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ON THE SOLVABILITY OF A THIRD-ORDER INTEGRAL m-POINT RESONANT BOUNDARY VALUE PROBLEM ON THE HALF-LINE WITH TWO DIMENSIONAL KERNEL

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Abstract

Existence results for a resonant third-order integral *m*-point boundary value problem on the half-line with dimension of the kernel of the linear differential operator equal to two are established. The tools that will be employed in this work are the Mawhin's coincidence degree theory, relevant algebraic methods and operators. An example will also be used to illustrate our result.

1. Introduction

Boundary value problems on an unbounded domain are encountered in the study of many physical phenomenons such as the study of unsteady flow

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of fluid through a semi-infinite porous media and radially symmetric solutions of nonlinear elliptic equations. They also arise in plasma physics and the study of drain flows, see [10].

Boundary value problems whose corresponding homogeneous boundary value problem has a non-trivial solution are said to be at *resonance*. Resonant problems can be expressed in abstract form as $Lu = Nu$, where the differential operator L is not invertible. Mawhin's continuation theorem [8] is used to study cases where L is linear. Many authors have recently considered the problem of existence of solutions for resonant boundary value problems when the dimension of the linear operator is either one or two, see [1, 6, 12, 9, 3, 7, 13]. However, to the best of our knowledge, only few authors in the literature have considered boundary value problems having integral boundary conditions with dimension of the kernel of the linear operator equal to two, see [11, 2]. Most of the works have considered second-order boundary value problems but not third-order boundary value problems.

This work considers the existence of solutions for the following resonant third-order boundary value problem having m-point integral boundary conditions on the half-line:

$$u'''(t) = f(t, u(t), u'(t), u''(t)), \quad t \in (0, +\infty), \tag{1.1}$$

$$u(0) = 0, \quad u'(0) = \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} u'(t)dt, \quad u''(+\infty) = \sum_{j=1}^{n} \beta_j \int_0^{\eta_j} u''(t)dt, \tag{1.2}$$

where $f : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ is an $L^1[0, +\infty)$-Carathéodory function, $0 < \xi_1 < \xi_2 < \cdots \le \xi_m < +\infty$, $0 < \eta_1 < \eta_2 < \cdots \le \eta_n < +\infty$, $\alpha_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$ and $\beta_j \in \mathbb{R}$, $j = 1, 2, \ldots, n$.

In Section 2, necessary lemmas, theorems and definitions are given, Section 3 is dedicated to stating and proving condition for existence of solutions. An example is given in Section 4 to corroborate the result obtained.

2. Preliminaries

In this section, we will give some definitions and lemmas that will be used in this work.

Take U, Z to be normed spaces, $L : dom\, L \subset U \to Z$ is a Fredholm mapping of zero index and $P : U \to U$, $Q : Z \to Z$ projectors that are continuous such that

$$Im\, P = \ker L,\ \ker Q = Im\, L,\ U = \ker L \oplus \ker P,\ Z = Im\, L \oplus Im\, Q.$$

Then

$$L\,|_{dom\, L \cap \ker P} : dom\, L \cap \ker P \to Im\, L$$

is invertible. The inverse of the mapping L will be denoted by $K_p : Im\, L \to dom\, L \cap \ker P$ while the generalized inverse, $K_{P,Q} : Z \to dom\, L \cap \ker P$ is defined as $K_{P,Q} = K_p(I - Q)$.

Definition 2.1 [15]**.** A map $w : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ is $L^1[0, +\infty)$-Carathéodory, if the following conditions are satisfied:

(i) for each $(d, e, f) \in \mathbb{R}^3$, the mapping $t \to w(t, d, e, f)$ is Lebesgue measurable;

(ii) for a.e. $t \in [0, \infty)$, the mapping $(d, e, f) \to w(t, d, e, f)$ is continuous on $\mathbb{R}^3$;

(iii) for each $k > 0$, there exists $\varphi_k(t) \in L_1[0, +\infty)$ such that, for a.e. $t \in [0, \infty)$ and every $(d, e, f) \in [-k, k]$, we have

$$|\, w(t, d, e, f)\,| \le \varphi_k(t).$$

Definition 2.2. Let $L : dom\, L \subset X \to Z$ be a Fredholm mapping, E be a metric space and $N : E \to Z$ be a nonlinear mapping. N is said to be L-*compact* on E if $QN : E \to Z$ and $K_{P,Q}N : E \to X$ are compact on E.

Also, N is L-completely continuous if it is L-compact on every bounded $E \subset U$.

Theorem 2.1 [13]**.** *Let U be the space of all bounded continuous vector-valued functions on* $[0, \infty)$ *and* $M \subset U$. *Then M is relatively compact on U if the following conditions hold*:

(i) *M is a bounded subset of U*;

(ii) *the functions from M are equicontinuous on any compact interval of* $[0, \infty)$;

(iii) *the functions from M are equiconvergent at* $+\infty$, *that is*, *if given an* $\varepsilon > 0$, *there exists a* $T = T(\varepsilon) > 0$ *such that* $|f(t) - f(\infty)| < \varepsilon$, *for all* $t > T$ *and* $f \in M$.

Theorem 2.2 [8]**.** *Let L be a Fredholm map of index zero and let N be L-compact on* $\overline{\Omega}$. *Assume that the following conditions are satisfied*:

(i) $Lx \neq \lambda Nx$ *for every* $(x, \lambda) \in [(dom\, L \ker L \cap \partial\Omega)] \times (0, 1)$,

(ii) $Nx \notin Im\, L$ *for every* $x \in \ker L \cap \partial\Omega$,

(iii) $\deg(QN|_{\ker L}, \ker L, 0) \neq 0$, *where* $Q : Z \to Z$ *is a projection with* $Im\, L = \ker Q$.

Then the abstract equation $Lu = Nu$ *has at least one solution in* $dom\, L \cap \overline{\Omega}$.

Let

$$U = \left\{u \in C^2[0, +\infty) : \lim_{t \to +\infty} \frac{u(t)}{1 + t^2}, \lim_{t \to +\infty} \frac{u'(t)}{1 + t} \text{ and } \lim_{t \to +\infty} u''(t) \text{ exists}\right\},$$

with norm

$$\| u \| = \max\{\| u \|_0, \| u \|_1, \| u \|_\infty\}$$

defined on U, where

$$\| u \|_0 = \sup_{t\in[0,+\infty)} \frac{|u(t)|}{1+t^2}, \quad \| u \|_1 = \sup_{t\in[0,+\infty)} \frac{|u'(t)|}{1+t}, \quad \| u \|_\infty = \sup_{t\in[0,+\infty)} |u''(t)|.$$

The space $(U, \|\cdot\|)$ by standard argument is a Banach space.

Let $Z = L^1[0, +\infty)$ with the norm $\| y \|_{L^1} = \int_0^{+\infty} |y(v)| dv$. Define $Lu = u'''$, with domain

$$dom\, L = \Bigg\{ u \in U : u'' \in L^1[0, +\infty), u(0) = 0, u'(0) = \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} u'(t)dt,$$

$$u''(+\infty) = \sum_{j=1}^{n} \beta_j \int_0^{\eta_j} u''(t)dt \Bigg\}.$$

Also, the nonlinear operator $N : U \to Z$ will be defined by

$$(Nu)t = f(t, u(t), u'(t), u''(t)), \quad t \in [0, +\infty),$$

hence, equations (1.1)-(1.2) may be written as

$$Lu = Nu.$$

In order to establish conditions for existence of solution of (1.1)-(1.2), we make the following assumptions:

(ϕ_1) $\sum_{j=1}^{n} \beta_j \eta_j = 1, \sum_{i=1}^{m} \alpha_i \xi_i = 1, \sum_{i=1}^{m} \alpha_i \xi_i^2 = 0,$

(ϕ_2) $G = \begin{vmatrix} Q_1 te^{-t} & Q_2 te^{-t} \\ Q_1 t^2 e^{-t} & Q_2 t^2 e^{-t} \end{vmatrix} := \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = g_{11} \cdot g_{22} - g_{12} \cdot g_{21} \neq 0,$

where

$$Q_1 y = \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \int_0^t \int_0^\tau y(v) dv d\tau dt, \quad Q_2 y = \sum_{j=1}^{n} \beta_j \int_0^{\eta_j} \int_\tau^{+\infty} y(v) dv d\tau.$$

By simple calculation, it can be shown that

$$\ker L = \{bt + ct^2 : b, c \in \mathbb{R}, t \in [0, +\infty)\}.$$

Lemma 2.1. *$Im\, L = \{y \in Z : Q_1 y = Q_2 y = 0\}$, where*

$$Q_1 y = \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \int_0^t \int_0^\tau y(v) dv d\tau dt, \quad Q_2 y = \sum_{j=1}^{n} \beta_j \int_0^{\eta_j} \int_\tau^{+\infty} y(v) dv d\tau.$$

Proof. Consider the problem

$$u'''(t) = y, \quad t \in [0, +\infty), \tag{2.1}$$

which has a solution $u(t)$ satisfying (1.2) such that

$$u(t) = u(0) + u'(0)t + \frac{1}{2}u''(0)t^2 + \int_0^t \int_0^s \int_0^\tau y(v) dv d\tau ds.$$

Integrating (2.1) from 0 to t, we have

$$u''(t) = u''(0) + \int_0^t y(v) dv. \tag{2.2}$$

At $t = +\infty$, (2.2) becomes

$$u''(+\infty) = u''(0) + \int_0^t y(v) dv + \int_t^\infty y(v) dv. \tag{2.3}$$

Since $\sum_{j=1}^{n} \beta_j \int_0^{\eta_j} dt = 1$, $u''(+\infty) = \sum_{j=1}^{n} \beta_j \int_0^{\eta_j} u''(+\infty) dt$. From (1.2), we have

$$\sum_{j=1}^{n} \beta_j \int_0^{\eta_j} u''(+\infty) dt = \sum_{j=1}^{n} \beta_j \int_0^{\eta_j} u''(t) dt.$$

Hence,

$$\sum_{j=1}^{n}\beta_j\int_0^{\eta_j}u''(0)dt+\sum_{j=1}^{n}\beta_j\int_0^{\eta_j}\int_0^{t}y(v)dvdt+\sum_{j=1}^{n}\beta_j\int_0^{\eta_j}\int_t^{\infty}y(v)dvdt$$

$$=\sum_{j=1}^{n}\beta_j\int_0^{\eta_j}u''(0)dt+\sum_{j=1}^{n}\beta_j\int_0^{\eta_j}\int_0^{t}y(v)dvdt$$

and

$$\sum_{j=1}^{n}\beta_j\int_0^{\eta_j}\int_\tau^{+\infty}y(v)dvd\tau=0.$$

Integrating (2.2) from 0 to t gives

$$u'(t)=u'(0)+u''(0)t+\int_0^t\int_0^\tau y(v)dvd\tau. \tag{2.4}$$

Applying boundary conditions (1.2), we have

$$u'(0)=\sum_{i=1}^{m}\alpha_i\int_0^{\xi_i}\left(u'(0)+u''(0)t+\int_0^t\int_0^\tau y(v)dvd\tau\right)dt$$

$$\Rightarrow u'(0)\left(1-\sum_{i=1}^{m}\alpha_i\xi_i\right)-\frac{u''(0)}{2}\sum_{i=1}^{m}\alpha_i\xi_i^2=\sum_{i=1}^{m}\alpha_i\int_0^{\xi_i}\int_0^t\int_0^\tau y(v)dvd\tau dt.$$

Since $\sum_{i=1}^{m}\alpha_i\xi_i=1$ and $\sum_{i=1}^{m}\alpha_i\xi_i^2=0$,

$$\sum_{i=1}^{m}\alpha_i\int_0^{\xi_i}\int_0^t\int_0^\tau y(v)dvd\tau dt=0$$

and

$$u(t)=bt+\frac{1}{2}ct^2+\sum_{i=1}^{m}\alpha_i\int_0^{\xi_i}\int_0^t\int_0^\tau y(v)dvd\tau dt,$$

where b and c are arbitrary constants and $u(t)$ is a solution to (2.1) satisfying (1.2). Hence $Im\, L = \{y \in Z : Q_1 y = Q_2 y = 0\}$. □

We next define the operator $Q : Z \to Z$ as

$$Qy = (R_1 y) \cdot t + (R_2 y) \cdot t^2,$$

where

$$R_1 y = \frac{1}{G}(m_{11}Q_1 y + m_{12}Q_2 y)e^{-t}, \quad R_2 y = \frac{1}{G}(m_{21}Q_1 y + m_{22}Q_2 y)e^{-t},$$

and m_{ij} is the algebraic cofactor of g_{ij}.

Lemma 2.2. *The following hold*:

(i) $L : dom\, L \subset U$ *is a Fredholm operator of index zero*;

(ii) *the generalized inverse* $K_p : Im\, L \to dom\, L \cap \ker P$ *may be written as*

$$K_p y = \int_0^t \int_0^s \int_0^\tau y(v)\, dv d\tau ds.$$

Also,

$$\| K_p y \| = \| y \|_{L^1}.$$

Proof. (i) We now make the following computations:

$$R_1((R_1 y)t) = \frac{1}{G}[m_{11}Q_1(((R_1 y)t)e^{-t}) + m_{12}Q_2(((R_1 y)t)e^{-t})]$$

$$= \frac{1}{G}[m_{11}Q_1 te^{-t} + m_{12}Q_2 te^{-t}](R_1 y)$$

$$= \frac{1}{G}[g_{22} \cdot g_{11} - g_{21} \cdot g_{12}](R_1 y)$$

$$= \frac{G}{G}(R_1 y) = (R_1 y), \tag{2.5}$$

$$R_1((R_2y)t^2) = \frac{1}{G}[m_{11}Q_1((R_2y)t^2e^{-t}) + m_{12}Q_2((R_2y)t^2e^{-t})]$$

$$= \frac{1}{G}[m_{11}Q_1t^2e^{-t} + m_{12}Q_2t^2e^{-t}](R_2y)$$

$$= \frac{1}{G}[g_{22} \cdot g_{21} - g_{21} \cdot g_{22}](R_2y) = 0, \tag{2.6}$$

$$R_2((R_1y)t^2) = \frac{1}{G}[m_{21}Q_1((R_1y)te^{-t}) + m_{22}Q_2((R_1y)te^{-t})]$$

$$= \frac{1}{G}[m_{21}Q_1te^{-t} + m_{22}Q_2te^{-t}](R_1y)$$

$$= \frac{1}{G}[-g_{12} \cdot g_{11} + g_{11} \cdot g_{12}](R_1y) = 0 \tag{2.7}$$

and

$$R_2((R_2y)t^2) = \frac{1}{G}[m_{21}Q_1((R_2y)t^2e^{-t}) + m_{22}Q_2((R_2y)t^2e^{-t})]$$

$$= \frac{1}{G}[m_{21}Q_1t^2e^{-t} + m_{22}Q_2t^2e^{-t}](R_2y)$$

$$= \frac{1}{G}[-g_{12} \cdot g_{21} - g_{11} \cdot g_{22}](R_2y)$$

$$= \frac{G}{G}(R_2y) = R_2y. \tag{2.8}$$

From (2.5), (2.6), (2.7) and (2.8), we have

$$\begin{aligned} Q^2y &= Q[(R_1y) \cdot t + (R_2y) \cdot t^2] \\ &= [R_1((R_1y) \cdot t + (R_2y) \cdot t^2)] \cdot t + [R_2((R_1y) \cdot t + (R_2y) \cdot t^2)] \cdot t^2 \\ &= (R_1((R_1y) \cdot t)) \cdot t + (R_1((R_2y) \cdot t^2)) \cdot t \\ &\quad + (R_2(R_1y) \cdot t) \cdot t^2 + (R_2((R_2y) \cdot t^2)) \cdot t^2 \\ &= (R_1y) \cdot t + 0 \cdot t + 0 \cdot t^2 + (R_2y) \cdot t^2 \\ &= (R_1y) \cdot t + (R_2y) \cdot t^2 = Qy, \end{aligned}$$

therefore, Qy is a projector.

Let $y \in Im\,L$. Then $Qy = (R_1 y) \cdot t + (R_2 y) \cdot t^2 = 0$. Next, we show that $\ker Q = Im\,L$. Let $y \in \ker Q$. Then $y \in Im\,L$ since $Qy = 0$. Conversely, if $y \in Im\,L$, then by $Qy = 0$, $y \in \ker Q$. Hence, $\ker Q = Im\,L$.

Let $y \in Z$ and $y = (I - Q)y + Qy$. $Q(I - Q)y = Qy - Q^2 y = Qy - Qy = 0$, thus $(I - Q)y \in \ker Q = Im\,L$ and $Qy \in Im\,Q$. Hence, $Z = Im\,L + \ker Q$. Setting $y = bt + ct^2 \neq 0$, from $y \in Im\,L$, we obtain the following equations:

$$\begin{cases} Q_1 bte^{-t} + Q_2 cte^{-t} = 0, \\ Q_1 bt^2 e^{-t} + Q_2 ct^2 e^{-t} = 0. \end{cases} \tag{2.9}$$

From (ϕ_2), $G \neq 0$, then (2.9) has a unique solution $b = c = 0$, implying that $Im\,L \cap Im\,Q = \{0\}$ and $Z = Im\,L \oplus Im\,Q$. Note that $\dim \ker Q = \dim \ker L = \operatorname{codim} Im\,L = 2$. Therefore, the index of $L = \dim \ker L - \operatorname{codim} Im\,L = 2 - 2 = 0$, thus L is a Fredholm mapping of index zero.

(ii) Let a continuous projector $P : U \to U$ be defined as

$$(Pu)(t) = u'(0)t + \frac{1}{2}u''(0)t^2, \quad t \in [0, +\infty).$$

Then the generalized inverse $K_p : Im\,L \to \operatorname{dom} L \cap \ker P$ may be written as

$$K_P y = \int_0^t \int_0^s \int_0^\tau y(v)\,dv d\tau ds.$$

For $y \in Im\,L$, we have

$$(LK_P)y(t) = (K_P y)''' = y(t)$$

and for $u \in dom\,L \cap \ker P$, we have

$$(K_P L)u(t) = (K_P)u'''(t) = \int_0^t \int_0^s \int_0^\tau u'''(v)\,dv d\tau ds$$

$$= u(t) - u(0) - u'(0)t - \frac{1}{2}u''(0)t^2$$

$$= u(t) - u(0) - Pu(t).$$

Since $u \in dom\, L \cap \ker P$, $Pu(t) = 0$, from the boundary condition $u(0) = 0$. Therefore,

$$(K_P L)u(t) = u(t).$$

Thus $K_P = (L|_{dom\, L \cap \ker P})^{-1}$. Also,

$$\| K_p y \|_0 = \sup_{t \in [0, +\infty)} \frac{1}{1 + t^2} | (K_p y) | \leq \int_0^{+\infty} | y(v) | dv = \| y \|_{L^1},$$

$$\| K_p y \|_1 = \sup_{t \in [0, +\infty)} \frac{1}{1 + t} | (K_p y)' | \leq \int_0^{+\infty} | y(v) | dv = \| y \|_{L^1}$$

and

$$\| K_p y \|_\infty = \sup_{t \in [0, +\infty)} | (K_P y)'' | \leq \int_0^{+\infty} | y(v) | dv = \| y \|_{L^1}.$$

Then

$$\| K_P y \| = \max\{\| K_p y \|_0, \| K_p y \|_1, \| K_p y \|_\infty\} = \| y \|_{L^1}.$$

Proof of Lemma 2.2 is complete. □

Lemma 2.3. *Let $\Omega \subset U$ be open and bounded with $dom\, L \cap \overline{\Omega} \neq \emptyset$. If f is a $L^1[0, +\infty)$-Carathéodory function, then the nonlinear operator N is L-compact on $\overline{\Omega}$.*

Proof. Let $u \in \overline{\Omega}$ and let $k > 0 \in \mathbb{R}$. Then $\| u \| \leq k$ since Ω is bounded. Since f is $L^1[0, +\infty)$-Carathéodory and for any $u \in \overline{\Omega}$, we have

$$\| Nu \|_{L^1} = \int_0^{+\infty} | f(v, u(v), u'(v), u''(v)) | dv \leq \| \varphi_k \|_{L^1}, \tag{2.10}$$

$$|Q_1Nu| = \left|\sum_{i=1}^{m}\alpha_i\int_0^{\xi_i}\int_0^{s}\int_0^{\tau} f(v, u(v), u'(v), u''(v))dvd\tau ds\right|$$

$$\leq \frac{\|\varphi_k\|_{L^1}}{2}\sum_{i=1}^{m}|\alpha_i|\xi_i^2 \tag{2.11}$$

and

$$|Q_2Nu| = \left|\sum_{j=1}^{n}\beta_j\int_0^{\eta_j}\int_{\tau}^{+\infty} f(v, u(v), u'(v), u''(v))dvd\tau\right|$$

$$\leq \|\varphi_k\|_{L^1}\sum_{j=1}^{n}|\beta_j|\eta_j. \tag{2.12}$$

Then

$$\|QNu\|_{L^1} = \int_0^{+\infty}|QNu(v)|dv$$

$$\leq \int_0^{+\infty}[(R_1Nu(v))\cdot v + (R_2Nu(v))\cdot v^2]dv$$

$$\leq \frac{1}{|G|}[|m_{11}|\cdot|QN_1u| + |m_{12}|\cdot|QN_2u|]$$

$$+ \frac{1}{|G|}[|m_{21}|\cdot|Q_1Nu| + |m_{22}|\cdot|Q_2Nu|]$$

$$\leq \frac{1}{|G|}\left[(|m_{11}| + |m_{21}|)\left(\frac{\|\varphi_k\|_{L^1}}{2}\sum_{i=1}^{m}|\alpha_i|\xi_i^2\right)\right.$$

$$\left. + (|m_{12}| + |m_{22}|)\|\varphi_k\|_{L^1}\sum_{j=1}^{n}|\beta_j|\eta_j\right]. \tag{2.13}$$

Then $QN(\overline{\Omega})$ is bounded.

We will use the following three steps to show that $K_P(I-Q)N(\overline{\Omega})$ is compact.

Step 1. Boundedness. Let $u \in \overline{\Omega}$. Then

$$\frac{|K_P(I-Q)Nu(t)|}{1+t^2} = \frac{1}{1+t^2}\left|\int_0^t\int_0^s\int_0^\tau (I-Q)Nu(v)\,dv\,d\tau\,ds\right|$$

$$\leq \frac{t^2}{2(1+t^2)}(\|\varphi_k\|_{L^1} + \|QNu\|_{L^1})$$

$$\leq \|\varphi_k\|_{L^1} + \|QNu\|_{L^1},$$

$$\frac{|(K_P(I-Q)Nu)'(t)|}{1+t} = \frac{1}{1+t}\left|\int_0^t\int_0^s (I-Q)Nu(v)\,dv\,ds\right|$$

$$\leq \frac{t}{1+t}(\|\varphi_k\|_{L^1} + \|QNu\|_{L^1})$$

$$\leq \|\varphi_k\|_{L^1} + \|QNu\|_{L^1}$$

and

$$|(K_P(I-Q)Nu)''(t)| = \left|\int_0^t (I-Q)Nu(v)\,dv\right|$$

$$\leq \|\varphi_k\|_{L^1} + \|QNu\|_{L^1}.$$

From (2.10) and (2.13), we see that $K_P(I-Q)N(\overline{\Omega})$ is bounded.

Step 2. Equicontinuity. Let $u \in \overline{\Omega}$, $t_1, t_2 \in [0, T]$ with $t_1 < t_2$ and $T \in (0, +\infty)$. Then

$$\left|\frac{K_P(I-Q)Nu(t_2)}{1+t_2^2} - \frac{K_P(I-Q)Nu(t_1)}{1+t_1^2}\right|$$

$$= \left|\frac{1}{1+t_2^2}\int_0^{t_2}\int_0^s\int_0^\tau (I-Q)Nu(v)\,dv\,d\tau\,ds\right.$$

$$- \frac{1}{1+t_1^2} \int_0^{t_1} \int_0^{s} \int_0^{\tau} (I-Q) Nu(v) dv d\tau ds \Bigg|$$

$$= \Bigg| \frac{1}{1+t_2^2} \int_0^{t_1} \int_0^{s} \int_0^{\tau} (I-Q) Nu(v) dv d\tau ds$$

$$+ \frac{1}{1+t_2^2} \int_{t_1}^{t_2} \int_0^{s} \int_0^{\tau} (I-Q) Nu(v) dv d\tau ds$$

$$- \frac{1}{1+t_1^2} \int_0^{t_1} \int_0^{s} \int_0^{\tau} (I-Q) Nu(v) dv d\tau ds \Bigg|$$

$$= \Bigg| \left(\frac{1}{1+t_2^2} - \frac{1}{1+t_1^2} \right) \int_0^{t_1} \int_0^{s} \int_0^{\tau} (I-Q) Nu(v) dv d\tau ds$$

$$+ \frac{1}{1+t_2^2} \int_{t_1}^{t_2} \int_0^{s} \int_0^{\tau} (I-Q) Nu(v) dv d\tau ds \Bigg|$$

$$\leq \left(\left| \frac{t_1^2 - t_2^2}{(1+t_2^2)(1+t_1^2)} \right| \frac{t_1^2}{2} + \left| \frac{1}{(1+t_2^2)} \right| \frac{t_2^2 - t_1^2}{2} \right) (\| \varphi_k \|_{L^1} + \| QNu \|_{L^1}),$$

$$\left| \frac{(K_P(I-Q)Nu)'(t_2)}{1+t_2} - \frac{(K_P(I-Q)Nu)'(t_1)}{1+t_1} \right|$$

$$= \left| \frac{1}{1+t_2} \int_0^{t_2} \int_0^{\tau} (I-Q) Nu(v) dv d\tau ds - \frac{1}{1+t_1} \int_0^{t_1} \int_0^{\tau} (I-Q) Nu(v) dv d\tau \right|$$

$$= \Bigg| \frac{1}{1+t_2} \int_0^{t_1} \int_0^{\tau} (I-Q) Nu(v) dv d\tau + \frac{1}{1+t_2} \int_{t_1}^{t_2} \int_0^{\tau} (I-Q) Nu(v) dv d\tau$$

$$- \frac{1}{1+t_1} \int_0^{t_1} \int_0^{\tau} (I-Q) Nu(v) dv d\tau \Bigg|$$

$$\leq \left(\left| \frac{t_1 - t_2}{(1+t_2)(1+t_1)} \right| t_1 + \left| \frac{1}{1+t_2} \right| (t_2 - t_1) \right) (\| \varphi_k \|_{L^1} + \| QNu \|_{L^1})$$

and

$$|(K_P(I-Q)Nu)''(t_2)-(K_P(I-Q)Nu)''(t_1)| \le \int_{t_1}^{t_2} |(I-Q)Nu(v)|dv.$$

Thus, $\left\{\frac{K_p(I-Q)Nu}{1+t^2} : u \in \overline{\Omega}\right\}$, $\left\{\frac{(K_p(I-Q)Nu)'}{1+t} : u \in \overline{\Omega}\right\}$, and

$$\{(K_P(I-Q)Nu)'' : u \in \overline{\Omega}\} \to 0$$

as $t_1 \to t_2$ on the compact interval $[0, T]$ hence, are equicontinuous on $[0, T]$.

Step 3. Equiconvergence at $+\infty$. Let $u \in \overline{\Omega}$. By L'Hospital rule, we have

$$\lim_{t\to\infty} \frac{K_P(I-Q)Nu(t)}{1+t^2} = \frac{1}{2}\int_0^\infty (I-Q)Nu(v)dv,$$

$$\lim_{t\to\infty} \frac{(K_P(I-Q)Nu)'(t)}{1+t} = \int_0^\infty (I-Q)Nu(v)dv$$

and

$$\lim_{t\to\infty} (K_P(I-Q)Nu)''(t) = \int_0^\infty (I-Q)Nu(v)dv.$$

Thus,

$$\frac{K_P(I-Q)Nu(t)}{1+t^2} \to \frac{1}{2}\int_0^\infty (I-Q)Nu(v)dv,$$

$$\frac{(K_P(I-Q)Nu)'(t)}{1+t} \to \int_0^\infty (I-Q)Nu(v)dv$$

and

$$(K_P(I-Q)Nu)''(t) \to \int_0^\infty (I-Q)Nu(v)dv,$$

hence, $K_P(I-Q)Nu(\overline{\Omega})$ is equiconvergent at ∞. By Definition 2.2 and Theorem 2.1, $K_P(I-Q)Nu(\overline{\Omega})$ is compact. Therefore, the nonlinear operator N is L-compact on $\overline{\Omega}$. This concludes the proof of Lemma 2.3. □

3. Existence Result

In this section, we establish the conditions for the existence of solutions to the problem (1.1) subject to (1.2).

Theorem 3.1. *Let f be a* $L[0, \infty)$*-Carathéodory function. If* (ϕ_1), (ϕ_2) *and the following conditions hold*:

(H_1) *There exist functions* $\delta_1(t), \delta_2(t), \delta_3(t), \delta_4(t), \delta_5(t) \in L^1[0, +\infty)$ *and a constant* $\sigma \in [0, 1)$ *such that for all* $(x, y, z) \in \mathbb{R}^3$ *either*

$$|f(t, u_1, u_2, u_3)| \le \delta_5(t) + \delta_4(t)\frac{|u_3|}{1+t^2} + \delta_3(t)\frac{|u_2|}{1+t} + \delta_2(t)|u_1| + \delta_1(t)\left(\frac{|u_3|}{1+t^2}\right)^\sigma, \tag{3.1}$$

$$|f(t, u_1, u_2, u_3)| \le \delta_5(t) + \delta_4(t)\frac{|u_3|}{1+t^2} + \delta_3(t)\frac{|u_2|}{1+t} + \delta_2(t)|u_1| + \delta_1(t)\left(\frac{|u_2|}{1+t}\right)^\sigma \tag{3.2}$$

or

$$|f(t, u_1, u_2, u_3)| \le \delta_5(t) + \delta_4(t)\frac{|u_3|}{1+t^2} + \delta_3(t)\frac{|u_2|}{1+t}$$

$$+ \delta_2(t)|u_1| + \delta_1(t)|u_1|^{\sigma}. \tag{3.3}$$

(H_2) *There exist constants* $B > 0, D > 0$ *such that for* $u \in dom\, L$ *if* $|u'(t)| > B$ *for* $t \in [0, D]$ *or* $|u''(t)| > B$ *for* $t \in [0, +\infty)$, *then either*

$$Q_1 Nu(t) \ne 0 \quad or \quad Q_2 Nu(t) \ne 0.$$

(H_3) *There exists a constant* $A > 0$ *such that if* $|b| > A$ *or* $|c| > A$, *then either*

$$bQ_1 N(bt + ct^2) + cQ_2 N(bt + ct^2) < 0 \tag{3.4}$$

or

$$bQ_1 N(bt + ct^2) + cQ_2 N(bt + ct^2) > 0 \tag{3.5}$$

holds where for $b, c \in \mathbb{R}$ *and* $b^2 + c^2 > A$.

Then the boundary value problem (1.1)-(1.2) *has at least one solution in* U, *provided*

$$(2 + D)(\|\delta_4\|_{L^1} + \|\delta_3\|_{L^1} + \|\delta_2\|_{L^1}) < 1.$$

Proof. We divide the proof into four steps:

Step 1. Let

$$\Omega_1 = \{u \in dom\, L \backslash \ker L : Lu = \lambda Nu, \text{ for } \lambda \in [0, 1]\}.$$

We will establish that Ω_1 is bounded. If $u \in \Omega_1$, then $Lu = \lambda Nu \ne 0$, $\lambda \ne 0$ and $Nu \in Im\, L$. Hence,

$$Q_1 Nu = Q_2 Nu = 0.$$

From (H_2), there exist $t_0 \in [0, D]$ and $t_1 \in [0, +\infty)$ such that $|u'(t_0)| \leq B$, $|u''(t_1)| \leq B$. Let

$$u'(0) = u'(t_0) - \int_0^{t_0} u''(v)dv. \text{ Then } |u'(0)| = \left| u'(t_0) - \int_0^{t_0} u''(v)dv \right|$$

and

$$|u'(0)| \leq B + \int_0^{t_0} |u''(v)|dv \leq B + D\|u''\|_\infty. \tag{3.6}$$

Also, from

$$u''(t) = u''(t_1) + \int_{t_1}^{t} u'''(v)dv \Rightarrow |u''(t)| = \left| u(t_1) + \int_{t_1}^{t} u'''(v)dv \right|,$$

we obtain

$$\|u''\|_\infty \leq B + \int_0^{+\infty} |Nu(v)|dv| \leq B + \|Nu\|_{L^1}. \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$|u'(0)| \leq B + D(B + \|Nu\|_{L^1}) \leq B(1 + D) + D\|Nu\|_{L^1}, \tag{3.8}$$

while from (3.7), we obtain

$$|u''(0)| \leq B + \|Nu\|_{L^1}. \tag{3.9}$$

Therefore,

$$\|Pu\| = \max\{\|Pu\|_0, \|Pu\|_1, \|Pu\|_\infty\} \leq B(2 + D) + (1 + D)\|Nu\|_{L^1}. \tag{3.10}$$

In addition, for $u \in \Omega_1$, $(I - P)u \in dom\, L \cap \ker P$, $LPu = 0$. Thus, from Lemma 2.2, we get

$$\|(I - P)u\| = \|K_P L(I - P)u\| \leq \|L(I - P)u\|_{L^1} = \|Lu\|_{L^1} \leq \|Nu\|_{L^1}. \tag{3.11}$$

Hence, from (3.10) and (3.11), we obtain

$$\| u \| = \| Pu + (I - P)u \| \leq \|Pu\| + \| (I - P)u \|$$

$$\leq B(2 + D) + (1 + D)\| Nu \|_{L^1} + \| Nu \|_{L^1}$$

$$\leq B(2 + D) + (2 + D)\| Nu \|_{L^1}$$

$$\leq (2 + D)(B + \| Nu \|_{L^1}). \tag{3.12}$$

If (3.1) holds, then from (3.12), we obtain

$$\| u \| \leq (2 + D)(B + \| \delta_5 \|_{L^1} + \| \delta_4 \|_{L^1} \| u \| + \| \delta_3 \|_{L^1} \| u \|$$

$$+ \| \delta_2 \|_{L^1} \| u \| + \| \delta_1 \|_{L^1} \| u \|^{\sigma}) \tag{3.13}$$

which implies

$$\| u \| \leq \frac{(2 + D)(\| \delta_5 \|_{L^1} + \| \delta_1 \|_{L^1} \| u \|^{\sigma} + B)}{1 - (2 + D)(\| \delta_4 \|_{L^1} + \| \delta_3 \|_{L^1} + \| \delta_2 \|_{L^1})}. \tag{3.14}$$

Since $\sigma \in [0, 1)$, there exists a constant $B_1 > 0$ such that (3.14) becomes $\| u \| \leq B_1$. Thus, Ω_1 is bounded.

If (3.2) and (3.3) hold, then Ω_1 can be shown to be bounded using similar argument.

Step 2. Let

$$\Omega_2 = \{u \in \ker L : Nu \in Im\, L\}.$$

For $u,\ Nu \in \Omega_2$, then $u(t) = bt + ct^2$ and $QNu = 0$. Hence

$$Q_1 N(bt + ct^2) = Q_2 N(bt + ct^2) = 0.$$

From (H_3), $|b| \leq A, |c| \leq A$, then $\| u \| = \max\{\| u \|_0, \| u \|_1, \| u \|_\infty\} = \max\{2A, 2A, A\} \leq 2A$, hence, Ω_2 is bounded.

Step 3. For $b, c \in \mathbb{R}$, $t \in [0, +\infty)$, we will define the isomorphism $J : \ker L \to Im\, Q$ by

$$J(bt + ct^2) = \frac{1}{G}[(m_{11}b + m_{12}c)t + (m_{21}b + m_{22}c)t^2]e^{-t}. \tag{3.15}$$

Suppose (3.4) holds. Let

$$\Omega_3 = \{u \in \ker L : \lambda Ju + (1 - \lambda)QNu = 0, \lambda \in [0, 1]\}.$$

Let $u \in \Omega_3$. Then $u(t) = bt + ct^2$. Since $\lambda Ju + (I - \lambda)QNu = 0$, we have

$$\begin{cases} m_{11}(b\lambda + (1-\lambda)Q_1N(bt + ct^2)) + m_{12}(c\lambda + (1-\lambda)Q_2N(bt + ct^2)) = 0, \\ m_{21}(b\lambda + (1-\lambda)Q_1N(bt + ct^2)) + m_{22}(c\lambda + (1-\lambda)Q_2N(bt + ct^2)) = 0. \end{cases}$$

In matrix form, we have

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} b\lambda + (1-\lambda)Q_1N(bt + ct^2) \\ c\lambda + (1-\lambda)Q_2N(bt + ct^2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

since

$$\det\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = m_{11}m_{22} - m_{21}m_{12} = g_{22}g_{11} + g_{12}g_{21} = -G \neq 0,$$

$$\begin{cases} b\lambda + (1-\lambda)Q_1N(bt + ct^2) = 0 \Rightarrow b\lambda = -(1-\lambda)Q_1N(bt + ct^2), \\ c\lambda + (1-\lambda)Q_2N(bt + ct^2) = 0 \Rightarrow c\lambda = -(1-\lambda)Q_2N(bt + ct^2). \end{cases} \tag{3.16}$$

From (3.16), we have

$$b\lambda = -(1-\lambda)Q_1N(bt + ct^2),$$

$$c\lambda = -(1-\lambda)Q_2N(bt + ct^2). \tag{3.17}$$

From (3.17), when $\lambda = 1, b = c = 0$. When $\lambda = 0$, $Q_1N(bt + ct^2) = Q_2N(bt + ct^2) = 0$, which contradicts (3.4) and (3.5). Hence, from (H_3), we get $|b| \leq A$ and $|c| \leq A$. For $\lambda \in (0, 1)$, if $|b| > A$ or $|c| > A$ by (3.5) and (3.16), we have

$$\lambda(b^2 + c^2) = -(1-\lambda)[bQ_1N(bt + ct^2) + cQ_2N(bt + ct^2)] < 0,$$

which contradicts $\lambda(a^2 + b^2) > 0$. Hence, from (H_3), we get $|b| \leq A$ and $|c| \leq A$, thus, $\|u\| \leq |b| + |c| \leq 2A$. Hence, Ω_3 is bounded.

Suppose (3.4) holds. Let

$$\Omega_3 = \{u \in \ker L : \lambda Ju - (I - \lambda)QNu = 0, \lambda \in [0, 1]\},$$

Ω_3 can be shown to be bounded using similar argument as above.

Step 4. Let $\Omega \supset U_{i=1}^{3}\overline{\Omega}_i$. We will now show that at least one solution of (1.1) and (1.2) exists in $dom\, L \cap \Omega$. We have shown in Step 1 and Step 2 that

(i) $Lu \neq \lambda Nu, \forall (u, \lambda) \in [(dom\, L \backslash \ker L) \cap \partial\Omega] \times [(0, 1)]$;

(ii) $Nu \notin Im\, L, \forall u \in \ker L \cap \partial\Omega$.

Finally, we show that

(iii) $\deg(QN|_{\ker L}, \Omega \cap \ker L, 0) \neq 0$.

Let $u \in \partial\Omega \cap \ker L$ and $H(u, \lambda) = \pm\lambda Ju + (1 - \lambda)QNu$. By the arguments of Step 3, $H(u, \lambda) \neq 0$, for all $(u, \lambda) \in (\ker L \cap \partial\Omega) \times [0, 1]$. Therefore, by the homotopy property,

$$\begin{aligned}\deg(QN|_{\ker L}, \Omega \cap \ker L, 0) &= \deg(H(\cdot, 0), \Omega \cap \ker L, 0)\\ &= \deg(H(\cdot, 1), \Omega \cap \ker L, 0)\\ &= \pm 1 \neq 0.\end{aligned}$$

Therefore, by Theorem 2.2, at least one solution of (1.1)-(1.2) exists in $dom\, L \cap \overline{\Omega}$. □

4. Example

Example 4.1. Consider the following boundary value problems:

$$u'''(t) = f(t, u(t), u'(t), u''(t)), \quad t \in (0, +\infty), \tag{4.1}$$

$$u(0) = 0, \; u'(0) = -\frac{1}{2}\int_0^2 u'(t)dt + 2\int_0^1 u'(t)dt,$$

$$u''(+\infty) = \frac{1}{3}\int_0^9 u''(t)dt - \int_0^2 u''(t)dt, \tag{4.2}$$

where

$$f(t, u(t), u'(t), u''(t)) = \begin{cases} (u(t) + \sin u'(0))e^{-15t}, & 0 \le t \le 2, \\ e^{-20t}u''(t) + e^{-2t} \sin \sqrt[5]{u''(t)}, & t > 2. \end{cases}$$

Here

$$\alpha_1 = -\frac{1}{2}, \; \alpha_2 = 2, \; \xi_1 = 2, \; \xi_2 = 1, \; \beta_1 = \frac{1}{3}, \; \beta_2 = -1, \; \eta_1 = 9, \; \eta_2 = 2,$$

$$\sum_{i=1}^{2} \alpha_i \xi_i = \alpha_1\xi_1 + \alpha_2\xi_2 = -\frac{1}{2}(2) + 2(1) = 1,$$

$$\sum_{i=1}^{2} \alpha_i \xi_i^2 = \alpha_1\xi_1^2 + \alpha_2\xi_2^2 = -\frac{1}{2}(2)^2 + 2(1)^2 = 0,$$

$$\sum_{j=1}^{2} \beta_j \eta_j = \beta_1\eta_1 + \beta_2\eta_2 = \frac{1}{2}(9) + (-1)(2) = 1,$$

$$G = g_{11}g_{22} - g_{21}g_{22} = (-0.1047)(-1.8367) - (-0.7926)(-0.2201)$$

$$= 0.0179 \neq 0.$$

Thus condition (ϕ_2) holds.

$$\delta_5 = 0, \; \delta_4 = \begin{cases} (1 + t^2)e^{-20t}, & 0 \le t \le 2, \\ 0, & t > 2, \end{cases} \quad \delta_3 = \begin{cases} (1 + t)e^{-15t}, & 0 \le t \le 2, \\ 0, & t > 2, \end{cases}$$

$$\delta_2 = \begin{cases} 0, & 0 \le t \le 2, \\ e^{-15t}, & t > 2, \end{cases} \quad \delta_1 = \begin{cases} 0, & 0 \le t \le 2, \\ e^{-2t}, & t > 2, \end{cases}$$

then

$$\| \delta_4 \|_{L^1} = \int_0^2 (1 + t^2) e^{-20t} dt = \left[-\frac{201 e^{-20t}}{4000} - \frac{t^2 e^{-20t}}{20} - \frac{t e^{-20t}}{200} \right]_0^2$$

$$= \frac{201}{4000} - \frac{1041}{4000} e^{-40} = 0.0503,$$

$$\| \delta_3 \|_{L^1} = \int_0^2 (1 + t) e^{-20t} dt = \left[-\frac{21 e^{-20t}}{400} - \frac{t e^{-20t}}{20} \right]_0^2$$

$$= \frac{21}{400} - \frac{61 e^{-40}}{400} e^{-40} = 0.0525,$$

$$\| \delta_2 \|_{L^1} = \int_2^{+\infty} e^{-15t} dt = -\frac{e^{-15t}}{15} \Bigg|_2^{+\infty} = \frac{e^{-30}}{15} = 2.0394 \times 10^{-8},$$

$$(2 + D)(\| \delta_4 \|_{L^1} + \| \delta_3 \|_{L^1} + \| \delta_2 \|_{L^1})$$

$$= 4(0.0503 + 0.0525 + 2.0394 \times 10^{-8}) = 0.4112 < 1.$$

Therefore, condition (H_1) holds.

Let $B = 243$. Then $Q_1 Nu \neq 0$ if $|u(t)| > B$, for any $t \in [0, 2]$ and $Q_2 Nu \neq 0$ if $|u''(t)| > B$, for $t \in [2, +\infty)$. Hence, (H_2) is satisfied.

Lastly, let $A = 32$, for any $b, c \in \mathbb{R}$, if $|b| > A, |c| > A$, then

$$aQ_1 N(bt + ct^2) + bQ_2 N(bt + ct^2) > 0.$$

Hence, (H_3) also holds. Since all the conditions of Theorem 2.2 hold, (4.1)-(4.2) has at least one solution.

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References

[1] D. Franco, G. Infante and M. Zima, Second order nonlocal boundary value problems at resonance, Math. Nachr. 284 (2011), 875-884.

[2] G. Cupini, C. Marcelli and F. Papalini, On the solvability of a boundary value problem on the real line, Bound. Value Probl. 26 (2011), 17 pp. doi: 10.1186/1687-2770-2011-26.

[3] J. Jeong, C. G. Kim and E. K. Lee, Solvability of nonlocal boundary value problems on a half-line with $\dim(\ker L) = 2$, Bound. Value Probl. 2014 (2014), 167-178.

[4] J. Mawhin, Topological degree methods in nonlinear boundary value problems, NSFCMBS, Regional Conference Series in Mathematics, American Mathematical Society Providence, RI, 1979.

[5] N. Kosmatov, A multi-point boundary value problem with two critical conditions, Nonlinear Anal. 5 (2006), 622-633.

[6] R. P. Agarwal and D. O'Regan, Infinite interval problems modeling phenomena which arise in the theory of plasma and electrical potential theory, Stud. Appl. Math. 111(3) (2003), 339-358.

[7] S. A. Iyase, On a third-order boundary value problem at resonance on the half-line, Arab. J. Math. 8 (2019), 43-53.

[8] S. A. Iyase and O. F. Imaga, Higher order boundary value problems with integral boundary conditions at resonance on the half-line, J. Nigerian Math. Soc. 382 (2019), 168-183.

[9] S. A. Iyase and O. F. Imaga, On a singular second-order multipoint boundary value problem at resonance, Int. J. Differ. Equ. 2017, Art ID 8579065, 6 pp. doi: 10.1155/2017/8579065.

[10] W. Feng and J. R. L. Webb, Solvability of three-point boundary value problems at resonance, Nonlinear Anal. 30 (1997), 3227-3238.

[11] W. Jiang, B. Wang and Z. Wang, Solvability of second order multi-point boundary-value problems at resonance on a half-line with $\dim \ker = 2$, Electron. J. Differential Equations 120 (2011), 1-11.

[12] X. Lin, Z. Du and F. Meng, A note on a third-order multi-point boundary value problem at resonance, Math. Nachr. 284 (2011), 1690-1700.

[13] X. Lin, Z. Du and Q. Zhang, Solvability of a second-order differential equation with integral boundary condition at resonance on an unbounded domain, Carpathian J. Math. 29(2) (2011), 201-208.

[14] X. Zhang, M. Feng and W. Ge, Existence result of second-order differential equations with integral boundary conditions at resonance, J. Math. Anal. Appl. 353 (2009), 311-319.

[15] Y. Liu, D. Li and M. Fang, Solvability for second-order m-point boundary value problems at resonance on the half-line, Electron. J. Differential Equations 13 (2009), 1-11.