

**SOLVABILITY OF SOME THIRD-ORDER BOUNDARY VALUE PROBLEMS  
AT RESONANCE WITH TWO DIMENSIONAL KERNELS ON THE  
HALF-LINE**

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**A THESIS SUBMITTED TO THE SCHOOL OF POSTGRADUATE STUDIES IN  
PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF  
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EMATICS IN THE DEPARTMENT OF MATHEMATICS, COLLEGE OF SCI-  
ENCE AND TECHNOLOGY, COVENANT UNIVERSITY, NIGERIA**

**OCTOBER, 2021**

## ACCEPTANCE

This is to attest that this thesis is accepted in partial fulfillment of the requirements for the award of the degree of Doctor of Philosophy in Industrial Mathematics in the Department of Mathematics, College of Science and Technology, Covenant University, Ota, Ogun State, Nigeria.

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## DECLARATION

I, **IMAGA, OGBU FAMOUS (16PCD01473)** declare that this research work was carried out by me under the supervision of Prof. Samuel A. Iyase of the Department of Mathematics, College of Science and Technology, Covenant University, Ota, Ogun State, Nigeria and Dr. Sheila A. Bishop of the Department of Mathematics, Faculty of Science, University of Lagos, Akoka, Lagos State, Nigeria. I attest that this thesis has not been presented either wholly or partially for the award of any degree elsewhere. All the scholarly information used in this thesis are duly cited and acknowledged.

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Signature and Date

## CERTIFICATION

We certify that this thesis titled "**SOLVABILITY OF SOME THIRD-ORDER BOUNDARY VALUE PROBLEMS AT RESONANCE WITH A TWO DIMENSIONAL KERNELS ON THE HALF-LINE**" is an original work carried out by **IMAGA, OGBU FAMOUS (16PCD01473)**, in the Department of Mathematics, College of Science and Technology, Covenant University, Ota, Ogun State, Nigeria, under the supervision of Prof. Samuel A. Iyase and Dr. Sheila A. Bishop. We have examined and found the work acceptable as part of the requirements for the award of Doctor of Philosophy (Ph.D) degree in Industrial Mathematics.

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## **DEDICATION**

This work is dedicated first to my Heavenly father, creator of heaven and earth for the grace that He gave me to finish this research work. I also dedicate this work to my family for all their support and unconditional love.

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## NOMENCLATURE

### List of Symbols

$\subset$	Subset
$\cap$	Intersection
$\cup$	Union
$\overline{\Omega}$	Closure of $\Omega$
$\partial\Omega$	Boundary of $\Omega$
$\in$	Member of a set
$\exists$	There exists
$\ \cdot\ $	Norm
$L^1[0, \infty)$	$L^1$ function space
$\varepsilon$	Epsilon

### Abbreviations

dim	Dimension
ker	Kernel
lim	Limit
Im	Image
deg	Degree
dom	Domain
ind	Index
coker	Co kernel
a.e	Almost every
BVP	Boundary value problem

## ABSTRACT

This research work considered the solvability for some resonant third order boundary value problems (BVPs) with integral and multi-point boundary conditions on the half-line when the differential operators have two-dimensional kernels. Problems where the differential operator is linear and problems where the differential operator is nonlinear as a result of the presence of the  $p$ -Laplacian operator were considered. For problems where the differential operator is linear, the coincidence degree theory of Mawhin was applied while for the nonlinear  $p$ -Laplacian operator, the Ge and Ren extension of the coincidence degree theory was applied. Conditions for the existence of solutions for the problems were proved using suitable operators like semi-projectors and algebraic methods. Examples were used to demonstrate the obtained results.

**Keywords:** *Coincidence degree, half-line, integral boundary conditions, multi-point boundary conditions,  $p$ -Laplacian, resonance.*