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Closed-Form Expressions for the Quantile Function of the Chi Square Distribution Using the Hybrid of Quantile Mechanics and Spline Interpolation

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Abstract

Chi square distribution is a continuous probability distribution primarily used in hypothesis testing, contingency analysis, and construction of confidence limits in inferential statistics but not necessarily in the modeling of real-life phenomena. The closed-form expression for the quantile function (QF) of Chi square is not available because the cumulative distribution function cannot be transformed to yield the QF and consequently places limitations on the use of the QF. Researchers have over the years proposed approximations that improve over time in terms of speed, computational efficiency, and precision, and so on. However, most of the available closed-form expressions (quantile approximation) fail at the extreme tails of the distribution. This paper used the Quantile mechanics approach to obtain second-order nonlinear ordinary differential equations whose solutions using the power series method yielded initial approximates in form of series for different values of the degrees of freedom. The initial approximate varies with the exact (R software) values which serve as the reference and the error between them was minimized by the natural cubic spline interpolation. The final approximates are correct up to an average of 8 decimal places, have small error, and is closer to the exact when compared with some other results from other researchers. The upper tail of the distribution was considered and excellent results were obtained which is a major improvement over the existing results in the literature. The approach used in this work is a hybrid of series expansions and numerical algorithms. Computer codes can be written for the application.

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