$$

$$\begin{aligned} T = {} & \frac{1}{2}[(m_1 + m_2 + m_3) l_1^2 \dot{\theta}_1^2 + (m_2 + m_3) l_2^2 \dot{\theta}_2^2 \\ & + m_3 l_3^2 \dot{\theta}_3^2] + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 (\cos\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2) \\ & + 2 m_3 l_1 l_3 \dot{\theta}_1 \dot{\theta}_3 (\cos\theta_1 \cos\theta_3) \\ & + m_3 l_2 l_3 \dot{\theta}_2 \dot{\theta}_3 (\cos\theta_2 \cos\theta_3 + \sin\theta_2 \cos\theta_3) \end{aligned} \tag{10}$$

The Lagrangian is given as

$$L = T - V \tag{11}$$

$$\begin{aligned} L = {} & \frac{1}{2}[(m_1 + m_2 + m_3) l_1^2 \dot{\theta}_1^2 + (m_2 + m_3) l_2^2 \dot{\theta}_2^2 + m_3 l_3 \theta_3^2) \\ & + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 (\cos\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2) + 2 m_3 l_1 l_3 \dot{\theta}_1 \dot{\theta}_3 (\cos\theta_1 \cos\theta_3) \\ & + m_3 l_2 l_3 \dot{\theta}_2 \dot{\theta}_3 (\cos\theta_2 \cos\theta_3 + \cos\theta_3 \sin\theta_2) \\ & + (m_1 + m_2 + m_3) g l_1 \cos\theta_1 + (m_2 + m_3) g l_2 \cos\theta_2 + m_3 g l_3 \cos\theta_3 \end{aligned} \tag{12}$$

The Euler – Lagrangian equation is given as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0$$

Evaluating the Euler-Lagrangian equation for $\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3$, for non-stationary values respectively gives:

$$\begin{aligned} & (m_1 + m_2 + m_3)(l_1 \ddot{\theta}_1 - g \sin\theta_1) + m_3 l_2 [\ddot{\theta}_2 \cos\theta_1 \cos\theta_2 \\ & + \ddot{\theta}_2 \cos\theta_1 \sin\theta_2 + \cos\theta_2 \sin\theta_1 + \sin\theta_1 \sin\theta_2 \\ & - \dot{\theta}_2 \cos\theta_1 \sin\theta_2 - \dot{\theta}_2 \cos\theta_2 \sin\theta_1 + \dot{\theta}_2 \cos\theta_1 \cos\theta_2 \\ & - \dot{\theta}_2 \sin\theta_1 \sin\theta_2] + 2 l_2 \dot{\theta}_2 [m_2 \ddot{\theta}_3 \cos\theta_1 \cos\theta_3 \\ & + m_3 \dot{\theta}_1 \dot{\theta}_3 \cos\theta_3 \sin\theta_1 - m_2 \dot{\theta}_3 \cos\theta_1 \sin\theta_3 \\ & - m_2 \dot{\theta}_3 \sin\theta_1 \cos\theta_3] = Q_1 \end{aligned} \tag{13}$$

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$$
\begin{aligned}
&(m_2+m_3)(l_2\ddot{\theta}_2+g\sin\theta_2)+m_3l_1[\ddot{\theta}_1\cos\theta_1\cos\theta_2\\
&+\ddot{\theta}_1\cos\theta_1\sin\theta_2-\dot{\theta}_1\cos\theta_1\sin\theta_2-\dot{\theta}_1\sin\theta_1\cos\theta_2\\
&+\dot{\theta}_1\cos\theta_1\cos\theta_2-\dot{\theta}_1\sin\theta_2\sin\theta_1\\
&+\dot{\theta}_1\dot{\theta}_2\cos\theta_1\sin\theta_2-\dot{\theta}_1\dot{\theta}_2\cos\theta_1\cos\theta_2]\\
&+m_3l_3[\ddot{\theta}_3\cos\theta_2\cos\theta_3+\ddot{\theta}_3\cos\theta_3\sin\theta_2\\
&-\dot{\theta}_3\cos\theta_2\sin\theta_3-\dot{\theta}_3\sin\theta_2\cos\theta_3\\
&+\dot{\theta}_3\cos\theta_2\cos\theta_3-\dot{\theta}_3\sin\theta_3\sin\theta_2\\
&+\dot{\theta}_2\dot{\theta}_3\sin\theta_2\cos\theta_3-\dot{\theta}_2\dot{\theta}_3\cos\theta_3\cos\theta_2]=Q_2 \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
&m_3(l_3\ddot{\theta}_3-g\sin\theta_3)+2m_2l_1(\ddot{\theta}_1\cos\theta_1\cos\theta_3\\
&-\dot{\theta}_1\cos\theta_1\sin\theta_3-\dot{\theta}_1\cos\theta_3\sin\theta_1)\\
&+m_3l_2(\ddot{\theta}_2\cos\theta_3\cos\theta_2+\ddot{\theta}_2\sin\theta_2\cos\theta_3\\
&+\dot{\theta}_2\dot{\theta}_3\cos\theta_2\sin\theta_3+\dot{\theta}_2\dot{\theta}_3\sin\theta_2\sin\theta_3\\
&+\dot{\theta}_2\cos\theta_2\cos\theta_3-\sin\theta_2\sin\theta_3\\
&-\cos\theta_2\sin\theta_3-\cos\theta_3\sin\theta_2)\\
&+2m_3l_1\dot{\theta}_1\dot{\theta}_3\cos\theta_1\sin\theta_3 = \mathrm{Q}_3 \qquad (15)
\end{aligned}
$$

Where, Q_1, Q_2, Q_3 are the (non-conservative) generalized forces.

3. Model Solution

From figure 1 above;

$$Sin\theta_1=\frac{x_1}{l_1}, Sin\theta_2=\frac{x_2}{l_2}, Sin\theta_3=\frac{x_3}{l_3}$$

$$\Rightarrow \theta_1=Sin^{-1}\frac{x_1}{l_1}, \theta_2=Sin^{-1}\frac{x_2}{l_2}, \theta_3=Sin^{-1}\frac{x_3}{l_3}$$

$$\Rightarrow \theta_1=Sin^{-1}\frac{x_1}{\sqrt{y_1^2+x_1^2}}, \theta_2=Sin^{-1}\frac{x_2}{\sqrt{y_2^2+x_2^2}}, \theta_3=Sin^{-1}\frac{x_3}{\sqrt{y_3^2+x_3^2}}$$

Table 1: The angular displacement, lengths of segments and different coordinates

(x_1,y_1)	(1,1)	(1,2)	(2,4)	(3,5)	(5,2)
θ_1	0.707	0.447	0.447	0.514	0.928
l_1	1.414	2.236	4.472	5.831	5.385
(x_2,y_2)	(7,4)	(8,6)	(10,7)	(10,9)	(11,11)
θ_2	0.868	0.800	0.819	0.743	0.707

l_2	8.062	10.000	12.207	13.454	15.556
(x_3,y_3)	(12,12)	(13,12)	(13,14)	(14,15)	(15, 15)
θ_3	0.707	0.735	0.680	0.682	0.707
l_3	16.971	17.692	19.105	20.518	21.213

In order to solve for the masses, m, m, and m, the values of the parameters in table 1 are substituted into equations (31), (32) and (33) to become

$$(m_1+m_2+m_3)(-g\sin\theta_1)+m_3l_2[\cos\theta_2\sin\theta_1+\sin\theta_1\sin\theta_2]=Q_1 \quad (16)$$

$$(m_2+m_3)(g\sin\theta_2)=Q_2 \quad (17)$$

$$m_3(-g\sin\theta_3)+2m_2l_1(-\cos\theta_2\sin\theta_3-\cos\theta_3\sin\theta_2)=Q_3 \quad (18)$$

For the purpose of this research work let Q_1, Q_2, Q_3 be assigned respectively the following values; 3,2,1.
When the three masses are at the position (1,1), (7,4) and (12,12) respectively, the absolute values of the masses can be evaluated from the following equations:

$$-0.121m_1-0.121m_2+15.044m_3=3 \quad (19)$$

$$0.149m_2+0.149m_3=2 \quad (20)$$

$$0.121m_3+0.08m_2=-1 \quad (21)$$

m_1= 6378.68, m_2= 64.01, m_3= 50.59
Similarly, on position (2,4), (10,7) and (13,14), the absolute values of the masses are:

m_1= 26.78, m_2= 11.5, m_3= 26.1

For position (1,2), (8,6) and (13,12), the absolute values of the masses are:

m_1= 4.94, m_2= 22.91, m_3= 37.2

For position (3,5), (10,9) and (14,15), the absolute values of the masses are:

m_1= 8.46, m_2= 6.35, m_3= 7.29

For position (5,2), (11,11) and (15,15), the absolute values of the masses are:

3.1 Periods and Frequency of different segments of the pendulum system

The period of the motion for a pendulum is how long it takes to swing back-and-forth, measured in seconds. The frequency of a pendulum is how many back-and forth swings there are in a second, measured in hertz [12]. Frequency F is the reciprocal of the period T: Looking at the three segments as different simple pendulums joined together, The periods of each segment, using the parameters obtained above, are obtained as follows:

The period of a simple pendulum is given as [20]:

$$T = 2\pi\sqrt{\frac{l}{g}} \tag{22}$$

Where l is the length and g is the acceleration due to gravity.

The periods for the three segments with different lengths, assuming a fixed initial position of the masses at (3,5), (10,9), (14,15) respectively, are shown in tables 2, 3, and 4 respectively.

Table 2: The values of the periods and frequency of first segment at given lengths.

Length of 1st Segment	Length	T_1 F_1
l_1	5.83	4.82 0.21
$2l_1$	11.66	6.85 0.15
$3l_1$	17.49	8.38 0.12
$4l_1$	23.32	11.39 0.09
$5l_1$	29.15	10.82 0.092
$6l_1$	34.98	11.87 0.084

Table 3: The values of the periods and frequency of second segment at given lengths

Length of 2nd Segment	Value	T_2 F_2
l_2	13.45	7.36 0.14
$2l_2$	26.9	10.41 0.096
$03l_2$	40.35	12.74 0.078
$4l_2$	53.8	14.72 0.068
$5l_2$	67.25	16.46 0.061
$6l_2$	80.7	18.03 0.055

Table 4: The values of the periods and frequency of third segment at given lengths

Length of 3rd Segment	Value	T_3 F_3
l_3	20.52	10.74 0.093
$2l_3$	41.04	15.19 0.066
$3l_3$	61.56	18.60 0.054
$4l_3$	82.08	21.48 0.047
$5l_3$	102.6	24.02 0.042
$6l_3$	123.12	26.31 0.038

4. Results and Discussion

To maintain balance during human activities, the required masses at the end of each segment of the human arm and their positions were calculated analytically. The values revealed that the position of the arm segments at every point in time, the mass at the end of each segment and the length of the segment are all important in the body mechanics analysis. From tables 2, 3, and 4, it can be seen that there is a general positive correlation between the length of the human arm segments and period but a negative correlation with the frequency. It was also observed from the study that the masses required at the end of the segments of the human arm depends on the position of that segment at a point in time, as shown the result from equations 21, 22, and 23. There is also a positive correlation between the angular displacement, angular acceleration and angular acceleration.

5. Conclusion

This paper analytically modelled the dynamics of human arm as a triple pendulum system in motion. The angular displacements were determined by the simulated positions of the three segments of the human arm. Each of the three simple pendulums that form the triple pendulum represents each of the three segments of the human arm, namely; the upper arm, the lower arm and the hand. With Euler - Lagrange equations, the equations of motion of the triple pendulum were obtained. The solution to these equations reveal the dynamics of the segments of human arm. The sensitivity and interrelationship of the parameters were studied. For a good body mechanics to be achieved, especially about the locomotion of human arm, the analytical results of this paper give a clue of how stability and balance of the whole human arm movement during an activity can be supported.

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