Journal of Physics: Conference Series

PAPER • **OPEN ACCESS**

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Existence results for a resonant third-order problem with two dimensional kernel on the half-line

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Abstract. The purpose of this research is to establish conditions that guarantee the solvability of a resonant third-order boundary value problem (BVP) on the half-line. The resonant problem is subject to integral and m-point boundary conditions while, the kernel of the differential operator is equal to two. The coincidence degree theory will be applied and an illustrative example will be considered to demonstrate the result.

1. Introduction

In this work, the Mawhin coincidence degree theory [4] will be used to study the existence of solutions for the resonant third-order problem of the form:

$$((\rho b'')(t))' = h(t, b(t), b'(t), b''(t)), t \in (0, +\infty) \tag{1.1}$$

$$b(0) = \sum_{j=1}^{m} \alpha_j \int_0^{\xi_j} b(t)dt, b'(0) = \sum_{k=1}^{n} \beta_k \int_0^{\eta_k} b'(t)dt, \lim_{t\to +\infty} \rho(t)b''(t) = 0 \tag{1.2}$$

where $h:[0,+\infty)\times\mathbb{R}^3 \to \mathbb{R}$ is a $L^1[0,+\infty)$-Caratheodory function, $0<\xi_1<\xi_2<\cdots<\xi_m<+\infty$, $0<\eta_1<\eta_2<\cdots<\eta_n<+\infty$, $\alpha_j\in\mathbb{R}$, $j=1,2,\cdots,m$ and $\beta_k\in\mathbb{R}$, $k=1,2,\cdots,n$, $\rho(t)>0$ on $[0,+\infty)$ and $\frac{1}{\rho}\in[0,+\infty)$.

The problem (1.1)-(1.2) is said to be at resonance if the solution of $((\rho u'')(t))' = 0$ subject to (1.2) is non-trivial. Though a lot of research has been carried out on resonant multi-point boundary value problems, see [1, 3, 5-8, 10, 14], only few authors in the literature have considered problems on the half-line having integral boundary conditions where the kernel of the differential operator is of dimension two, see [2,11].

2. Preliminaries

We are going to state definitions and theorems, then prove lemmas that will be used in this work.
Taking U, Z to be normed spaces, $T: dom\, T \subset U \to Z$ a Fredholm mapping of zero index. Also, let $A:U\to U$ and $B:Z\to Z$ be projectors that are continuous such that
$Im\, A = \ker T, \ker B = Im\, T, U = \ker T \oplus \ker A, Z = Im\, T \oplus Im\, B.$
Then,

$$T|_{dom\,T\cap\ker A}: dom\,T\cap\ker A\to Im\,T$$

is invertible. We denote the inverse of the mapping T by $K_A: Im\,T\to dom\,T\cap\ker A$ while the generalized inverse, $K_{A,B}: Z\to dom\,T\cap\ker A$ is defined as $K_{A,B}=K_A(I-B)$.

Definition 2.1. [3] If $\Omega\subset U$ is open and bounded and $dom\,T\cap\bar{\Omega}\neq\varnothing$ then a mapping $N:U\to Z$ is L-compact on $\bar{\Omega}$ if $BN(\bar{\Omega})\subset Z$ is bounded and $K_A(I-B)N:\bar{\Omega}\to U$ is compact.

Theorem 2.1. [4] Let N be L-compact on $\bar{\Omega}$ and T a index zero-Fredholm map, then at least one solution of $Tb=Nb$ exists in $dom\,T\cap\bar{\Omega}$, if

(i) $Tb\neq\lambda Nb \quad \forall(b,\lambda)\in[(dom\,T\ \ker T\cap\partial\Omega]\times(0,1)$,

(ii) $Nb\notin Im\,T \quad \forall b\in\ker T\cap\partial\Omega$,

(iii) $deg(AN|_{\ker T},\ker T,0)\neq 0$, where $B:Z\to Z$ is a projection such that $Im\,T=\ker B$.

Theorem 2.2. [15] Let $N\subset V$ where V implies the space of all bounded and continuous functions on the interval $[0,+\infty)$, then N is relatively compact on V if:

(i) N is a bounded;

(ii) on any compact interval of $[0,+\infty)$, every $f\in N$ is equicontinuous;

(iii) every $f\in N$ is equiconvergent at $+\infty$.

Let

$$U=\left\{b\in C^2[0,+\infty): b,b',\rho(b'')\in AC[0,+\infty), \lim_{t\to+\infty} e^{-t}|b^{(i)}(t)| \text{ exist}, i=0,1,2\right\}$$

and the norm on U is $\|b\|=\max\{\|b\|_\infty,\|b'\|_\infty,\|b''\|_\infty\}$ where $\|b^i\|_\infty=\sup_{t\in[0,+\infty)} e^{-t}|b^i|, i=0,1,2$.

Then, the space $(U,\|\cdot\|)$ is a Banach Space.

Let $Z=L^1[0,+\infty)$ with the norm $\|y\|_{L^1}=\int_0^{+\infty}|y(v)|\,dv$.

Also, define $Tb=(\rho b'')'$, with domain

$$domT=\Big\{b\in U:(\rho b'')'\in L^1[0,+\infty), b(0)=\sum_{j=1}^{m}\alpha_j\int_0^{\xi_j}b(t)dt\ ,$$

$$b'(0)=\sum_{k=1}^{n}\beta_k\int_0^{\eta_k}b'(t)dt, (\rho b'')(+\infty)=0\Big\},$$

and the operator \$N:U \to Z\$ be defined as $(Nb)t=h(t,b(t),b'(t),b''(t)),\quad t\in[0,+\infty)$, hence, equations (1.1)-(1.2) in abstract form is $Tb=Nb$.

We made the following assumptions:

(ϕ_1) $\sum_{k=1}^{n}\beta_k\eta_k=1$, $\sum_{j=1}^{m}\alpha_j\xi_j=1$ and $\sum_{j=1}^{m}\alpha_j\xi_j^2=0$,

(ϕ_2) $W=(B_1e^{-t}\cdot B_2te^{-t}-B_2e^{-t}\cdot B_1te^{-t}):=w_{11}\cdot w_{22}-w_{12}\cdot w_{21}\neq 0$ where

$B_1y=\sum_{k=1}^{n}\beta_k\int_0^{\eta_k}\int_0^{s}\frac{1}{\rho(\tau)}\int_\tau^{+\infty}y(v)dvd\tau ds$, and $B_2y=\sum_{j=1}^{m}\alpha_j\int_0^{\xi_j}\int_0^{x}\int_0^{s}\frac{1}{\rho(\tau)}\int_\tau^{+\infty}y(v)dvd\tau dsdx$.

It is easily seen that $\ker T=\{a+ct: a,c\in\mathbb{R}, t\in[0,+\infty)\}$.

Lemma 2.1. $imT=\{y\in Z: B_1y=B_2y=0\}$.

Proof. Consider

$$(\rho b''(t))'=y, t\in[0,+\infty)\ , \tag{2.1}$$

integrating (2.1) from τ to $+\infty$ gives

$$(\rho b''(t)) = (\rho b'')(+\infty) + \int_{\tau}^{+\infty} y(v)dv \ . \tag{2.2}$$

At $t=+\infty$, $(\rho b'')(+\infty)=0$, hence (2.2) becomes

$$b''(t) = \frac{1}{\rho(\tau)}\int_0^{\tau} y(v)dv \ . \tag{2.3}$$

Integrating (2.3) with respect to t from 0 to t gives

$$b'(t) = b'(0) + \int_0^{s}\frac{1}{\rho(\tau)}\int_0^{\tau} y(v)dvd\tau \ . \tag{2.4}$$

Applying boundary condition (1.2) and in view of $\sum_{k=1}^{n}\beta_k\eta_k = 1$, we have

$$\sum_{k=1}^{n}\beta_k\int_0^{\eta_k}\int_0^{s}\frac{1}{\rho(\tau)}\int_0^{\tau} y(v)dvd\tau ds = 0 \ .$$

Integrating (2.4) from 0 to t gives

$$b(t) = b(0) + b'(0)t + \int_0^{x}\int_0^{s}\frac{1}{\rho(\tau)}\int_0^{\tau} y(v)dvd\tau ds \ .$$

Applying boundary condition (1.2), and in view of $\sum_{j=1}^{m}\alpha_j\xi_j = 1$ and $\sum_{j=1}^{m}\alpha_j\xi_j^2 = 0$, we have

$$\sum_{j=1}^{m}\alpha_j\int_0^{\xi_j}\int_0^{x}\int_0^{s}\frac{1}{\rho(\tau)}\int_{\tau}^{+\infty} y(v)dvd\tau dsdx = 0$$

and

$$b(t) = a + ct + \int_0^{t}\int_0^{s}\frac{1}{\rho(\tau)}\int_0^{\tau} y(v)dvd\tau ds \ ,$$

where a, c are arbitrary constants and $b(t)$ is a solution to (2.1) satisfying (1.2). End of proof.
We next define the operator $B: Z \to Z$ as

$$By = (R_1y) + (R_2y)\cdot t$$

where

$$R_1y = \frac{1}{W}(m_{11}B_1y + m_{12}B_2y)e^{-t}, \ R_2y = \frac{1}{W}(m_{21}B_1y + m_{22}B_2y)e^{-t} \ ,$$

and m_{ij} is the algebraic cofactor of w_{ij}.

Lemma 2.2. The following conditions hold:

(i) $T: domT \subset U$ is a Fredholm operator whose index is zero;

(ii) the generalized inverse $K_A: imT \to domT \cap \ker A$ may be written as

$$K_Ay = \int_0^{t}\int_0^{s}\frac{1}{\rho(\tau)}\int_{\tau}^{+\infty} y(v)dvd\tau ds \ .$$

Also,

$$\| K_p y\| = \left\{\left\|\frac{1}{\rho}\right\|_{L^1}, \left\|\frac{1}{\rho}\right\|_{\infty}\right\}\| y\|_{L^1} \ .$$

Proof. Prove of (i) is obvious.
ii) Let a continuous projector $A: U \to U$ be defined as

$$(Ab)(t) = b(0) + b'(0)t, \ t \in [0,+\infty) \ .$$

Then $K_A: imT \to domT \cap \ker A$ may be written as

$$K_A y = \int_0^t \int_0^s \frac{1}{\rho(\tau)} \int_\tau^{+\infty} y(v)dvd\tau ds \ .$$

For $y \in imT$ we have

$$(TK_A)y(t) = (K_A y)''' = y(t)$$

and for $b \in dom\, T \cap ker\ A$, we have

$$(K_A T)b(t) = (K_A)b'''(t) = \int_0^x \int_0^s \frac{1}{\rho(\tau)} \int_\tau^{+\infty} (\rho b''(v))' dvd\tau ds = b(t) - Ab(t) \ .$$

Since $u \in dom\, T \cap ker\ A$, $Ab(t) = 0$ then $(K_A T)b(t) = b(t)$. Thus $K_A = (T\,|_{dom\,T \cap \ker A})^{-1}$ ..
Also, it is easily seen that

$$\| K_A y\| = \max\{\| K_A y\|_\infty, \| (K_A y)'\|_\infty, \| (K_A y)''\|_\infty\} \max\left\{\left\|\frac{1}{\rho}\right\|_{L^1}, \left\|\frac{1}{\rho}\right\|_\infty\right\}\| y\|_{L^1} \, .$$

End of proof of Lemma 2.2.

Lemma 2.3. Let $\Omega \subset U$ be open and bounded with $dom\,T \cap \bar{\Omega} \neq \emptyset$. If h is a L^1-Caratheodory function, then, the nonlinear operator N is L -compact on $\bar{\Omega}$.

Proof. Let $b \in \bar{\Omega}$ and let $q > 0$ then, $\| b\| \le q$ since Ω is bounded. Since h is L^1-Caratheodory and for any $b \in \bar{\Omega}$, where $t \in [0, +\infty)$, then

$$\| Nb\|_{L^1} \le \int_0^{+\infty} |h(v, b(v), b'(v), b''(v))| dv \le \| \varphi_q\|_{L^1} \, . \tag{2.5}$$

Hence,

$$\| BNb\|_{L^1} \le \left\| \frac{1}{\rho} \right\|_{L^1} \| \varphi_q\|_{L^1} \frac{1}{|W|} \Big[(|m_{11}| + |m_{21}|)\Big(\sum_{k=1}^{n} |\beta_k| \eta_k\Big) + (|m_{12}| + |m_{22}|)\Big(\sum_{j=1}^{m} |\alpha_k| \xi_j^2\Big)\Big] . \tag{2.6}$$

Then, $BN(\bar{\Omega})$ is bounded. We will use the following three steps to prove that $K_A(I - B)N(\bar{\Omega})$ is compact.

Step 1: <u>Boundedness</u>. Let $b \in \bar{\Omega}$, then

$$\sup_{t \in [0, +\infty)} e^{-t} |K_A(I - B)Nb(t)| \le \left(\| \varphi_q\|_{L^1} + \| BNb\|_{L^1}\right) \left\| \frac{1}{\rho} \right\|_{L^1} , \tag{2.7}$$

$$\sup_{t \in [0, +\infty)} e^{-t} |(K_A(I - B)Nb)'(t)| \le \left(\| \varphi_q\|_{L^1} + \| BNb\|_{L^1}\right) \left\| \frac{1}{\rho} \right\|_{L^1} \tag{2.8}$$

and

$$\sup_{t \in [0, +\infty)} e^{-t} |(K_A(I - B)Nb)''(t)| \le \left(\| \varphi_q\|_{L^1} + \| BNb\|_{L^1}\right) \left\| \frac{1}{\rho} \right\|_\infty . \tag{2.9}$$

From (2.5), (2.6), (2.7), (2.8) and (2.9), we see that $K_A(I - B)N(\bar{\Omega})$ is bounded.

Step 2: <u>Equi</u>-continuity. Let $b \in \bar{\Omega}$, $t_1, t_2 \in [0, G]$ with $t_1 < t_2$ and $G \in (0, +\infty)$, then $\left| e^{-t_2} K_A(I - B)Nb(t_2) - e^{-t_1} K_A(I - B)Nb(t_1) \right|$, $\left| e^{-t_2} (K_A(I - B)Nb)'(t_2) - e^{-t_1} (K_A(I - B)Nb)'(t_1) \right|$, and $| e^{-t_2} (K_A(I - B)Nb)''(t_2) - e^{-t_1} (K_A(I - B)Nb)''(t_1) | \to 0$ as $t_1 \to t_2$.

Thus, $K_A(I-B)Nb(\bar{\Omega})$ is equi-continuous on the compact set $[0,G]$.

Step 3: Equi-convergence at $+\infty$**.** Let $b\in\bar{\Omega}$. We have,

$\left|e^{-t}K_A(I-B)Nb(t)-\lim_{t\to+\infty}e^{-t}K_A(I-B)Nb(t)\right|$, $\left|e^{-t}(K_A(I-B)Nb)'(t)-\lim_{t\to+\infty}(K_A(I-B)Nb)'(t)\right|$ and

$|e^{-t}(K_A(I-B)Nb)''(t)-\lim_{t\to+\infty}e^{-t}(K_A(I-B)Nb)''(t)|\ \to 0$ as $t\to\infty$

Then, $K_A(I-B)Nb(\bar{\Omega})$ is equi-convergence at $+\infty$. Thus, $K_A(I-B)Nb(\bar{\Omega})$ is compact, therefore, the nonlinear operator N is L-compact on $\bar{\Omega}$.

3. Existence Result

Subject to (1.2), the conditions that guarantee the existence of solutions to the present problem (1.1) are as follows:

Theorem 3.1. Suppose h is an $L^1[0,+\infty)$ - Caratheodory well-defined function. If (ϕ_1), (ϕ_2) and the following hold

(H_1)] The functions $\nu_i(t)\in L^1[0,+\infty)$, $i=1,2,\dots,5$ and a constant $\sigma\in[0,1)$ exist such that for all $(b_1,b_2,b_3)\in\mathbb{R}^3$ then

$$|h(t,b_1,b_2,b_3)\le e^{-t}[\nu_1(t)|b_1|+\nu_2(t)|b_2|+\nu_3(t)|b_3|+\nu_4(t)|b_3|^{\sigma}+\nu_5(t)]\ . \tag{3.1}$$

(H_2) The constants $L>0$ and $l_0>0$ exist in such a way that for $b\in dom\,T$ if $|b(t)|>L$ for $t\in[0,l_0]$ or $|b'(t)|>L$ for $t\in[0,+\infty)$, then either

$$B_1Nb(t)\neq 0 \text{ or } b_2Nb(t)\neq 0$$

(H_3) A constant $D>0$ exist such that if $|a|>0$ or $|c|>D$, then either

$$B_1N(a+ct)+B_2N(a+ct)<0 \tag{3.2}$$

or

$$B_1N(a+ct)+B_2N(a+ct)>0 \tag{3.3}$$

where $a,c\in\mathbb{R}$ satisfying $|a|+|c|>D$.

Then at least one solution of the problem (1.1) and (1.2) exists provided

$$\Delta(\|\nu_1\|_{L^1}+\|\nu_2\|_{L^1}+\|\nu_3\|_{L^1})<1$$

Where $\Delta=\max\left\{(2+l_0)\left\|\frac{1}{\rho}\right\|_{L^1},(1+l_0)\left\|\frac{1}{\rho}\right\|_{L^1}+\left\|\frac{1}{\rho}\right\|_{\infty}\right\}$.

This was proved using Theorem 2.1.

4. Example

Consider

$$((e^tb''(t))'=\begin{cases}0 & t\in[0,2],\\ -\frac{e^{-t}}{20}+\frac{e^{-t}}{29}b'(t)+\frac{e^{-2t}}{14}\sin b''(t)+e^{-t}\sqrt[3]{b'(t)}, & t\in(2,+\infty),\end{cases} \tag{4.1}$$

$$b(0)=-\frac{1}{2}\int_0^2 b'(t)dt+2\int_0^1 b'(t)dt,\ b'(0)=\frac{1}{3}\int_0^3 b'(t)dt,\ \rho b''(+\infty)=0. \tag{4.2}$$

We have $l_0 = 2$, $\rho = e^t$, $\alpha_1 = -\frac{1}{2}$, $\alpha_2 = 2$, $\xi_1 = 2$, $\xi_2 = 1$, $\beta_1 = \frac{1}{3}$, $\eta_1 = 3$, $\sum_{j=1}^{2} \alpha_j \xi_j = 1$, $\sum_{j=1}^{2} \alpha_j \xi_j^2 = 0$, $\sum_{k=1}^{1} \beta_k \eta_k = 1$ and $W = -0.0473 \neq 0$.

Hence, (ϕ_1) and (ϕ_2) hold. It can be shown that all the conditions of Theorem 3.1 hold, then (4.1)-(4.2) has at least one solution.

Acknowledgment
We are grateful to Covenant University for their support.

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