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Abstract: The paper investigates the stability of the SIR mathematical model of transmission of an infectious disease with delay. First, the study investigates local stability of the positive steady state of an infectious disease model by analyzing the linearised system where more general stability criteria with delay and model parameters are obtained. Secondly, the study shows that the model exhibits Hopf bifurcation on choosing the delay as a bifurcation parameter. Conditions for existence of qualitative behaviour for positive steady state are identified. Finally, numerical simulation of results and biological interpretations were verified using MATLAB software for the delay model. The study supplements theoretical improvement to earlier results obtained in the literature.

Keywords: Characteristic equation, Differential equations, Hopf bifurcation, Reliable Jacobian Matrix, Stability analysis.

1.0: Introduction:

For several years, the outbreak of infectious diseases has been a great challenge and has been studied by various researchers. It has been observed that the ability to make predictions concerning infectious diseases enabled scientists to evaluate inoculation or isolation plans, which further have a significant effect in reducing the mortality rate of a particular population or species. Transmission of infectious diseases, such as tuberculosis, cholera, measles, smallpox, and their dynamics have been modeled by many scientists and mathematicians along this line. It is also evident that the application of mathematics to proffer solutions, regulate, control and reduce epidemic outbreaks has proven very successful [1-6]. Kermack & McKendrick [7] earlier developed theoretical papers on infectious disease models using differential equation model. Ever since then, different deterministic and stochastic models have been formulated and applied to a variety of infectious diseases. For example, see [4, 8]. Different reasons have been adduced to spread and transmit infectious diseases [9-10]. However, in modeling infectious disease transmission models, it is sometimes necessary to divide the total population into various groups and subgroups depending on different infection rate, multiple infection, among others. An example of such compartmentalisation is the SIR model where S is the total susceptible group to the disease, I is the total infected group to the disease and R is the total recovered group. In some cases, the infected or recovered population is categorised into subgroups [11-15], while in other cases, there may be multiple infections [16]. Recently, many variations of SIR model involving complex diseases and infection mechanisms have been studied. The role of infectious disease in stabilizing the population is significant, hence the need for

a realistic mathematical model. For instance, Liu et al., [17] in their mathematical model, considered the spread of disease in favour of bilinear law (i.e., $\lambda(\mathrm{I}) = \beta f(\mathrm{I})\mathrm{S}$). In this case, $f(\mathrm{I})$ becomes unbounded when I is large making the model equation complicated and unrealistic. In their investigation, [18-21], gave realistic reasons in favour of nonlinear transmission rate of infectious disease, especially when such disease is transmitted from one person to another susceptible person. Since the process of infection is not instantaneous, there is the need to include time delay for more realistic disease models.
The qualitative character of solutions of delay systems drives system models to exhibit complex characteristics such as periodic orbits, sustained oscillation, chaotic attractors, and classifications induced by different bifurcation analysis and stability switches [22]. These qualitative features make possible parameter classification for delay-independence stability and delay-dependence stability that are complicated and challenging in stability analysis. However, applications of model dynamics resulting from the inclusion of time delay and nonlinearity in mathematical formulations provide realistic results in infectious disease transmission models. It is now obvious that stability behaviour of systems are commonly studied property and the mathematical formulations of underlying structure of stability of an infectious disease models attracted a lot of research attention in investigating the relative stability of the qualitative behaviour dynamics in science, technology and engineering. For points around equilibriums, linear terms will dominate the higher-order terms, and Taylor's formula of order one is employed for linear stability. Since around equilibrium point, qualitative behavior of nonlinear systems is similar to that of the linearised systems, therefore, finite dimensional linear method is employed to determine the stability of equilibrium point of nonlinear system. For a more realistic model, the study employed intracellular delay based on existing SIR mathematical model of [8, 12], investigated the qualitative behaviour of the delayed system using linearization techniques. The distribution of roots on right-hand plane coupled with conditions for stability switch were derived using Hopf bifurcation. Numerical simulations and biological interpretations of equilibrium results are discussed using MATLAB software. The paper for study is structured such that Section 1 gives general introduction of the study. Section 2 provides the underlying assumptions and existence of solution of the considered model. Section 3 provides the stability analysis of the model at specified point and the Hoph bifurcation discussed. The numerical simulations and biological interpretations of the nontrivial stability are discussed in section 4 and finally, the paper is remarkably concluded.

2.0: The Mathematical Infection Disease Model

For investigation of qualitative behavior of delay systems, this study is motivated by susceptible infection (SI) epidemic model of open system with variable size studied by Golam, Panja & Mondal, 2020 and of the form,

$$\left.\begin{aligned}\frac{dS}{dt} &= S\alpha(S) - \beta(S,I)\\ \frac{dI}{dt} &= \beta(S,I) - \delta I\end{aligned}\right\}, \tag{1}$$

where $S(t_0) = S_0 > 0, I(t_0) = I_0 > 0$.
The dependent variables S and I are the population sizes of susceptible and infected of the infectious diseases. In this model, $\alpha(\mathrm{S})$ represents the susceptible intrinsic growth rate of the susceptible population, $\beta(\mathrm{S},\mathrm{I})$ represents the transmission rate of the disease expressing nonlinear mass action and δ is the addition of the death rate due to the presence of the disease μ and the natural death rate γ i.e., $\delta = \mu + \gamma$. Based on the preceding model, this study considered a more general SI model which included the recovered population, represented by R as in the works of [7, 12] of the form

$$\left.\begin{aligned}\mathrm{S}'(\mathrm{t}) &= \pi - \beta SI - \mu S\\ \mathrm{I}'(\mathrm{t}) &= \beta SI - (\gamma+\mu)I\\ \mathrm{R}'(\mathrm{t}) &= -\mu R\end{aligned}\right\}. \tag{2}$$

On defining the delay term, τ the associated delay differential equation becomes

$$\left.\begin{aligned} \frac{\mathrm{dS}}{\mathrm{dt}} &= \pi - \beta SI - \mu S \\ \frac{\mathrm{dI}}{\mathrm{dt}} &= \beta S(t-\tau)I - (\gamma+\mu)I \\ \frac{\mathrm{dR}}{\mathrm{dt}} &= -\mu R, \end{aligned}\right\}, \tag{3}$$

where,
$S(t_0) = S_0 > 0,\ I(t_0) = I_0 > 0,\ R(t_0) = R_0 > 0$ and $t_0 \in [0, \tau]$. Let $\phi = (S_0, I_0, R_0)$ from where $\mathbb{R}^3_{+0} = \{\mathrm{S}, \mathrm{I}, \mathrm{R} | \mathrm{S}_0 \geq 0, \mathrm{I}_0 \geq 0, \mathrm{R}_0 \geq 0\}$.
The dependent variable S is the total susceptible group to the disease, I is the total infected group to the disease and R is the total recovered group. From equations (2) and (3), π is the recruitment into the population, β is disease transmission rate, μ is natural death rate while γ is death rate due to presence of the disease. The equation (3) is a delay model consisting of three components and a discrete disease transmission time delay describing the time between infection and the time the susceptible individual gets infected. Motivated by the models above, this study investigated the stability of transmission of an infectious disease delay model of system (3) with intracellular delay term τ. Based on the preceding assumptions and motivations from equations (1) and (2), the study investigates model (3) for local asymptotic stability using methods of characteristic and algebraic tools. The study, which incorporates intracellular delay terms, followed the work of [23, 24]. For general investigation of local stability, the study provides a detailed analysis of the delay equation (3), which improves the work of previous authors such as Song & Xiao, 2017.
The study investigates the stability of solutions of model (3). The study equally analyse and provides conditions for stability of solutions of model (3). The effect of delay on stability is verified using MATLAB software to confirm the theoretical results of the complex dynamics.
For preliminary analysis of model (3), we explore some basic properties of solutions to system of equations (3). The theorems below established the positivity and boundedness of solutions of model (3).

Theorem 1: For positive initial data, solutions of equations (3) are positive for all $t \geq 0$.
Proof: From the first equation of system (3),

$$\frac{\mathrm{dS}}{\mathrm{dt}} = \pi - \beta SI - \mu S$$

We claim that $S(t) > 0$ for all $t > 0$. Suppose not, there exist $t_1 > 0$ and $\varepsilon_1 > 0$ such that $S(t) > 0$ for $t < t_1, S(t) = 0$ for $t = t_1$ and $S(t) < 0$ when $t \in [t_1, t_1 + \varepsilon_1)$. Thus

$$\begin{aligned} \frac{\mathrm{dS}}{\mathrm{dt}} &= \pi - \beta S(t_1)I(t_1) - \mu S(t_1) \\ &= \pi > 0. \end{aligned}$$

This is a contradiction. Hence $S(t)$ is positive for all $t > 0$.
From the second equation of equations (2),
$\frac{\mathrm{dI}}{\mathrm{dt}} = \beta S(t-\tau)I - (\gamma+\mu)I,$
$\frac{\mathrm{dI}}{I} = [\beta S(t-\tau) - (\gamma+\mu)]dt,$
where on integrating and based on the initial value, we have
$I(t) = I_0(t)e^{\int_0^t [\beta S(v-\tau) - (\gamma+\mu)]dv}$
This implies $I(t) > 0$ for all $t > 0$ since $I_0(t) > 0$. Hence $I(t)$ is positive for all $t > 0$.
Also, from the third equation of equations (2), $\mathrm{R}'(\mathrm{t}) = -\mu R$,
$\frac{\mathrm{dR}}{R} = -\mu dt$, and on integrating based on the initial value, we have
$\mathrm{R(t)} = R_0 e^{-\mu t}$.
This implies $R(t) > 0$ for all $t > 0$. Hence $R(t)$ is positive for all $t > 0$.

Theorem 2: The solutions of model (3) are ultimately bounded.
Proof: Let $N = S + I + R$.

$$\frac{dN(t)}{dt} = \pi - \mu S - (\gamma + \mu)I - \mu R + \beta I\big(S(t-\tau) - S(t)\big)$$

Since coefficients of $\beta S(t-\tau)$ and $\beta S(t)$ are equal, solutions need not approach the origin but a positive limit.

Thus, $\frac{dN(t)}{dt} \leq \pi - \mu S - (\gamma + \mu)I - \mu R$

Choose a constant $\xi > 0$ such that

$$\frac{dN(t)}{dt} + \xi N \leq \pi - \mu(S - \xi) - (\gamma + \mu)(I - \xi) - \mu(R - \xi)$$

Define $\xi = \min\{\mu, \gamma + \mu, \gamma\}$ such that

$\frac{dN(t)}{dt} + \xi N \leq \pi$. Therefore, $Ne^{\xi t} \leq \int \pi e^{\xi t} dt + C$ yields

$N(t) = \frac{\pi}{\xi} + \left(N_0 - \frac{\pi}{\xi}\right) e^{-\xi t}$ and $\lim_{t\to\infty} N(t) \leq \frac{\pi}{\xi}$. Hence the result.

For the phase space of this study, let $\tau \in [0, \infty)$ be a positive number and the Banach space C is the vector space of continuous functions mappingthe interval $[-\tau, 0]$ into $\mathbb{R}^n_+$ with the norm

$$\| . \| = \sup_{t_0 \epsilon [-\tau, 0]} |\phi(t_0)|, \ \phi \in C.$$

The following definition is employed for the investigation of stability analysis in this study.

Definition: Let $\mathrm{T}: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear map. The flow of T is $\mathrm{T}(\mathrm{T}_0, \mathrm{t}) = T_0 e^{tT}$. Let $\{\lambda_j\}$ be the characteristic values of T. Then $\{e^{\lambda_j t}\}$ are the characteristic values ofe^{tT}. Suppose $R_e \lambda_j < 0$ for all j, then $|e^{\lambda_j t}| = e^{R_e \lambda t} \longrightarrow 0$ as $t \longrightarrow \infty$. In this case, the origin 0 is asymptotically stable. If there is λ_j with $\lambda_j > 0$, say j, we say the point 0 is unstable.

Since stability analysis of delay systems is simpler to investigate in a complex plane, the study implored some relevant results in spectral theory for easy analysis of characteristic roots.

Theorem 3: If $T: \mathrm{E} \longrightarrow \mathrm{E}$ is a bounded linear operator on a Banach space and let $\sigma(T)$ be the spectrum, $\sigma(T) = \{\lambda \in \mathbb{C} | \mathrm{T} - \lambda \mathrm{I}$ is not invertible on the complexification of $E\}$. Then $\sigma(T)$ is nonempty, compact and for $\lambda \in \sigma(T)$, $|\lambda| \leq \|T\|$. The spectral radius defined by $r(T) = \sup\{|\lambda| \ | \lambda \in \sigma(T)\}$ is given by the spectral radius formula of the form

$$r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}.$$

From the theorem above, the existence of finite roots in the right half complex plane $\mathbb{C}$ is assured. The following lemma gives better techniques to investigate the asymptotic stability of linearised systems by computing the characteristic roots.

Lemma 1: If $|a| < b$, then the roots of the linear delay differential equation

$$\frac{dx(t)}{dt} = ax(t-\tau) - bx(t) \text{ approach } 0 \text{ as } t \longrightarrow \infty.$$

The primary interest in investigating the stability analysis of system (3) is to identify the value of the delay for which the real part of the principal root $R_e(\lambda_c)$ becomes positive. The characteristic roots at the critical delay $\lambda_c(\tau_c)$ passes through and continue in the positive real half plane if the criteria $\frac{d}{d\tau}R_e\lambda_c(\tau)\Big|_{\tau=\tau_c} > 0$ is satisfied. The following theorem supports the objective of bifurcation for the study.

Theorem 4: (Rouche's Theorem): Let A be an open set in $\mathbb{C}$, the set of complex numbers, F a metric space, f a continuous complex valued function in $A \times F$, such that for each $\alpha \in F$, $Z \longrightarrow f(z,\alpha)$ is analytic in A. Let $B \subset A$ be an open set in A whose closur such that $\bar{B}$ in $\mathbb{C}$ is compact and contained in A, and $\alpha_0 \in F$ be such that no root of $f(z,\alpha)$ is on the frontier of B. Then there exists a neighbourhood W of α_0 in F such that;

i. For any $\alpha \in W$, $f(z,\alpha)$ has no zero on the frontier of B;

ii. For any $\alpha \in W$, the sum of the orders of the roots $f(z,\alpha)$ belonging to B is independent of α.

2.1 Linearization For Autonomous Constant Delay Differential Equations

Thus, around equilibrium qualitative, the behaviour of nonlinear system will be similar to that of the linearised system. For linearization of autonomous constant delay system in $\mathbb{R}^n$, we consider

$$\dot{x}(t) = f\big(x(t), x(t-\tau_1), x(t-\tau_2), \ldots, x(t-\tau_m)\big).$$

Let $f(x, v_1, \ldots, v_m)\colon \mathbb{R}^{\boldsymbol{n}} \times \mathbb{R}^{\boldsymbol{mn}} \longrightarrow \mathbb{R}^{\boldsymbol{n}}$ satisfy $f(0,0,\ldots,\ 0) = 0$, so $x = 0$ is a steady state i.e.,

$$\dot{x}(t) = \mathrm{A}_0 x(t) + \textstyle\sum_{j=1}^{m} A_j x\big(t-\tau_j\big),$$

where, $A_0 = f_x$ and $\mathrm{A_j} = f_{vj}$ are $\mathrm{n} \times \mathrm{n}$ matrices evaluated at the steady state, which are essentially Jacobian matrices for each delay, making linearization a variational equation (Humphries, 2016).

3. Stability Analysis of an Infectious Disease Model

In order to determine the behaviours of the system, system (3) is rewritten with the following expression

$$\left.\begin{aligned} \frac{\mathrm{dS}}{\mathrm{dt}} &= \pi - \beta SI - \mu S = \psi\big(S,\ I, P\big) \\ \frac{\mathrm{dI}}{\mathrm{dt}} &= \beta S(t-\tau)I - (\gamma+\mu)I = \varphi\big(S,\ I, P\big) \\ \frac{\mathrm{dR}}{\mathrm{dt}} &= -\mu R = \phi\big(S,\ I, P\big) \end{aligned}\right\}, \tag{4}$$

The equilibrium of system (3) can be obtained by solving the equations

$$\frac{\mathrm{dS}}{\mathrm{dt}} = 0,\ \frac{\mathrm{dI}}{\mathrm{dt}} = 0 \text{ and } \frac{\mathrm{dR}}{\mathrm{dt}} = 0. \tag{5}$$

Direct solutions of system (3) yield two kinds of equilibriums. The trivial equilibrium $E_0(0,0,0)$ which describes the absence of the disease and the positive interior equilibrium at $E^*\big(S^*,\ I^*,\ R^*\big)$ representing

the presence of the disease. For disease free equilibrium (DFE), $(\text{i.e.}, \text{I} \neq 0)$, system (3) becomes system (2) in the absence of delay (Egbetade *et al*,, 2018).

3.1 Stability Analysis of an Infectious Disease Equilibrium with Delay

For stability analysis of positive equilibrium point at $E^*(S^*, I^*, R^*)$, we solve system (4) for when $\text{I} \neq 0$ and obtained the equilibrium point $E^*(S^*, I^*, R^*) = \left(\frac{\gamma+\mu}{\beta}, \frac{\pi\beta-\mu(\gamma+\mu)}{\beta(\gamma+\mu)}, 0\right)$. The linearised system of (3) at $E^*(S^*,\ I^*,\ R^*) = \left(\frac{\gamma+\mu}{\beta}, \frac{\pi\beta-\mu(\gamma+\mu)}{\beta(\gamma+\mu)}, 0\right)$ yields

$$\left.\begin{aligned} \frac{d\psi}{dt} &= -\beta I^* S - \mu S - \beta S^* I \\ \frac{d\varphi}{dt} &= \beta I^* \mathrm{S}(t-\tau) + \left(\beta S^* - (\gamma+\mu)\right)\mathrm{I} \\ \frac{d\phi}{dt} &= -\mu R \end{aligned}\right\}. \tag{6}$$

Since linearisation in DDE is a variational equation, the $\text{n} \times \text{n}$ matrices evaluated at the steady state for endemic equilibrium $E^*(S^*,\ I^*,\ R^*) = \left(\frac{\gamma+\mu}{\beta},\ \frac{\pi\beta-\mu(\gamma+\mu)}{\beta(\gamma+\mu)},\ 0\right)$ of equation (4) yields

$$\frac{d}{dt}\begin{pmatrix}\psi(t)\\ \varphi(t)\\ \phi(t)\end{pmatrix} = J^*{}_A\begin{pmatrix}\psi(t)\\ \varphi(t)\\ \phi(t)\end{pmatrix} + J^*{}_B\begin{pmatrix}\psi(t-\tau)\\ \varphi(t-\tau)\\ \phi(t-\tau)\end{pmatrix} \tag{7}$$

where $J^*{}_A$ and $J^*{}_B$ are partitioned matrices of order 3 and the associated nonzero reliable Jacobian matrix of the linearised system (7) is given by

$$J^*{}_A = \begin{pmatrix} -\left(\frac{\pi\beta-\mu(\gamma+\mu)}{(\gamma+\mu)}+\mu\right) & -(\gamma+\mu) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\mu \end{pmatrix} \text{ and } J^*{}_B = \begin{pmatrix} 0 & 0 & 0 \\ \frac{\pi\beta-\mu(\gamma+\mu)}{(\gamma+\mu)} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For stability analysis of positive equilibrium of equations (4) at $E^*\left(\frac{\gamma+\mu}{\beta}, \frac{\pi\beta-\mu(\gamma+\mu)}{\beta(\gamma+\mu)}, 0\right)$, we consider the associated reliable Jacobian matrix of the characteristic equation of the form

$$\left|\lambda I - J_A - e^{-\lambda\tau} J_B\right| = 0. \tag{8}$$

Equation (8) yields the characteristic equation of the form

$$\begin{vmatrix} \lambda + \left(\frac{\pi\beta-\mu(\gamma+\mu)}{(\gamma+\mu)}+\mu\right) & (\gamma+\mu) & 0 \\ -\left(\frac{\pi\beta-\mu(\gamma+\mu)}{(\gamma+\mu)}\right)e^{-\lambda\tau} & \lambda & 0 \\ 0 & 0 & \lambda+\mu \end{vmatrix} = 0. \tag{9}$$

Thus, the associated characteristic polynomial of equations (8) and (9) is a quasi-polynomial (transcendental) function and is expressed as:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3\lambda e^{-\lambda\tau} + a_4 e^{-\lambda\tau} = 0, \tag{10}$$

where the parameters $a_i, i = 1, \ldots, 4$ are given as

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} \mu + \frac{\pi\beta}{(\gamma+\mu)} \\ \frac{\mu\pi\beta}{(\gamma+\mu)} \\ \pi\beta - \mu\gamma - \mu^2 \\ \mu\pi\beta - \mu^2\gamma - \mu^3 \end{pmatrix}, \tag{11}$$

However, unlike polynomial functions in ODE equation, equation (10) becomes complicated and difficult to handle. Firstly, it is a quasi-polynomial (i.e., transcendental) function with infinitely many characteristic roots. Secondly, since it is a transcendental function of which Routh-Hurwitz criterion cannot be applied; and thirdly, some known general tests can applied for negative root test. Hence, the need to investigate the stability of equation of (10). For stability analysis of equation (10), we assume that conditions for asymptotic stability for $\tau = 0$ are satisfied. That is at $\tau = 0$, equation (10) becomes

$$\lambda^3 + a_1\lambda^2 + (a_2 + a_3)\lambda + a_4 = 0. \tag{12}$$

The derived Routh-Hurwitz criterion for negative root test is given by

$$a_1 > 0, a_2 - \frac{a_3}{a_1} > 0, a_3 > 0. \tag{13}$$

Equation (12) yields exactly the associated characteristic polynomial equation of system (2), with the parameter values $a_2 + a_3 \approx a_2$ and $a_2 \approx a_3$ for system of (3) at $\tau = 0$.

By Rouche's Theorem and negative root test, we subject equation (10) to Hoph bifurcation analysis following the ideas in the work of [25] while other aspects of epidemiology of related interest include [26-28]. The existence of a solution can be proved using Mawhin's coincidence degree theory and other well-known theorems for highly nonlinear versions of the considered model [29-31]. For positive root test, we let $\lambda(\tau) = \eta(\tau) + i\omega(\tau), (\omega \epsilon \mathbb{R})$ be the eigenvalues of equation (10), where $\eta(\tau)$ and $i\omega(\tau)$ depend on the intracellular delay. Since the positive equilibrium of equation (3) is stable when $\tau = 0$, it follows that $\eta(0) < 0$ when $\tau = 0$. By continuity, if $\tau > 0$ is sufficiently small we have $\eta(\tau) < 0$ and the positive steady state E^* remains stable. If $\eta(\tau_0) = 0$ for $\tau_0 > 0$ such that $\lambda = i\omega(\tau_0)$ is a purely imaginary root of (11), then the positive steady state E^* losses its stability and eventually becomes unstable where $\eta(\tau)$ is positive. If $\lambda(\tau) = \eta(\tau_0) + i\omega(\tau_0)$ is the continuation of the root of $i\omega$, it is necessary to confirm that the root continue into the positive half plane as τ increases past τ_k. Thus, the criterion for nondegeneracy to occur is $\frac{d}{d\tau} Re(\lambda)\Big|_{\tau=\tau_k,\, \lambda=i\omega_c} > 0$. By the procedure for positive root test, we analyse the positive equilibrium by replacing $\lambda = i\omega$ in equation (10) to get

$$(i\omega)^3 + a_1(i\omega)^2 + a_2(i\omega) + a_3(i\omega)e^{-(i\omega)\tau} + a_4 e^{-(i\omega)\tau} = 0$$

which yields,

$$-i\omega^3 - a_1\omega^2 + ia_2\omega + ia_3\omega(cos\omega t - isin\omega t) + a_4(cos\omega t - isin\omega t) = 0. \quad (14)$$

Separating equation (14) above into real and imaginary parts, we have

$$a_1\omega^2 = a_3\omega sin\omega t + a_4 cos\omega t \quad (15)$$

$$-\omega^3 + a_2\omega = a_4 sin\omega t - a_3\omega cos\omega t, \quad (16)$$

and adding the squares of (15) and (16) yields

$$\omega^6 + ({a_1}^2 - 2a_2)\omega^4 + ({a_2}^2 - {a_3}^2)\omega^2 - {a_4}^2 = 0. \quad (17)$$

For the reduced form of (17), let $z = \omega^2$, $\alpha = {a_1}^2 - 2a_2$, $\beta = {a_2}^2 - {a_3}^2$, $\gamma = {a_4}^2$.

The reduced form of equation (17) therefore yields

$$\mathrm{L(z)} = \mathrm{z}^3 + \alpha\mathrm{z}^2 + \beta\mathrm{z} - \gamma = 0. \quad (18)$$

Assume that $\gamma = {a_4}^2 \geq 0$, we therefore make the following claim.

Claim1.

If $\gamma = 0$ and $\beta > 0$, equation (18) has no positive roots. But

$$\frac{\mathrm{dL(z)}}{\mathrm{dz}} = 3\mathrm{z}^2 + 2\alpha\mathrm{z} + \beta.$$

For the roots of (19),

$$3z^2 + 2\alpha z + \beta = 0, \quad (19)$$

where

$$z_{1,2} = \frac{-\alpha \pm \sqrt{\alpha^2 - 3\beta}}{3}. \quad (20)$$

If $\beta > 0$, then $\alpha^2 - 3\beta < \alpha^2$ i.e., $\sqrt{\alpha^2 - 3\beta} < \alpha$. Therefore, neither z_1 nor z_2 is positive and stability is assured. Hence, the claim that equation (10) have all negative real roots for delay $\tau \geq 0$, and hence the system is stable. Equation (18) is a polynomial with positive coefficients and cannot induces positive real roots for $\gamma = 0$. So, the introduction of delay cannot lead to a bifurcation. Hence, equation (18) does not have purely imaginary roots for all delay. Hence the proposition below:

Proposition 1: Given the characteristic equation of infectious disease model of system (3),

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3\lambda e^{-\lambda\tau} + a_4 e^{-\lambda\tau} = 0.$$

Suppose that,

i. $a_1 > 0, a_4 > 0, \left(a_2 + a_3 - \frac{a_4}{a_1}\right) > 0,$

ii. for $\gamma \geq 0$ and $\beta > 0, a_3 > 0,$

then the steady state E^* of system (3) is absolutely stable in which we say E^* is asymptotically stable for all $\tau \geq 0$.

We now consider the effect of positive intracellular delay, τ, on the stability of positive equilibrium E^* of equation (10) by studying the distribution of the roots. In other words, we seek conditions on τ under which equation (10) has a pair of imaginary roots. For distribution of positive roots in the complex plane of equation (10), we let $\lambda(\tau) = \eta(\tau) + i\omega(\tau)$ be the eigenvalues of equation (10) near $\tau = \tau_k$ satisfying $\eta(\tau_0) = 0$ and $\omega(\tau_0) = \omega_0$. From equations (15) and (16), we have

$$a_3 \cos\omega_0\tau_k = a_4\left[\frac{a_1\omega_0}{a_3} - \frac{a_4 cos\omega_0\tau_k}{a_3\omega_0}\right] + {\omega_0}^3 - a_2\omega_0$$

$$a_3 \cos\omega_0\tau_k + \frac{{a_4}^2 cos\omega_0 t}{a_3\omega_0} = \frac{a_1a_4\omega_0}{a_3} + {\omega_0}^3 - a_2\omega_0$$

$$\tau_k = \frac{1}{\omega_0}\left[arc\cos\left(\frac{a_1a_4{\omega_0}^2 + a_3{\omega_0}^4 - a_2a_3{\omega_0}^2}{{a_3}^2{\omega_0}^2 + {a_4}^2}\right)\right] + \frac{2k\pi}{\omega_0},\ k = 0,1,2,3,\ \ldots. \quad (21)$$

At $\tau = \tau_0$, we have

$$\tau_0 = \frac{1}{\omega_0}\left[\frac{a_1a_4{\omega_0}^2 + a_3{\omega_0}^4 - a_2a_3{\omega_0}^2}{{a_3}^2{\omega_0}^2 + {a_4}^2}\right] \quad (22)$$

Thus at $\tau = \tau_0$, equation (10) has two simple complex conjugate roots $\pm i\omega_0$ while all other roots lies in the complex half plane. From the analysis of the reduced form of equations (17) through (19), proposition 2 indicates that if the parameter values satisfy conditions (i) and (ii), then the infective (positive) steady state of equations (3) is asymptotically stable for all delay values. If in addition, any of the conditions (say condition (ii)) in proposition 2 is not satisfied, then the stability of the positive state depends on the delay values and the delay induces oscillations. For example, from equation (20), if $\beta < 0$, then $\sqrt{\alpha^2 - 3\beta} > \alpha$. Therefore,

$z_1 = \frac{1}{3}\left(-\alpha + \sqrt{\alpha^2 - 3\beta}\right) > 0.$

It follows that equation (20) and hence (19) has a positive root ω_0. This implies that equation (10) has two simple complex conjugate roots $\pm i\omega_0$ at a sequence of critical values τ_k. Since equation (10) has a pair of purely imaginary roots $\pm i\omega_0$ at a sequence of critical values τ_k, from the analysis above, and for simplicity, we can rewrite ${\omega_0}^+$ and ${\omega_0}^-$ as ω_0. Thus, we can now derive a pair of purely imaginary root $i\omega$ from equation (10) with $\tau = \tau^{\pm}$. Let $\{\tau_k^{\pm}\}_{k=0}^{\infty} = \{\tau_k\}_{k=0}^{\infty}$ such that $\tau_0 < \tau_1 < \tau_2 < \ldots < \tau_k < \ldots$, where, $\tau_0 = \min\{\tau^+, \tau^-\}$. Hence proposition 2 below

Proposition 2: Given the characteristic equation of infectious disease model of equation (3),

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3\lambda e^{-\lambda\tau} + a_4 e^{-\lambda\tau} = 0.$$

Suppose that,

i. $a_1 > 0$, $a_4 > 0$, $(a_1 + a_3) > \frac{a_4}{a_1}$, If either

ii. $\gamma < 0$, or

iii. $\gamma \geq 0$ and $\beta < 0$ is satisfied,

then the infective (positive) steady state E^* of model (3) is asymptotically stable when $\tau < \tau_0$ and unstable when $\tau > \tau_0$.

Thus, at $\tau = \tau_0$, a family of periodic function bifurcates at E^* as τ passes through the critical value τ_0. Thus delay model undergoes Hopf bifurcation at certain value of the delay if the parameters satisfy the conditions in (ii) and (iii).

Due to Rouche's theorem and the continuity of τ, from functional delay differential equation theory for every $\tau_k, k = 0, 1, \ldots$, there exists a $\varepsilon > 0$ such that $\lambda(\tau)$ is continuously differentiable in τ for $|\tau - \tau_k| < \varepsilon$.

Denote $\lambda(\tau) = \eta(\tau) + i\omega(\tau)$ to be the characteristic roots of equation (10) near $\tau = \tau_k$ satisfying $\eta(\tau_0) = 0$ and $\omega(\tau_0) = \omega_0$. On substituting $\lambda(\tau)$ into the left-hand side of (10), and taking the derivative of λ with respect to τ yields the Hopf bifurcation with the transversality condition given by $\left[\frac{dRe(\lambda)}{d\tau}\right|^{-1}\Big]_{\omega=\omega_0,\tau=\tau_0} > 0, \ (k = 0,1,2,\ldots)$.

From equation (10), we have

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3\lambda e^{-\lambda\tau} + a_4 e^{-\lambda\tau} = 0,$$

and on taking the derivative of λ with respect to τ, we have

$$\frac{d\lambda}{d\tau}\left[3\lambda^2 + 2a_1\lambda + a_2 + a_3 e^{-\lambda\tau} - a_3\tau\lambda e^{-\lambda\tau} - a_4\tau e^{-\lambda\tau}\right] = a_3\lambda^2 e^{-\lambda\tau} + a_4\lambda e^{-\lambda\tau}$$

$$\frac{d\lambda}{d\tau} = \frac{a_3\lambda^2 e^{-\lambda\tau} + a_4\lambda e^{-\lambda\tau}}{3\lambda^2 + 2a_1\lambda + a_2 + a_3 e^{-\lambda\tau} - a_3\tau\lambda e^{-\lambda\tau} - a_4\tau e^{-\lambda\tau}}$$

$$\left[\frac{d\lambda}{d\tau}^{-1}\right] = \frac{(3\lambda^2 + 2a_1\lambda + a_2)e^{\lambda\tau} + (a_3 - a_3\tau\lambda - a_4\tau)}{a_3\lambda^2 + a_4\lambda},$$

and on replacing $\lambda = i\omega_0$, we have

$$\left[\frac{d\lambda}{d\tau}^{-1}\right] = Re\left[\frac{(-3{\omega_0}^2 + 2a_1 i\omega_0 + a_2)(\cos\omega_0\tau + i\sin\omega_0\tau)(a_3 - a_3\tau i\omega_0 - a_4\tau)}{-a_3{\omega_0}^2 + a_4 i\omega_0}\right],$$

and on taking the complex conjugate of the denominator, we get

$$\left[\frac{d\lambda}{d\tau}^{-1}\right] = Re\left\{\frac{A+B+C}{D}\right\},$$

where,

$A = (3a_3{\omega_0}^2 - a_2a_3 + 2a_1a_4{\omega_0}^2)\,{\omega_0}^2\cos\omega_0\tau_k$

$B = (2a_1a_3\omega_0 - a_3a_4{\omega_0}^2 + a_2a_4)\,\omega_0\sin\omega_0\tau_k$, $C = -{a_3}^2$ and $D = {a_4}^2{\omega_0}^2 + {a_3}^2{\omega_0}^6$.

If $(3a_3{\omega_0}^2 - a_2a_3 + 2a_1a_4{\omega_0}^2)\,{\omega_0}^2\cos\omega_0\tau_k$

$+(2a_1a_3\omega_0 - a_3a_4{\omega_0}^2 + a_2a_4)\,\omega_0\sin\omega_0\tau_k - {a_3}^2 > 0$ and $(A + B + C) > 0$ are satisfied.

Hence lemma 2 below follows:

Lemma 2: If $(3a_3{\omega_0}^2 - a_2a_3 + 2a_1a_4{\omega_0}^2)\,{\omega_0}^2\cos\omega_0\tau_k$

$+(2a_1a_3\omega_0 - a_3a_4{\omega_0}^2 + a_2a_4)\,\omega_0\sin\omega_0\tau_k - {a_3}^2 > 0$ is satisfied, then

$$\left|\frac{dRe(\lambda(\tau))}{d\tau}\right|\Big|_{\omega=\omega_0,\tau=\tau_k} = \frac{d(\eta(\tau))}{d\tau} > 0 \text{ is satisfied.}$$

In fact, when $\tau = \tau_k$, substituting $\lambda(\tau)$ into the left-hand side of the first equation of (10) and taking derivative with respect to τ, we have

$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{3\lambda^2+2a_1\lambda+a_2+(a_3-a_3\tau\lambda-a_4\tau)e^{-\lambda\tau}}{a_3\lambda^2+a_4\lambda}$ which leads to $\left[\left|\frac{dRe(\lambda(\tau))}{d\tau}\right|^{-1}\right]_{\omega=\omega_0,\tau=\tau_k} > 0.$

From choice of suitable Lyapunov function, we can discuss the global stability of system (3) around the interior equilibrium point.

3.2 Numerical Examples of Positive Equilibrium of Delay Infectious DiseaseModel

From equation (11) and the parameter values for $\beta = 0.31, \gamma = 0.6$ and $\mu = 0.1$ (Egbetade *et al.*, 2018), we have

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 1.491 \\ 0.139 \\ 0.904 \\ 0.0904 \end{pmatrix}.$$

On substituting and simplifying equation (10), equation (17) therefore yields

$\omega^6 + ({a_1}^2 - 2a_2)\omega^4 + ({a_2}^2 - {a_3}^2)\omega^2 - {a_4}^2 = 0,$

and with the parameter values of

$\alpha = {a_1}^2 - 2a_2 = 1.9448, \beta = {a_2}^2 - {a_3}^2 = 0.7978, \gamma = {a_4}^2 = 0.00817,$

we have $\omega^6 + 1.9448\omega^4 + 0.7978\omega^2 - 0.00817 = 0.$

Substituting the parameter values in equation (17) yields the required value of $\omega_0 = 0.5972,$

Therefore, $\omega_0 = 0.5972 \in \mathbb{R}_+$ and $\tau_0 = 1.9510.$

The graphical representations of the data are in the following figures 1-3:

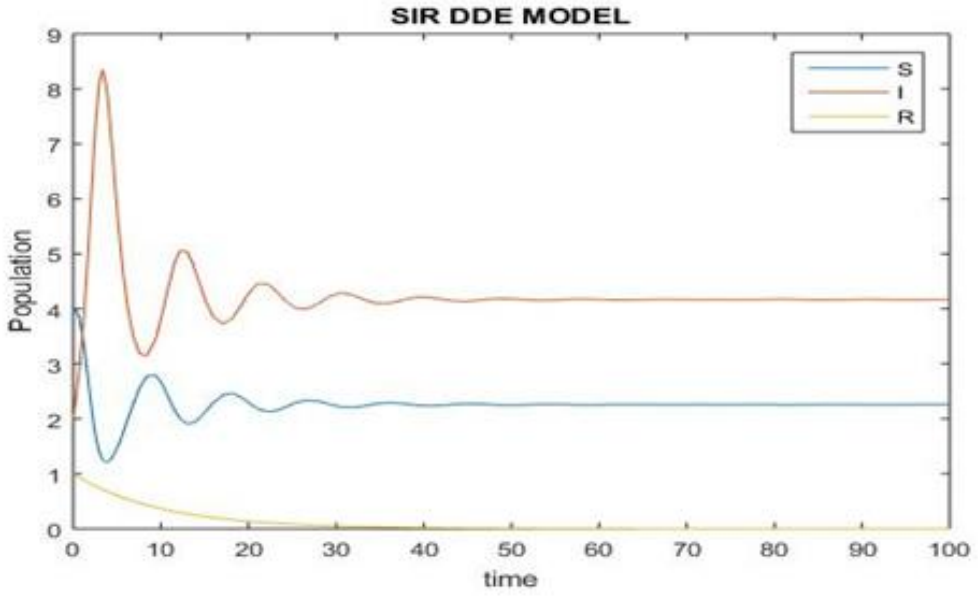

Figure 1: The Plot shows the stability of the Delay model when $\tau = 1.3 < 1.951$(calculated). Note the difference in the dynamics when compared with the ODE model.

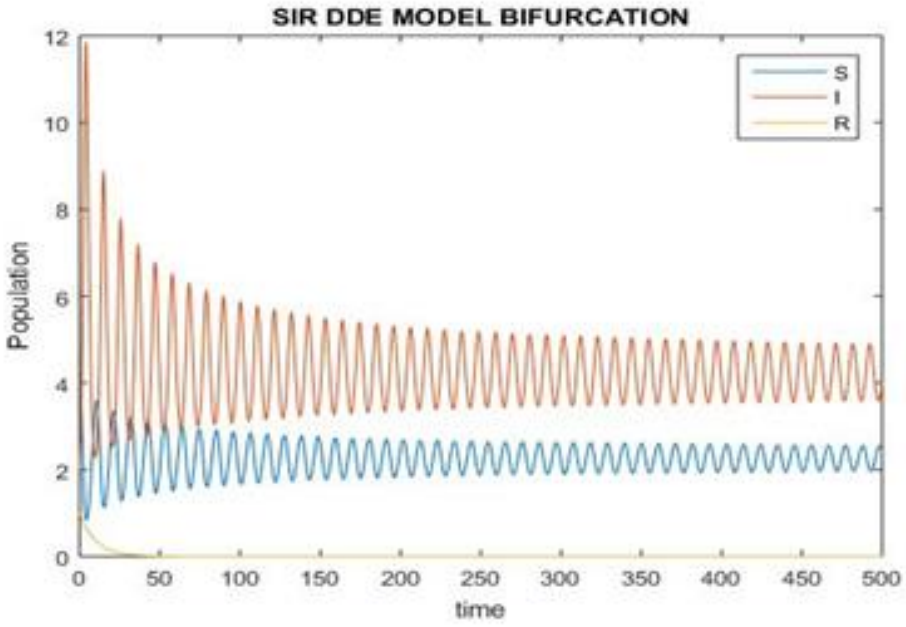

Figure 2: The Plot showing the bifurcation of the delay model at when $\tau = 1.951$

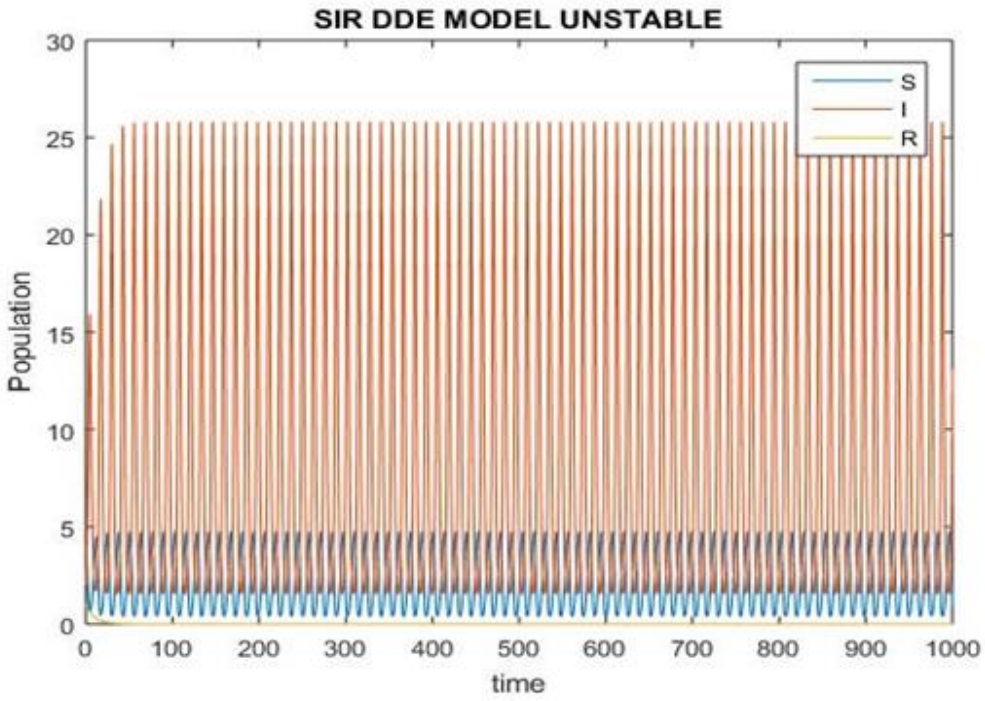

Figure 3: Plot showing the instability of the Delay model when $\tau = 2.5 > 1.951$.

4.0: Conclusion

The study investigated the dynamical behaviour of a disease infection model with a discrete delay term using Hopf bifurcation. The study established that under certain conditions, there exists a critical valueτ_0 for positive stability of delayed disease model and showed that some families of periodic solution appear when the delay passes through some certain critical values as can be observed from figures 1-3. If $\tau \in [0, \infty)$, the infected steady state of the model is asymptotically stable i.e., the disease keeps a steady state. When the delay term τ switches, through some critical values $\tau = \tau_k, k = 1, 2, ...$, the positive equilibrium loses its stability and a Hopf bifurcation occurs. However, a chaotic or aperiodic phenomenon may occur when the delay is large, as observed in figure 3.

Acknowledgments

The authors appreciate all forms of supports from Covenant University.

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