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Stability Analysis of a Nontrivial Solution for Delayed Nicholson Blowflies' Model with Linear Harvesting Function

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Abstract: The study investigates stability analysis of Nicholson-blowflies' equation with a linear harvesting function, where sufficient conditions are obtained for nontrivial equilibrium of a delayed model using the corresponding characteristic equation and Hopf bifurcation analysis. The Hopf bifurcation is studied for the qualitative character of the dynamical system, and conditions for the existence of periodic oscillation are identified. The periodic orbit of the model is equally investigated, from which further chaotic results are obtained using numerical example via MATLAB software. The study supplements theoretical improvement to earlier results obtained in the literature.

Keywords: Characteristic equation, Delay differential equation, Hopf bifurcation, Stability analysis.

1.0: Introduction

The application of differential equations (ordinary differential equation (ODEs) and partial differential equations (PDEs)) to model dynamical systems can be traced and dated to Malthus, Verhulst, Lotka and Volterra. However, it has been observed that ODEs and PDEs models cannot completely capture the qualitative dynamics observed in natural systems. These rich variety dynamics cannot easily be determined experimentally. Thus, the introduction of a time delay to dynamical systems cannot be ignored. The application of delay differential equations (DDEs) to modeling has proved to be very useful, especially in biological sciences. For instance, the application of DDEs to population dynamics has helped to reduce epidemic outbreaks [1-5]. Several authors in this regard have formulated and studied models involving population growth and interactions ranging from ODEs to PDEs models, discrete to continuous models, and from deterministic to stochastic models. These studies have proved very successful. For example, in his exponential law, Malthus [6] modeled a single species population where population density is negatively affected per capita growth rate, which is largely due to environmental degradation. Verhulst (1836), in his Logistic Growth Model modelled a single species population with a positive growth rate and a carrying capacity of single species population. Huthchinson [7], in his model, introduced the delay term called logistic delay equation. Nicholson [8], motivated by Huthchinson's equation through his data on population fluctuations of sheep blowfly *Lucillia cuprina*, modified Hutchinson equation.

Recent studies of population dynamics have generated remarkable outcomes where great results have been uncovered and applied in various fields of learning. For instance, a single species model is given by the form

$$\frac{dx(t)}{dt} = -\delta\big(x(t)\big)x(t) + P\big(x(t-\tau)\big)x(t-\tau). \tag{1}$$

where $x(t)$ is the population density at time t and $t-\tau$, and τ is the delayed birth term. The function $P(x)$ is assumed continuous, positive and decreasing i.e., the per capita growth rate of the population density decreases when population sizes increases. The positive nondecreasing function $\delta(x)$ represents the per capita death rate which may increases due to intraspecific competition. Equation (1) provides more general population dynamics of a single species model.

The qualitative behaviour of solutions of (1) can drive the system model to exhibit complex characteristics, including periodic orbits, sustained oscillation, and classification resulting from Hopf bifurcation analysis. These qualitative features of equation (1) make possible parameter classification of stability for delay-independent and delay-dependenct giving rise to complicated and challenging instability analysis. However, the applications of model dynamics resulting from the inclusion of time delay and nonlinearity in mathematical modeling provide for a more realistic results in various areas of study, including biological sciences [9, 10]. For instance, several authors [11, 12] described population density of the Australian sheep blowfly with experimental data of Nicholson which is a particular case of equation (1) and proposed nonlinear autonomous delay independent system of the form

$$\frac{dx(t)}{dt} = -\delta x(t) + Px(t-\tau)e^{-ax(t-\tau)}, \tag{2}$$

where $x(t)$ denotes population density at time t, P is the highest per capita daily egg production, $\frac{1}{a}$ is the size at which the population reproduces at its optimum rate, δ is the per capita adult death rate and τ the generation time. Previous studies on the qualitative properties if (2) assumed $P > \delta$ for the existence of equilibriums (trivial and nontrivial). Further qualitative bahaviour of (2) such as the positivity, boundedness of solutions among others have been studied by several authors in the literature. Berezansky, Braverman and Idels [13], in their study, modified equation (2) by defining a linear harvesting function of the form

$$\frac{dx(t)}{dt} = -\delta x(t) + Px(t-\tau)e^{-ax(t-\tau)} - Hx(t-\sigma), \tag{3}$$

where $\delta, P, \tau, a, H, \sigma \in (0, +\infty)$.

In their further study, Berezansky *et al*., [13], attributed all form of artificial deaths of the population to a linear harvesting function and investigated the qualitative behaviour of equation (3). On replacing σ by τ, for mathematical convenience, the model (3) becomes

$$\frac{dx(t)}{dt} = -\delta x(t) + Px(t-\tau)e^{-ax(t-\tau)} - Hx(t-\tau). \tag{4}$$

Xu & Peiluan [14], investigated the stability property of equation (4) and showed that the trivial steady state loses stability when Hopf bifurcation occurs resulting to chaotic phenomena when the delay is large enough. Majid Bani-Yaghoub [15], categorised the qualitative features of equation (4) solutions as consisting of periodic orbits, oscillation, separatrix loop, and equilibrium bifurcations under suitable parameters values. The need for further stability behavour and local Hopf bifurcation of (4) enabled the system model to experience complicated qualitative behaviour, ranging from stability to instability and then stability, making possible parameter regions for delay stability. Stability, therefore, remained the most commonly studied property in the behaviour of dynamical systems. Hence, this study investigated the stability character of nontrivial solutions of equation (4) and employed Hopf bifurcation analysis for the nontrivial steady state.

The whole paper is partitioned as follows. In section 1, a detailed introduction of the study is given. Section 2 contains the methodology for the study, while section 3 investigated the nontrivial stability analysis at the specific equilibrium point and the Hoph bifurcation discussed. In section 4, numerical simulations and biological interpretations of the nontrivial stability are discussed. Finally, section (5) concludes the paper.

For the phase space of this study, let $\tau \in [0, \infty)$ be a positive number and the Banach space C denotes vector space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}_+^n$ with the norm

$$\|.\| = \sup_{t_0 \epsilon [-\tau,0]} |\phi(t_0)|, \ \phi \in C.$$

The following definition is employed for the investigation of stability analysis in this study.

Let $\mathrm{T}: \mathbb{R}^{\mathrm{n}} \longrightarrow \mathbb{R}^{\mathrm{n}}$ be a linear map. The flow of T is $\mathrm{T}(\mathrm{T_0}, \mathrm{t}) = T_0 e^{tT}$. Let $\{\lambda_j\}$ be the characteristic values of T. Then $\{e^{\lambda_j t}\}$ are the characteristic values of e^{tT}. Suppose $R_e\lambda_j < 0$ for all j, then $|e^{\lambda_j t}| = e^{R_e \lambda t} \longrightarrow 0$ as $t \longrightarrow \infty$. In this case, the point 0 is asymptotically stable. If there is λ_j with $\lambda_j > 0$, say j, we say the point 0 is unstable.

Since stability analysis of delay systems is simpler to investigate in a complex plane, the study implored some relevant results in spectral theory for easy analysis of characteristic roots.

Theorem 1: If $T: \mathrm{E} \longrightarrow \mathrm{E}$ is a bounded linear operator on a Banach space and let $\sigma(T)$ be the spectrum, $\sigma(T) = \{\lambda \in \mathbb{C} | \mathrm{T} - \lambda \mathrm{I} \text{ is not invertible on the complexification of } E\}$. Then $\sigma(T)$ is nonempty, compact and for $\lambda \in \sigma(T)$, $|\lambda| \leq \|T\|$. The spectral radius defined by $r(T) = \sup\{|\lambda| \ |\lambda \in \sigma(T)\}$ is given by the spectral radius formula of the form $r(T) = \lim_{n \to \infty} \|T^n\|^{\frac{1}{n}}$.

From the theory above, the existence of finite roots in the right half complex plane $\mathbb{C}$ is assured.

Theorem 2: If $\mathrm{Re}\, \lambda < \beta$ for every solution $x(t)$ of associated transcendental equation (4), then there exists a constant $\mathrm{M} > 0$ such that, for each $\phi \in \mathrm{C}\ ([\mathrm{t_0} - \tau, \mathrm{t_0}], \mathbb{R})$, the solution of linear DDE satisfies

$$\|x(t,\phi)\| \le M\|\phi\|e^{\beta(t-t_0)}.$$

Thus, the behaviour of solutions of delay system is given an upper bound by the location of the characteristic root with the largest real part. Moreso, the solutions $x(t)$ of the linearised DDE (4) decay exponentially to zero. That is, by linearisation theory, $x(t)$ is asymptotically stable if all eigenvalues of the linearised system have negative real parts. The lemma below gives better techniques to investigate the asymptotic stability of linearised systems by computing the characteristic roots.

Lemma 1: If $|a| < b$, then the roots of the linear delay differential equation

$$\frac{dx(t)}{dt} = ax(t-\tau) - bx(t)$$

approach 0 as $t \longrightarrow \infty$.

The primary objective in the analysis of linearised DDEs is to identify the value of the delay for which the real part of the principal root $R_e(\lambda_c)$ becomes positive. A standard method for proving that such a delay τexists is to decompose the characteristic equation into real and imaginary equations and establish the existence of τ_c (critical delay) such that the principal root at the critical delay $\lambda_c(\tau_c)$ is purely imaginary. The eigenvalue of such principal root at the critical delay $\lambda_c(\tau_c)$ passes through and continue in the positive real half plane if the criteria

$$\frac{d}{d\tau} R_e\lambda_c(\tau)\Big|_{\tau=\tau_c} > 0 \text{ holds.}$$

The folloing theorem plays useful role for Hoph bifurcation analysis for this study.

Theorem 2 (Rouche's Theorem): Let A be an open set in $\mathbb{C}$, the set of complex numbers, F metric space, f a continuous complex valued function in $A \times F$, such that for each $\alpha \in F$, $Z \longrightarrow f(z,\alpha)$ is analytic in A. Let $B \subset A$ be an open set in A whose closur such that $\bar{B}$ in $\mathbb{C}$ is compact and contained in A, and $\alpha_0 \in F$ be such that no root of $f(z,\alpha)$ is on the frontier of. Then there exists a neighbourhood W of α_0 in F such that;

i. For any $\alpha \in W$, $f(z,\alpha)$ has no zero on the frontier of B;

ii. For any $\alpha \in W$, the sum of the orders of the roots $f(z,\alpha)$ belonging to B is independent of α.

2.0: Stability Analysis of Nontrivial Steady State

The system represented by model (4) above possesses a nontrivial equilibrium point when $P > \delta + H$. Thus for $Pe^{-ax^*} = \delta + H$, $x^* = \frac{1}{a} ln \frac{P}{\delta+H}$.

From the general form and with linear harvesting function, equation (1) becomes

$$\dot{x}(t) = -\delta(x(t))x(t) + P(x(t-\tau))x(t-\tau) - H(x(t-\tau))x(t-\tau). \quad (5)$$

The function $P(x)$ is assumed continuous, positive and decreasing. The per capita growth rate of the population drops as the population size increases. The functions $\delta(x)$ and $H(x)$ are nondecreasing positive, where $\delta(x)$ and $H(x)$ are as defined in equations (2) and (4).

Theorem 3: Let P, δ and H be as defined above. Suppose there exists a zero at x^* such that sign $\big(P(x) - (\delta + H)(x)\big) = -\text{sign}\,(x - x^*)$ with $(P' - H')(x^*) < \delta'(x^*)$, then x^* is a positive steady state and the trivial steady state is unstable. If, in addition,

$$(P' - H')(x^*)x^* > -2\delta(x^*) - \delta'(x^*)x^*, \tag{6}$$

then x^* is linearly stable for all τ. On the other hand, there exists critical delay $\tau_c > 0$ such that x^* is stable for $\tau < \tau_c$ and unstable for $\tau > \tau_c$.

Proof:

From the hypothesis of the theorem, the zero of x^* is the only positive steady state when $\big(P(x) - (\delta + H)(x)\big) = 0$ (i.e. $P - H = \delta$)if and only if $x = x^*$. This is the point at which

$$P(x^*) = (\delta + H)(x^*).$$

Linearising equation (5) about the steady state at x^* yields the followings equation

$$\dot{x}(t) = \big((P - H)(x^*) + (P' - H')(x^*)x^*\big)x(t - \tau) - (\delta(x^*) + \delta'(x^*)x^*)x(t). \tag{7}$$

The associated characteristic equation of (6) yields

$$\lambda = \gamma x(t - \tau) - \beta x(t) \tag{8}$$

where,

$\gamma = \delta(x^*) + (P' - H')(x^*)x^*$ and $\beta = \delta(x^*) + \delta'(x^*)x^*$.

From equation (7), it is clear that $(P' - H')(x^*) < \delta'(x^*)$, $\gamma < \beta$. For $|\gamma| < |\beta| = \beta$, the roots of equation (8)possesses negative real parts (Lemma 1). Since $\gamma < \beta$, condition (6) holds if and only if $\gamma > -\beta$. If this is not the case, then $\gamma < -\beta$. From equation (8), it is clear that for $\tau = 0$ the only characteristic root is $\lambda = \gamma - \beta < 0$. Thus, by continuation of the location of roots for small delays, the system is stable. The derived polynomial for the characteristic equation is therefore$\sigma = \sqrt{\gamma^2 - \beta^2}$, which clearly has a positive real root. By Hoph bifurcation Theorem, we suppose that there is $\tau = \tau_c$ for which equation (8) has two purely imaginary roots. As τ increases past τ_c, the positive root enters the right half-plane. Since the derived polynomial has degree 1, the Sturm sequence analysis indicates that this root can not exist. Thus, the steady state is stable for $\tau < \tau_c$ and unstable for $\tau > \tau_c$.

We next consider the stability of equation (4) on considering the particular case of theorem (6), where $\delta, P, \tau, a, H, \in (0, +\infty)$, and for $P > \delta + H$, the nontrivial equilibrium of equation (4) will occur only if $Pe^{-ax^*} = \delta + H$, i.e., $x^* = \frac{1}{a} ln \frac{P}{\delta + H}$.

By applying Taylor's expansion of first order at $x = x^*$ and treating $x(t)$ and $x(t - \tau)$ as separate variables, the linearised equation (4) at $x = x^*$ yields the form

$$\dot{x}(t) = -\delta x(t) + (\delta + H)\alpha x(t - \tau) - Hx(t - \tau) \tag{9}$$

where $\alpha = 1 - \mathrm{l_n}\frac{P}{\delta+H}$.

The nontrivial stability exists and occurs if $Pe^{-ax^*} = \delta + H$ or $x^* = \frac{1}{a}ln\frac{P}{\delta+H}$

From the particular case of Theorem (3), x^* is stable if and only if

$$\frac{d}{dx}(Pe^{-ax})\Big|_{x=x^*} > -\frac{2(\delta+H)}{x^*}$$

i.e., $Pe^{-ax^*} < \frac{2(\delta+H)}{ax^*} \Longrightarrow \frac{P}{e^{a\left(\frac{1}{a}\ln\frac{P}{\delta+H}\right)}} < \frac{2(\delta+H)}{a\left(\frac{1}{a}\ln\frac{P}{\delta+H}\right)}$. Hence the condition $\frac{P}{\delta+H} < e^2$ holds.

Thus $P < (\delta + H)e^2$ is linearly stable at the nontrivial point regardless of the value of τ.

Therefore, $\frac{d}{dx}(Pe^{-ax})\Big|_{x=x^*} > -\frac{2(\delta+H)}{x^*}$ is equivalent to $P < (\delta + H)e^2$.

From the analysis of Hopf bifurcation, there exists critical delay $\tau_c > 0$ such that x^* is stable for $\tau < \tau_c$ and unstable for $\tau > \tau_c$. Suppose $P > (\delta + H)e^2$ and $\alpha = 1 - l_n\frac{P}{\delta+H}$. This means that the lower unit of α is given by $\alpha = 1 - \mathrm{l_n}\frac{P}{\delta+H} = 1 - \frac{(\delta+H)l_n e^2}{\delta+H} = -1 < 0$ (where $P > (\delta + H)e^2$). Hence α is negative.

2.1.The Hopf Bifurcation

This section discusses the Hopf bifurcation of (4) by choosing the delays as a bifurcation parameter. Thus, for $P > (\delta + H)e^2$ there is a critical delay τ_c for which the characteristic equation has a purely imaginary root. That is, as τ increases past τ_c, a root enters the right half plane.Substituting the solution $x(t) = x_0e^{\lambda t}$, where $x_0 \in \mathbb{R}$ a constant and λ the characteristic roots, theassociated characteristic equation of (9) yields

$$\lambda = -\delta + \big((\delta + H)\alpha - H\big)e^{-\lambda\tau} = 0 \tag{10}$$

By Rouche's Theorem, the transcendental characteristic equation (10) has roots with positive real parts if and only if it has purely imaginary roots.

Let $\lambda = i\omega_c$ be the purely imaginary root. On substituting $\lambda = i\omega_c$ in equation (10) yields separated real and imaginary parts of equation (10) of the form

$$[(\delta + H)\alpha - H]\cos\omega_c\tau = \delta \tag{11}$$

$$[(\delta + H)\alpha - H]\sin\omega_c\tau = -\omega_c, \text{ where } \omega_c\tau \in \left(\frac{\pi}{2}, \pi\right) \tag{12}$$

Squaring each of the equations of (11) and (12) and summing the results yields

$$[(\delta + H)\alpha - H]^2 = \delta^2 + \omega_c{}^2. \tag{13}$$

Equation (13) is a polynomial equation that is independent of the trigonometric function and the delay term, and is expressed by equality of even polynomials. Clearly, if the condition

$$|(\delta + H)\alpha - H| > |\delta| \tag{14}$$

holds, then equation (4) has a pair of purely imaginary roots $\pm i\omega_c$ at a sequence of critical values τ_k, $k = 0,1,2, \ . \ . \ ..,$

where

$$\omega_c \tau_k = \pm\sqrt{[(\delta + H)\alpha - H]^2 - \delta^2}. \quad (15)$$

For $\omega_c \in \mathbb{R}_+$ in equation (11), the periodic critical values becomes

$$\tau_k = \frac{1}{\omega_c}\left[\arccos\frac{\delta}{[(\delta+H)\alpha-H]} + 2k\pi\right], k \in \mathbb{Z}, k = 0,1,2, \ . \ . \ .. \quad (16)$$

At $k = 0$, the equation (16) yields the critical value of the form

$$\tau_c = \frac{1}{\omega_c}\left[\arccos\frac{\delta}{[(\delta+H)\alpha-H]}\right]. \quad (17)$$

Also, at $\tau = 0$, the characteristic equation (10) becomes

$$\lambda = [(\delta + H)\alpha - H] - \delta. \quad (18)$$

If the condition

$$[(\delta + H)\alpha - H] < \delta \quad (19)$$

holds, then equation (4) is stable.

Lemma 3: If conditions (14) and (19) hold, then the dynamical system of equation (4) admits a pair of imaginary roots $\lambda = \pm i\omega_c$ when $\tau = \tau_k$, $k = 0,1,2, \ . \ . \ ..$

Proof: Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be the continuation of the root of the characteristic equation (10) near $\tau = \tau_k$ satisfying $\alpha(\tau_k) = 0$ and $\omega(\tau_k) = \omega_c$. For every $\tau_k(k = 0,1,2, \ . \ . \ .)$ there exists a $\varepsilon > 0$ such that $\lambda(\tau)$ is continuously differentiable in τ for $|\tau - \tau_k| < \varepsilon$.
On substituting $\lambda(\tau)$ into the left hand side of equation (10), we have

$$\lambda(\tau) = -\delta + \big((\delta + H)\alpha - H\big)e^{-\lambda\tau}. \quad (20)$$

Theorem 4: If conditions (14) and (19) hold, the equilibrium point x^* of equation (4) is asymptotically stable for $\tau \in [0, \tau_c)$ but unstable for $\tau \geq \tau_c$. System (4) undergoes a Hopf bifurcation at the equilibrium point x^* such that at $\tau = \tau_k$, $k = 0, 1, 2, \ . \ . \ . \ .$

Proof: If $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ is the continuation of the root of $i\omega$, it is necessary to confirm that the root continue into the positive half plane as τ increases past τ_k. Thus, the criterion for nondegeneracy to occur is $\frac{d}{d\tau}Re(\lambda)\Big|_{\tau=\tau_k,\, \lambda=i\omega_c} > 0$.

Taking the derivative of λ with respect to τ in equation (20), we have

$$\frac{d\lambda}{d\tau} = -\left(\lambda + \tau\frac{d\lambda}{d\tau}\right)[(\delta + H)\alpha - H]e^{-\lambda\tau}$$

$$\frac{d\lambda}{d\tau}\big(1 + \tau[(\delta + H)\alpha - H]e^{-\lambda\tau}\big) = -\lambda[(\delta + H)\alpha - H]e^{-\lambda\tau}$$

$$\frac{d\lambda}{d\tau} = \frac{\lambda(H-(\delta+H)\alpha)e^{-\lambda\tau}}{1+(\tau\alpha(\delta+H)-\tau H)e^{-\lambda\tau}}. \quad (21)$$

For non-transversality test, we take the inverse of equation (21) to get

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{1+(\tau\alpha(\delta+H)-\tau H)e^{-\lambda\tau}}{\lambda e^{-\lambda\tau}(H-(\delta+H)\alpha)}, \text{ which simplifies to}$$

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{e^{\lambda\tau}}{\lambda(H-(\delta+H)\alpha)} + \frac{(\tau\alpha(\delta+H)-\tau H)}{\lambda(H-(\delta+H)\alpha)}$$

From the above analysis, it then follows together with equation (11) that

$$R_e\left[\frac{d\lambda}{d\tau}\right]^{-1}\bigg|_{\lambda=i\omega_c} = R_e\left\{\frac{cos\omega_c\tau_k + isin\omega_c\tau_k}{[H-\alpha(\delta+H)]i\omega_c}\right\}, \lambda = i\omega_c$$

$$= \frac{sin\omega_c\tau_k}{[H-\alpha(\delta+H)]\omega_c} = \frac{[H-\alpha(\delta+H)]\omega_c sin\omega_c\tau_k}{[H-\alpha(\delta+H)]^2\omega_c^2}, \text{ from equation (12).}$$

$$= \frac{\omega_c^2}{[H-\alpha(\delta+H)]^2\omega_c^2} = \frac{1}{[H-\alpha(\delta+H)]^2} > 0$$

Thus, $\text{sign}\left\{R_e\left[\frac{d\lambda}{d\tau}\right]\bigg|_{\tau=\tau_k,\,\lambda=i\omega_c}\right\} = \text{sign}\left\{R_e\left[\frac{d\lambda}{d\tau}\right]^{-1}\bigg|_{\tau=\tau_k,\,\lambda=i\omega_c}\right\} > 0.$

Hence, $\frac{d}{d\tau}Re(\lambda)\Big|_{\tau=\tau_k,\,\lambda=i\omega_c} > 0.$

Theorem 5: For equation (3) the following hold:

i. If $\delta < P \leq (\delta + H)$, then $x = x^*$ is asymptotically stable.

ii. If $P > (\delta + H), x = x^*$ is asymptotically stable for $\tau \in [0,\ \tau_c)$ and unstable for $\tau >\ \tau_c$

iii. If $P > (\delta + H)e^2$, equation (3) undergoes Hopf bifurcation at $x^*$for $\tau = \tau_k, k = 0, 1,. . .$

3.0: Numerical Example ofStability Analysis of Positive Equilibrium

Given $\frac{dx(t)}{dt} = -0.2x(t) + 4.0x(t-\tau)e^{-3x(t-\tau)} - 1.2x(t-\tau)$, if conditions (14) and (19) hold, then (4) has a unique equilibrium at $x^* = 0.3269$. Here we get $\alpha = -0.0498$, $\omega_c \approx 1.2539$, $\tau_c \approx 1.3789$. Thus, the equilibrium at $x^* = 0.3499$ is stable when $\tau < \tau_c$ which is illustrated by the computer simulations (see Figures 1-6). When τ passes through the critical value $\tau_c \approx 1.3789$, the equilibrium at $x^* = 0.3499$ loses its stability and a Hopf bifurcation occurs, i.e., a family of periodic solution bifurcates from the equilibrium at $x^* = 0.3499$ which are depicted in Figures 5-6. When τ is large, a chaotic or aperiodic phenomenon will appear. The above analyses are displayed by the Figures below:

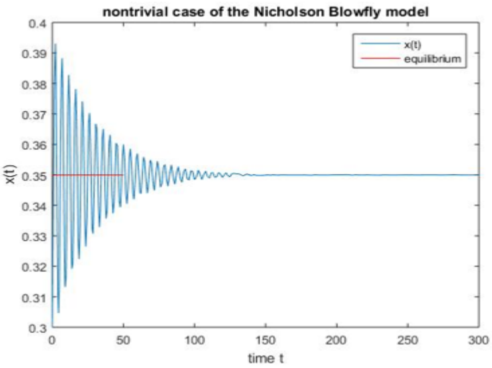

Figure 1: Equilibrium at at $\tau < \tau_c$

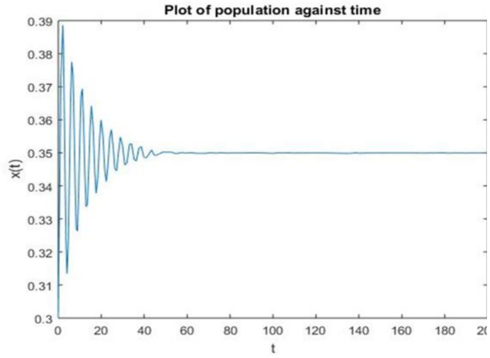

Figure 2:. Equilibrium at at $\tau < \tau_c$

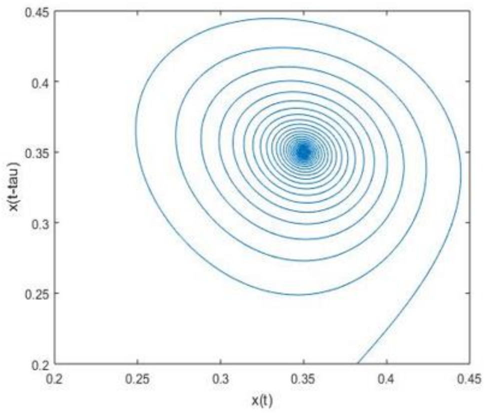

Figure 3. Equilibrium at $\tau < \tau_c$

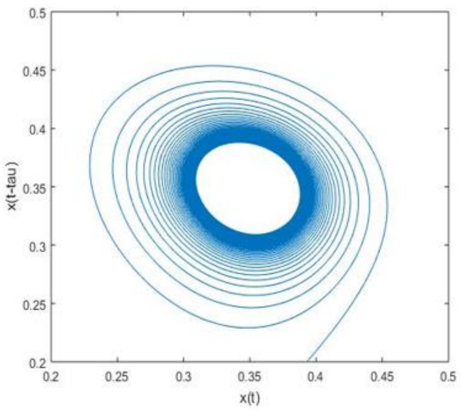

Figure 4. Equilibrium at $\tau < \tau_c$

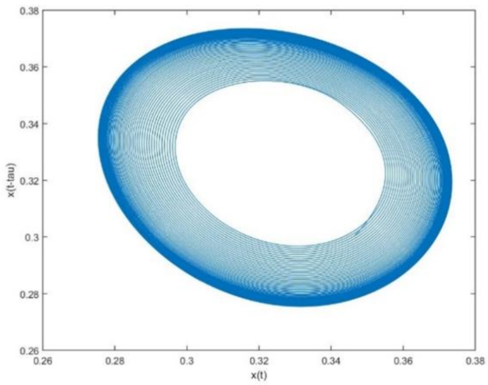

Figure 5: Aperiodic solution $\tau > \tau_c$

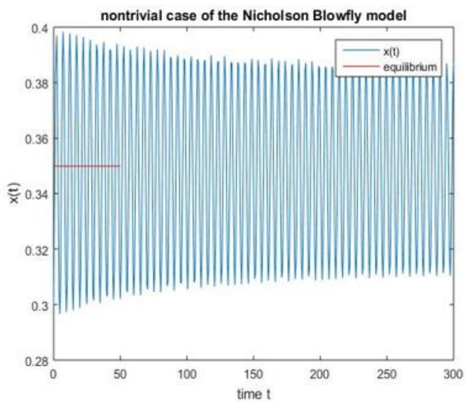

Figure 6: Aperiodic solution $\tau > \tau_c$

4.0: Conclusion

It is observed under certain conditions, there exists a critical value τ_c of the delay for which the stability of the Nicholson blowflies delay system can be investigated. If $\tau \in [\tau, \tau_c)$, the zero equilibrium of the Nicholson blowfly's system is asymptotically stable,which means that the size of the population remains in a steady state. However, when the delay τ passes through some critical values $\tau = \tau_k, k = 0, 1, 2, . . .$, the zero equilibrium of the system loses its stability and a Hopf bifurcation occurs. Moreover, it is shown that chaotic phenomena appearfor large delay (i.e., nondegeneracy). The obtained results complement the conclusions of previous authors in the literature [16-18] while other aspects of epidemiology of related interest include [19-21]. The existence of a solution can be proved using Mawhin's coincidence degree theory and other well-known theorems for highly nonlinear versions of the considered model [22-24]. The numerical simulations show the existence of chaos for assured parametervalues. The associated parameter bifurcation diagrams are plotted for certain values of the parameters.The irregular and unpredictable behaviour of various species can be controlled by introducing an approximate number of harvesting function so that the system can be balanced dynamically.

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