Heliyon 6 (2020) e04826

Contents lists available at ScienceDirect

Heliyon

journal homepage: www.cell.com/heliyon

Research article

Higher-order p-Laplacian boundary value problem at resonance on an unbounded domain

Samuel A. Iyase, Kanayo S. Eke *

Department of Mathematics, Covenant University, Canaanland, KM 10 Idiroko Road, P. M. B. 1023, Ota, Ogun State, Nigeria

ARTICLE INFO

Keywords:
Mathematics
Higher-order
Resonance
p-Laplacian
Unbounded domain
Coincidence degree

ABSTRACT

In this work, we employ the extension of Mawhin's coincidence degree by Ge and Ren to investigate the solvability of the p-Laplacian higher-order boundary value problems of the form

$$(w(t)\phi_p(x^{(n-1)}(t)))' = h(t,\ x(t), \cdots,\ x^{(n-1)}(t)),\ 0 < t < \infty,$$

$$x^{(n-2)}(0) = \frac{(n-2)!}{\eta^{n-2}}x(\eta),\ x^{(n-1)}(0) = x^{(i)}(0) = 0,\ i = 1, 2,\ \cdots,\ n-3,\ n \geq 3,$$

$$x^{(n-2)}(\infty) = \int_0^\eta x^{(n-2)}(s)dA(s),$$

Where $\eta > 0$, $h : [0,\ \infty) \times \mathbb{R}^n \to \mathbb{R}$ is a Caratheodory's function with $A(0) = 0$, $A(\eta) = 1$, $w \in C[0,\ \infty)$, $w(t) > 0$ for all $t \geq 0$, $\phi_p(s) = |s|^{p-2}s$.

1. Introduction

In this paper we obtain existence results for the following higher-order p-Laplacian boundary value problems

$$\left(w(t)\phi_p\left(x^{(n-1)}(t)\right)\right)' = h\left(t,\ x(t), \cdots,\ x^{(n-1)}(t)\right),\ 0 < t < \infty, \tag{1.1}$$

subject to the following boundary conditions

$$x^{(n-2)}(0) = \frac{(n-2)!}{\eta^{n-2}}x(\eta),\ x^{(n-1)}(0) = x^{(i)}(0) = 0,\ i = 1, 2,\ \cdots,\ n-3,\ n \geq 3,$$

$$x^{(n-2)}(\infty) = \int_0^\eta x^{(n-2)}(s)dA(s). \tag{1.2}$$

where $\eta > 0$, $h : [0,\ \infty) \times \mathbb{R}^n \to \mathbb{R}$ is a Caratheodory's function with respect to $L^1[0,\ \infty)$. $A : [0,\ \eta] \to [0, \infty)$ is a non-decreasing function with $A(0) = 0$, $A(\eta) = 1$, $w \in C[0, \infty)$ with $w(t) > 0$ for all $t \geq 0$, $\phi_p(s) = |s|^{p-2}s$, $p > 1, \phi_p^{-1} = \phi_q$, $q > 1$.

The study of boundary value problems with a p-Laplacian is important because they have applications such as in non-Newtonian mechanics, nonlinear elasticity, bloodflow models, glaciology etc. Higher-order boundary value problems at resonance when p = 2 have received much attention in the literature e.g [2, 4, 5, 7, 10]. Most of the results were obtained using Mawhin's coincidence degree arguments [11]. However when $p \neq 2$, the differential operator is no longer linear and Mawhin's continuation theorem can no longer be applied directly. On the other hand, higher-order p-Laplacian boundary value problems on unbounded domains at resonance have not received much attention in the literature. Most of the studies in this direction have concentrated mainly on bounded domains. For some recent results on boundary value problems with a p-Laplacian, see [6, 8, 9, 12, 13, 15, 16] and the references therein.

The boundary value problem 1.1), 1.2) is a problem at resonance if $Lx = (w(t)\phi_p(x^{(n-1)}(t)))' = 0$ has nontrivial solutions under the boundary conditions (1.2). If $Lu = 0$ has only the trivial solutions then 1.1), 1.2) is then said to be at nonresonance. In general, resonance problems can be written in the abstract form $Lx = Nx$ where L is not invertible. This paper is organized as follows. In section 2 we recall some preliminary definitions, theorems and Lemmas. Section 3 will be devoted to stating and proving the main existence results and in section 4 we provide an example to validate our result.

* Corresponding author.
E-mail address: kanayo.eke@covenantuniversity.edu.ng (K.S. Eke).

https://doi.org/10.1016/j.heliyon.2020.e04826
Received 24 June 2019; Received in revised form 27 May 2020; Accepted 26 August 2020

2. Preliminaries

In this section we introduce the theoretical foundations of the methods we shall utilize in the subsequence sections.

Definition 2.1: Let X and Z be Banach spaces with norms $||\cdot||_X$, $||\cdot||_Z$ respectively. An operator $L: X\cap dom\ L \to Z$ is said to be quasi-linear if

(i) $ImL = L(X\cap domL)$ is a closed subset of Z,
(ii) $kerL = \{x\in X\cap domL : Lx = 0\}$ is linearly homeomorphic to $\mathbb{R}^n$.

Definition 2.2: Let X be a Banach space and $X_1 \subset X$ a subspace. A mapping $Q: X \to X_1$ is called a semi-projector if

(i) $Q^2x = Qx,\ x\in X$,
(ii) $Q(\lambda x) = \lambda\, Qx,\ x\in X,\ \lambda\in\mathbb{R}$.

Definition 2.3: $N_\lambda : \overline{\Omega} \to Z,\ \lambda\in[0,\ 1]$ is said to be L-compact in $\overline{\Omega}$ if there exists $Z_1 \subset Z$ with $dimZ_1 = dim\ ker\ L$ and the operators Q and R such that the following conditions hold

$$\text{(a) } ker\ Q = ImL, \tag{2.1}$$

$$\text{(b) } QN_\lambda x = 0,\ \lambda\in(0,1),\ \text{if and only if } QNx = 0 \tag{2.2}$$

$$\text{(c) } R(\cdot, 0) \text{ is the zero operator}, \tag{2.3}$$

$$\text{(d) } R(.\ ,\ \lambda)|_{\Sigma_\lambda} = (1 - P)|_{\Sigma\lambda}, \tag{2.4}$$

Where $\Sigma_\lambda = \{x\in\overline{\Omega}\ :\ Lx = N_\lambda\ x\}$,

$$\text{(e) } L[P + R(.\ ,\lambda)] = (1 - Q)N_\lambda, \tag{2.5}$$

where $P: X\to X$ is a projector and Q is a semi-projector such that $ImP = ker\ L$ and
$ImQ = Z_1$.

We shall use the following inequalities [14] in the context of the p-Laplacian $\phi_p(s),\ p>1$,

$$\text{(i) } \phi_p(u + v) \le \phi_p(u) + \phi_p(v),\ \text{if } 1<p\le 2. \tag{2.6}$$

$$\text{(ii) } \phi_p(u + v) \le 2^{p-2}(\phi_p(u) + \phi_p(v)),\ \text{if } p>2. \tag{2.7}$$

Definition 2.4: The map $h: [0,\ \infty) \times \mathbb{R}^n \to \mathbb{R}$ is L^1- Caratheodory if the following conditions hold,

(i) for each $x\in\mathbb{R}^n$, the mapping $t\to h(t,\ x)$ is Lebesgue measurable,
(ii) for a.e. $t\in [0,\infty)$, the mapping $x\to h(t,\ x)$ is continuous on $\mathbb{R}^n$,
(iii) for each $r > 0$, there exists a $\psi_r \in L^1[0,\infty)$ such that for a.e. $t\in[0,\infty)$ and every x such that $|x|\le r$ we have $|h(t,\ x)| \le \psi_r(t)$.

In what follows we shall use the following result of Ge and Ren.

Theorem 2.1 [6]**:** Let X and Z be Banach spaces with norms $||\cdot||_X, ||\cdot||_Z$ and $V\subset X$ be an open and bounded non empty set. Suppose $L: X\cap dom\ L\to Z$ is a quasi-linear operator and $N_\lambda : \overline{V}\to Z, \lambda\in[0,\ 1]$ is L-compact. In addition, if the following conditions are satisfied;

(1) $Lx \ne N_\lambda x,\ \ x\in\partial V\cap\ dom\ L,\ \lambda\in(0,\ 1)$,
(2) $deg\ (JQN,\ V\cap ker\ L,\ 0) \ne 0$,

where $N = N_1$ and $J: ImQ \to ker\ L$ is a homeomorphism with $J(0) = 0$ and deg is the Brouwer degree. Then the abstract equation $Lx = Nx$ has at least one solution in $dom\ L\cap\overline{V}$.

Let $D_\infty = \{x: [0,\ \infty)\to\mathbb{R}\ \ x$ is continuous on $[0,\ \infty)$ and $lim_{t\to\infty}x(t)$ exists$\}$. For $x\in D_\infty$ define $||x||_\infty = Sup_{t\in[0,\infty)}|x(t)|$.

Then D_∞ is a Banach space (see [1, 3]).

Lemma 2.1: Let $X = \{x: [0,\ \infty) \to\mathbb{R}:\ x,\ w\phi_p(x^{(n-1)})\in A\ C\ [0,\ \infty),\ lim_{t\to\infty}e^{-t}|x^{(i)}(t)|$ exists,

$$0\le i\le n-1,\ \left(w(t)\phi_p\left(x^{(n-1)}(t)\right)\right)'\in L^1[0,\infty)\}, \tag{2.8}$$

With norm,

$$||x|| = \max_{0\le i\le n-1}\left(sup_{t\in[0,\ \infty)}\ e^{-t}|x^{(i)}(t)|\right). \tag{2.9}$$

Then X is a Banach space.

Proof: First we show that $||\cdot||$ defines a norm. Clearly $||x|| > 0$ and $||x|| = 0$ if and only if $(sup_{t\in[0,\infty)}\ e^{-t}|x^{(i)}(t)| = 0$ if and only if $x^{(i)} = 0,\ i = 0,1,2,3,\cdots,\ n-1$ .Let $\lambda\in\mathbb{R}$. Then,

$$\begin{aligned}||\lambda x|| &= \max_{0\le i\le n-1}\left(sup_{t\in[0,\infty)}e^{-t}|(\lambda x)^{(i)}(t)|\right)\\ &= |\lambda|\max_{0\le i\le n-1}\left(sup_{t\in[0,\infty)}e^{-t}|x^{(i)}(t)|\right)\\ &= |\lambda|||x||.\end{aligned}$$

$$\begin{aligned}||x+y|| &= \max_{0\le i\le n-1}\left(sup_{t\in[0,\infty)}\ e^{-t}|(x(t) + y(t))^{(i)}|\right)\\ &= \max_{0\le i\le n-1}\left(sup_{t\in[0,\infty)}\ e^{-t}|(x^{(i)}(t) + y^{(i)}(t))|\right)\\ &\le \max_{0\le i\le n-1}\left(sup_{t\in[0,\infty)}\ e^{-t}|x^{(i)}(t)|\right) + \max_{0\le i\le n-1}\left(sup_{t\in[0,\infty)}\ e^{-t}|y^{(i)}(t)|\right)\\ &\le ||x|| + ||y||.\end{aligned}$$

Next we prove that X is complete.

Let $\{x_k\}$ be a Cauchy sequence in X. Then $\{e^{-t}x_k\}$ and $\{e^{-t}x_k'\}$ are Cauchy sequences in D_∞. Thus there exists y_0^* and y_1^* in D_∞ such that

$$lim_{k\to\infty}\ sup_{t\in[0,\infty)}|\ e^{-t}x_k - y_0^*| = lim_{k\to\infty}\ sup_{t\in[0,\infty)}\ |e^{-t}x_k' - y_1^*| = 0.$$

Let $x_0^*(t) = e^ty_0^*$ then

$$lim_{k\to\infty}\ sup_{t\in[0,\infty)}\ e^{-t}|x_k(t) - x_0^*(t)| = 0.$$

Set $x_1^*(t) = e^ty_1^*(t)$ then $lim_{k\to\infty}sup_{t\in[0,\infty)}\ e^{-t}|x_k'(t) - x_1^*(t)| = 0$.

Since the convergence to $x_0^*(t)$ and $x_1^*(t)$ is uniform on [0, M] for any $M>0$, then $x_0^*(t)$ is differentiable on [0, M] and $(x_0^*)' = x_1^*$. Since M is arbitrary x_0^* is differentiable on $[0,\ \infty)$. Following the same reasoning we obtain

$$x_i^* = (x_0^*)^{(i)},\quad i = 0,\ 1,\ 2,\ \cdots,\ n-1\ .$$

Hence,

$$lim_{k\to\infty}\ sup_{t\in[0,\infty)}\ e^{-t}|x_k^{(i)} - x_i^*| = 0.$$

Now

$$\begin{aligned}&lim_{k\to\infty}||x_k - x_0^*||\\ &= lim_{k\to\infty}\max_{0\le i\le n-1}\left(sup_{t\in[0,\infty)}\ e^{-t}|(x_k - x_0^*)^{(i)}|\right)\\ &= lim_{k\to\infty}\max_{0\le i\le n-1}\left(sup_{t\in[0,\infty)}\ e^{-t}|(x_k)^{(i)} - (x_i^*)^{(i)}|\right) = 0.\end{aligned}$$

Also

$$w(t)\phi_p\left(\left(x_0^*(t)\right)^{(n-1)}\right) = w(t)\phi_p(e^ty_0^*)^{(n-1)}\in\ A\ C\ [0,\ \infty),$$

and

$$w(t)\phi_p\left(\left(x_0^*\right)^{(n-1)}\right)' = \left(w(t)\phi_p(e^ty_0^*)^{(n-1)}\right)'\ \in L^1[0,\infty).$$

Thus $x_0^* \in X$ and hence X is a Banach space.Let $Z = L^1[0,\infty)$ with the norm,

$$\left\|y\right\|_1 = \int_0^\infty |y(t)|dt, \; y \in Z. \tag{2.10}$$

To derive the compactness of the operator R we use the following compactness criterion.

Theorem 2.2 [2]**:** Let X be the space of all bounded continuous vector valued functions on $[0,\infty)$ and $V \subset X$. Then V is relatively compact if the following conditions hold,

(i) V is bounded in X,
(ii) the functions from V are equicontinuous on any compact interval of $[0,\infty)$,
(iii) the functions from V are equiconvergent at infinity.

Let $L:\; dom\, L \subset X \to Z$ be defined by

$$L\,x(t) = \left(w(t)\phi_p(x^{(n-1)}(t))\right)', \; t \in [0,\infty), \tag{2.11}$$

Where

$$domL = \left\{ \begin{array}{c} x \in X : x^{(n-2)}(0) = \dfrac{(n-2)!}{\eta^{n-2}} x(\eta), \\ x^{(n-1)}(0) = x^{(i)}(0) = 0, \; i = 1,2\cdots,n-3, \\ x^{(n-2)}(\infty) = \displaystyle\int_0^\eta x^{(n-2)}(s)dA(s) \end{array} \right\} \tag{2.12}$$

We define $N_\lambda : X \to Z$ by

$$N_\lambda x(t) = \lambda h\left(t, \; x(t), \; \cdots, \; x^{(n-1)}(t)\right). \tag{2.13}$$

Then 1.1), 1.2) takes the abstract form,

$$Lx = N_\lambda x \quad \text{when} \quad \lambda = 1. \tag{2.14}$$

Lemma 2.2: If $x^{(n-2)}(\infty) = \int_0^\eta x^{(n-2)}(s)dA(s), \; x^{(n-2)}(0) = \frac{(n-2)!}{\eta^{n-2}} x(\eta),$

$x^{(n-1)}(0) = x^{(i)}(0) = 0, \; i = 1,2,\cdots, \; n-3,$

$A(\eta) = 1, \;\; A(0) = 0$ then

(i) $ker\; L = \{x \in dom\, L : \; x(t) = ct^{n-2}, \; c \in R, \; t \in (0,\infty)\}$.
(ii) $ImL = \left\{ y \in Z : \int_0^\infty \phi_q\left(\frac{1}{w(s)}\right) \phi_q\left(\int_0^s y(\tau)d\tau\right)ds - \int_0^\eta \int_0^t \phi_q\left(\frac{1}{w(s)}\right) \phi_q\left(\int_0^s y(\tau)d\tau\right)dsdA(t) = 0 \right\}.$
(iii) $L:\; domL \to Z$ is a quasi-linear operator.

Proof: It is easily verified that $ker\; L = \{x \in domL : \; x(t) = ct^{n-2}\}$ hence (i) holds. To prove (ii), let $y \in Z$ and consider the equation,

$$\left(w(t)\phi_p(x^{(n-1)}(t))\right)' = y(t) \tag{2.15}$$

We prove that (2.15) has a solution $x(t)$ that satisfies the boundary conditions (1.2) if and only if

$$\int_0^\infty \phi_q\left(\frac{1}{w(s)}\right)\phi_q\left(\int_0^s y(\tau)d\tau\right)ds - \int_0^\eta\int_0^t \phi_q\left(\frac{1}{w(s)}\right)\phi_q\left(\int_0^s y(\tau)d\tau\right)dsdA(t) = 0.$$

From (2.15) we obtain

$$x(t) = x(0) + \frac{1}{(n-2)!}\int_0^t (t-s)^{n-2}\phi_q\left(\frac{1}{w(s)}\right)\phi_q\left(\int_0^s y(\tau)d\tau\right)ds + ct^{n-2}. \tag{2.16}$$

Then,

$$x^{(n-2)}(t) = \int_0^t \phi_q\left(\frac{1}{\omega(s)}\right)\phi_q\left(\int_0^s y\left(\tau\right)d\tau\right)ds + x^{(n-2)}(0).$$

$$\begin{aligned} x^{(n-2)}(\infty) &= x^{(n-2)}(0) + \int_0^\infty \phi_q\left(\frac{1}{w(s)}\right)\phi_q\left(\int_0^s y(\tau)d\tau\right)ds \\ &= \int_0^\eta \left[x^{(n-2)}(0) + \int_0^s w_q\left(\frac{1}{w(v)}\right)\phi_q\left(\int_0^v y(\tau)d\tau\right)dv\right]dA(s) \\ &= x^{(n-2)}(0)A(\eta) + \int_0^\eta\int_0^s \phi_q\left(\frac{1}{w(v)}\right)\phi_q\left(\int_0^v y(\tau)d\tau\right)dv\, dA(s), \end{aligned}$$

and hence,

$$\int_0^\infty w_q\left(\frac{1}{w(s)}\right)\phi_q\left(\int_0^s y(\tau)d\tau\right)ds - \int_0^\eta\int_0^t \phi_q\left(\frac{1}{w(s)}\right)\phi_q\left(\int_0^s y(\tau)d\tau\right)dv\, dA(t) = 0. \tag{2.17}$$

If (2.17) holds then,

$$x(t) = ct^{n-2} + \frac{1}{(n-2)!}\int_0^t (t-s)^{n-2}\phi_q\left(\frac{1}{w(s)}\right)\phi_q\left(\int_0^s y(\tau)d\tau\right)ds. \tag{2.18}$$

where c an arbitrary constant is the solution of (2.15). Thus $dim\; ker\; L = 1 < \infty$, $ImL \subset Z$ is closed. Hence L is a quasi-linear operator.

Lemma 2.3: If h is a L^1-Caratheodory's function then $N_\lambda :\; \varpi \to Z$ is L-compact in ϖ, $\omega \subset X$ an open and bounded subset with the origin $0 \in \omega$.

Proof: Let $Q:\; Z \to Z$ be defined by

$$Qy(t) = \rho\left[\int_0^\infty \phi_q\left(\frac{1}{w(s)}\right)w_q\left(\int_0^s y(\tau)d\tau\right)ds - \int_0^\eta\int_0^t \phi_q\left(\frac{1}{w(s)}\right)w_q\left(\int_0^s y(\tau)d\tau\right)ds\, dA(t)\right] \tag{2.19}$$

where

$$\rho(t) = re^{-t},$$

$$r = \frac{1}{\int_0^\infty \phi_q\left(\frac{1}{w(s)}\right)\phi_q\left(\int_0^s y\left(\tau\right)d\tau\right)ds - \int_0^\eta\int_0^t \phi_q\left(\frac{1}{w(s)}\right)\phi_q\left(\int_0^s y(\tau)d\tau)dsdA(t)\right.}, \tag{2.20}$$

and

$$\int_0^\infty \phi_q\left(\frac{1}{w(s)}\right)\phi_q\left(\int_0^s y(\tau)d\tau\right)ds - \int_0^\eta\int_0^t \phi_q\left(\frac{1}{w(s)}\right)\omega_q\left(\int_0^s y(\tau)d\tau\right)ds\ dA(t) \neq 0.$$

Then

$$Q^2y = Q(Qy) = r\times\frac{1}{r}(Qy) = Qy,\ y\in\ Z,$$

and $Q(\lambda y) = \lambda Qy,\ y\in Z,\ \lambda\in R.$

Therefore Q is a semi-projector with

$dimkerL = dim\ ImQ = 1.$

We define $R(x,\lambda):\ X\times[0,\ 1]\to X$ by

$$R(x,\lambda) = \frac{1}{(n-2)!}\int_0^t (t-s)^{n-2}\phi_q\left(\frac{1}{w(s)}\right)\phi_q\left(\int_0^s (I-Q)N_\lambda x(\tau)d\tau\right)ds - \frac{1}{(n-2)!}\int_0^\eta (\eta-\ )^{n-2}\phi_q\left(\frac{1}{w(s)}\right)\phi_q\left(\int_0^s (I-Q)N_\lambda x(\tau)d\tau\right)ds \tag{2.21}$$

(2.1), 2.2) and (2.3) are easily verified. We verify (2.4) and (2.5).

Let $P: X\to kerL$ be defined by

$$Px(t) = \frac{x^{(n-2)}(0)t^{n-2}}{(n-2)!},t\ \in(0,\infty). \tag{2.22}$$

For $x\ \in\Sigma_\lambda = \{x\ \in\Omega:\ Lx =\ N_\lambda x\}$

$$\left(w(t)\phi_p\left(x^{(n-1)}(t)\right)\right)' = \lambda h\left(t,\ x(t),\ x'(t),\cdots,\ x^{(n-1)}(t)\right) \tag{2.23}$$

Therefore

$$\begin{aligned}R(x,\lambda)(t) &= \frac{1}{(n-2)!}\int_0^t (t-s)^{n-2}\phi_q\left(\frac{1}{w(s)}\right)\phi_q\left(\int_0^s (I-Q)N_\lambda x(\tau)d\tau\right)ds - \frac{1}{(n-2)!}\int_0^\eta (\eta-s)^{n-2}\phi_q\left(\frac{1}{w(s)}\right)\phi_q\left(\int_0^s (I-Q)N_\lambda x(\tau)d\tau\right)ds\\
&= \frac{1}{(n-2)!}\int_0^t (t-s)^{n-2}\phi_q\left(\frac{1}{w(s)}\right)\phi_q\left(\int_0^s \left(\mathrm{w}(\tau)\phi_p\left(x^{(n-1)}(\tau)\right)\right)'d\tau\right)ds - \frac{1}{(n-2)!}\int_0^\eta (\eta-s)^{n-2}\phi_q\left(\frac{1}{w(s)}\right)\phi_q\left(\int_0^s \left(w(\tau)\phi_p\left(x^{(n-1)}(\tau)\right)\right)'d\tau\right)ds\\
&= \frac{1}{(n-2)!}\int_0^t (t-s)^{n-2}x^{(n-1)}(s)ds - \frac{1}{(n-2)!}\int_0^\eta (\eta-s)^{n-2}x^{(n-1)}(s)ds\\
&= x(t) - \left(x(0) + x'(0)t + \cdots + \frac{x(n-2)(0)}{(n-2)!}t^{n-2}\right) - x(\eta) + x(0) + x'(0)\eta + \cdots + \frac{x^{(n-2)}(0)}{(n-2)!}\eta^{n-2}\end{aligned}$$

From the boundary conditions $x^{(n-1)}(0) = x^{(i)}(0) = 0,\ \ i = 1,\ 2,\cdots,\ n-3$ we obtain

$$R(x,\lambda)(t) = x(t) - \frac{x^{(n-2)}(0)}{(n-2)!}t^{n-2} - x(\eta) + \frac{x^{(n-2)}(0)}{(n-2)!}\eta^{n-2}$$

and from $x^{(n-2)}(0) = \frac{(n-2)!}{\eta^{n-2}}x(\eta)$ we derive

$$R(x,\ \lambda)(t) = x(t) - \frac{x^{(n-2)}(0)}{(n-2)!}\ t^{n-2} = (I-\ P)x(t). \tag{2.24}$$

Also

$$\begin{aligned}L[P+R(x,\lambda)](t) &= \left\{w(t)\phi_p\left[\frac{x^{(n-2)}(0)}{(n-2)!}t^{n-2} + \frac{1}{(n-2)!}\int_0^t (t-s)^{n-2}\phi_q\left(\frac{1}{w(s)}\right)\phi_q\left(\int_0^s (I-Q)N_\lambda x(\tau)d\tau\right)ds - \frac{1}{(n-2)!}\int_0^\eta (\eta-s)^{n-2}\phi_q\left(\frac{1}{w(s)}\right)\phi_q\left(\int_0^s N_\lambda(I-Q)N_\lambda x(\tau)d\tau\right)ds\right]^{(n-1)}\right\}'\\
&= \left[w(t)\phi_p\left(\phi_q\left(\frac{1}{w(s)}\right)\right)\phi_q\left(\int_0^t (I-Q)N_\lambda x(\tau)d\tau\right)\right]'\\
&= [(I-Q)N_\lambda x](t)\end{aligned} \tag{2.25}$$

This verifies (2.4) and (2.5). Next, we show that $R(x,\lambda)$ is compact. Let $r = \sup\{||x||:x\in\Omega\}$. Since $h:[0,\infty)\times\mathbb{R}^n\to\mathbb{R}$ is L^1 Caratheodory there exists $\psi_r\in L^1[0,\infty)$ such that for $x\in\Omega$ and a. e. $t\in[0,\ \infty)$

$$\left|h\left(t,\ x(t),\ x'(t),\cdots,\ x^{(n-1)}(t)\right)\right| \le \psi_r(t). \tag{2.26}$$

$$\begin{aligned}&|QN_\lambda\ x|\\
&\le re^{-t}\left[\int_0^\infty \phi_q\left(\frac{1}{\omega(s)}\right)\phi_q\left(\int_0^s \lambda\left|h\left(\tau,\ x(\tau),\ x'(\tau),\ \cdots,\ x^{(n-1)}(\tau)\right)d\tau\right|\right)ds + \int_0^\eta\int_0^t \phi_q\left(\frac{1}{\omega(s)}\right)\phi_q\left(\int_0^s \lambda\left|h\left(\tau,\ x(\tau),\ x'(\tau),\cdots,\ x^{(n-1)}(\tau)\right)d\tau\right|\right)ds\ dA(t)\right]\\
&\le re^{-t}\left[\left|\left|\phi_q\left(\frac{1}{w}\right)\right|_1\phi_q(||\phi_r||_1) + \left|\left|\phi_q\left(\frac{1}{w}\right)\right|\right|_1\phi_q(||\psi_r||_1)\ (A(\eta) - A(0))\right]\right.\\
&\le (r+1)\left|\left|\phi_q\left(\frac{1}{w}\right)\right|\right|_1\phi_q(||\psi_r||_1).\end{aligned}$$

Thus

$$||QN_\lambda||_1 \le (r\ +\ 1)\left|\left|\phi_q\left(\frac{1}{w}\right)\right|\right|_1\phi_q(||\psi_r||_1). \tag{2.27}$$

For $x\in\Omega$ and setting

$$A_n = \max_{0\le i\le n-2}\left(sup_{t\in[0,\infty)}\ e^{-t}t^{n-2-i}\right). \tag{2.28}$$

We have $$e^{-t}|R(x,\lambda)(t)| \le e^{-t}\left[\int_0^t (t-s)^{n-2}\phi_q\left(\frac{1}{\omega}\right)\times \phi_q\left(\int_0^s\left|(I-Q)\lambda h\left(\tau, x(\tau), x'(\tau), \cdots, x^{(n-1)}(\tau)\right)d\tau\right|\right)ds\right]$$

$$+\int_0^\eta (\eta-s)^{n-2}\phi_q\left(\frac{1}{\omega(s)}\right)\phi_q\left(\int_0^s\left|(I-Q)\lambda\, h\left(\tau, x(\tau), x'(\tau), \cdots, x^{n-1}(\tau)\right)d\tau\right|\right)ds\Bigg]$$

$$\le Sup_{t\in[0,\infty)}e^{-t}t^{n-2}\left\|\phi_q\left(\frac{1}{\omega}\right)\right\|_1\phi_q(||\psi_r||_1+||QN_\lambda||_1)$$

$$+\,sup_{t\in[0,\infty)}e^{-t}t^{n-2}\left\|\phi_q\left(\frac{1}{\omega}\right)\right\|_1\phi_q(||\psi_r||_1+||QN_\lambda||_1)$$

$$=2\,sup_{t\in[0,\infty)}e^{-t}t^{n-2}\left[\left(\left\|\phi_q\left(\frac{1}{\omega}\right)\right\|_1\right)\phi_q(||\psi_r||_1+||QN_\lambda||_1)\right]\le 2A_n\left[\left(\left\|\phi_q\left(\frac{1}{\omega}\right)\right\|_1\right)\phi_q(||\psi_r||_1+||QN_\lambda||_1)\right].$$

For $1 \le i \le n-2$ we have

$$e^{-t}\left|R(x,\ \lambda)^{(i)}(t)\right|$$
$$=e^{-t}\left|\frac{1}{(n-2-i)}\int_0^t (t-s)^{n-2-i}\phi_q\left(\frac{1}{\omega(s)}\right)\phi_q\left(\int_0^s (I-Q)N_\lambda x(\tau)d\tau\right)ds\right|$$
$$\le \max_{0\le i\le n-2} sup_{t\in[0,\infty)}\ e^{-t}t^{n-2-i}\left||\phi_q\left(\frac{1}{\omega}\right)|\right|_1\phi_q(||\psi_r||_1+\ ||\ Q\ N_\lambda||_1)$$
$$\le A_n\left||\phi_q\left(\frac{1}{\omega}\right)|\right|_1\phi_q(||\psi_r||_1+\ ||\ Q\ N_\lambda||_1).$$

For $i = n-1$ we have

$$e^{-t}\left|R(x,\lambda)^{(n-1)}(t)\right| \le \left||\phi_q\left(\frac{1}{\omega}\right)|\right|_\infty\phi_q(||\psi_r||_1+\ ||\ Q\ N_\lambda||_1).$$

Therefore

$$||R(x,\ \lambda)||\le max\left(2A_n\left\|\phi_q\left(\frac{1}{w}\right)\right\|_1,\ \left\|\phi_q\left(\frac{1}{w}\right)\right\|_\infty\right)\phi_q(||\psi_r||_1+\ ||\ Q\ N_\lambda||_1)$$
$$=\ L_n. \tag{2.29}$$

Thus $R(.\ ,\ \lambda)$ is uniformly bounded in X.

For any $t_1,\ t_2\in[0,\ D],\ D\in(0,\infty),\ \ t_1<\ t_2$ we have

$$|e^{-t_2}R(x,\ \lambda)(t_2)\ -\ e^{-t_1}\ R(x,\lambda)(t_1)\ |$$
$$=\left|\int_{t_1}^{t_2}[e^{-\tau}R(x,\ \lambda)(\tau)]'d\tau\right|$$
$$=\left|\int_{t_1}^{t_2}[-e^{-\tau}R'(x,\ \lambda)(\tau)]'d\tau\ +\ \int_{t_1}^{t_2}[e^{-\tau}R'(x,\ \lambda)(\tau)]'d\tau\right|$$
$$\le 2\ (t_2-\ t_1)||R(x,\ \lambda)||\to\ 0\ as\ t_1\to\ t_2.$$

For $1\le i\le n-2$ we have

$$\left|e^{-t_2}R^{(i)}(x,\ \lambda)(t_2)\ -\ e^{-t_1}R^{(i)}(x,\lambda)(t_1)\ \right|$$
$$=\left|\int_{t_1}^{t_2}[e^{-s}R^{(i)}(x,\ \lambda)(s)]'ds\right|$$
$$=\left|\int_{t_1}^{t_2}\left[-e^{-s}R^{(i)}(x,\ \lambda)(s)\ +\ e^{-s}R^{(i+1)}(x,\ \lambda)(s)\right]ds\right|$$
$$\le 2\ (t_2-\ t_1)||R(x,\ \lambda)||\to\ 0\ as\ t_1\to t_2.$$

For $i = n-1$ we have

$$\left|e^{-t_2}\phi_pR^{(n-1)}(x,\ \lambda)(t_2)-\ e^{-t_1}\phi_p\ R^{(n-1)}(x,\ \lambda)(t_1)\right.$$

$$=\left|\frac{e^{-t_2}}{\omega(t_2)}\int_0^{t_2}(I-Q)N_\lambda x(\tau)d\tau-\frac{e^{-t_1}}{\omega(t_1)}\int_0^{t_1}(I-Q)N_\lambda x(\tau)d\tau\right|$$
$$\le\left|\ \frac{e^{-t_2}}{\omega(t_2)}\ -\frac{e^{-t_1}}{\omega(t_1)}\right|\int_0^{t_2}|(I-Q)N_\lambda x(\tau)|d\tau\ +\frac{e^{-t_1}}{\omega(t_1)}\ \left|\int_{t_1}^{t_2}|(I-Q)N_\lambda x(\tau)|d\tau\right|$$
$$\le\left\|\frac{1}{\omega}\right\|_\infty^2|\omega(t_1)e^{-t_2}-\ \omega(t_2)e^{-t_1}|\int_0^{t_2}(|\psi_r(s)|\ +\ |QN_\lambda(s)|)ds$$
$$+\left\|\frac{1}{\omega}\right\|_\infty\int_{t_1}^{t_2}(|\psi_r(s)|\ +\ |QN_\lambda(s)|)ds$$
$$\le\left\|\frac{1}{\omega}\right\|_\infty^2\left|\omega(t_1)e^{-t_2}-\omega(t_2)e^{-t_1}\right|\left(\left||\psi_r\right||_1+\left||QN_\lambda\right||_1\right)$$
$$\left.+\left\|\frac{1}{\omega}\right\|_\infty\int_{t_1}^{t_2}(|\psi_r(s)|\ +\ |QN_\lambda(s)|)ds\to\ 0\ as\ \ t_1\to t_2\right|.$$

This shows that

$$\left|e^{-t_2}R^{(n-1)}(x,\lambda)(t_2)\ -\ e^{-t_1}\ R^{(n-1)}(x,\ \lambda)(t_1)\ \right|\ \to 0\ \ as\ \ t_1\to t_2.$$

Thus $R(x,\ \lambda)$ is equicontinuous on every compact subset of $[0,\ \infty)$. Next we show that $R(x,\ \lambda)(\bar{\Omega})$ is equiconvergent at infinity. We have

$$e^{-t}|\ R(x,\ \lambda)(t)|\le 2e^{-t}t^{n-1}\left[\phi_q(||\psi_r||_1+\ |QN_\lambda||_1)\left(\left\|\ \phi_q\left(\frac{1}{\omega}\right)\right\|_1\right]\right.$$
$$\to 0\ as\ \ t\ \to\infty.$$

For $1\le i\le n-2$ we have

$$e^{-t}|\ R^{(i)}(x,\ \lambda)(t)|\le e^{-t}t^{n-2-i}\left\|\ \phi_q\left(\frac{1}{\omega}\right)\right\|_1\phi_q(||\psi_r||_1\ +\ ||QN_\lambda||_1)$$

$\to 0 \ as \ t \to \infty.$

For $i = n-1$ we have

$$e^{-t}\left|R^{(n-1)}(x,\lambda)(t)\right| \le e^{-t}\left\|\phi_q\left(\frac{1}{\omega}\right)\right\|_\infty \phi_q[\|\psi_r\|_1 + \|QN_\lambda\|_1] \to 0 \ as \ t \to \infty.$$

Thus $R(x, \lambda)(\varpi)$ is equiconvergent at infinity. Hence from the above computations $R(x, \lambda)$ is compact. The continuity of $R(x, \lambda)$ follows from the Lebesgue dominated convergence theorem. Thus N_λ is L-compact in ϖ.

3. Existence result

In what follows we shall assume the following conditions

(H_1) There exist positive functions $a_i \in L^1[0, \infty)$ $i = 0, 1, 2, \cdots, n-1$ with

$$2^{q-2}A_n\left\|\phi_q\left(\frac{1}{\omega}\right)\right\|_1\left(\sum_{i=1}^{n-1}\|a_i\|_1\right)^{q-1} < 1 \ if \ p \le 2, \tag{3.1}$$

or

$$A_n\left\|\phi_q\left(\frac{1}{\omega}\right)\right\|_1\left(\sum_{i=1}^{n-1}\|a_i\|_1\right)^{q-1} < 1 \ if \ p > 2, \tag{3.2}$$

such that

$$|h(t, x_1, x_2, \cdots, x_n| \le a_0(t) + \sum_{i=1}^{n} a_i\phi_p(e^{-t}|x_i|), \ x \in \mathbb{R}. \tag{3.3}$$

(H_2) There exists a constant $M_n > 0$ such that for $|x^{(n-2)}(t)| > M_n$, for all $t \in [0, \infty)$ we have $QN_\lambda x \neq 0$.

(H_3) $r = \int_0^\infty \phi_q\left(\frac{1}{w(s)}\right)\phi_q\left(\int_0^s e^{-\tau}d\tau\right)ds - \int_0^\eta\int_0^t \phi_q\left(\frac{1}{w(s)}\right)\phi_q\left(\int_0^s e^{-\tau}d\tau\right)ds\, dA(t) \neq 0.$

(H_4) There exists constant $M_n^* > 0$ such that for $x(t) = ct^{n-2} \in kerL \ with \ |c| > \frac{M_n^*}{(n-2)!}$ we have either

$$cQN_\lambda x > 0 \tag{3.4}$$

or

$$cQN_\lambda x < 0. \tag{3.5}$$

Theorem 3.1: If (H_1)- (H_4) is satisfied then the boundary value problem 1.1), 1.2) has at least one solution.

Proof: Our objective here is to construct an open bounded set $V \subset X$ that satisfies the assumptions of theorem 2.1. Let

$$V_1 = \{x \in domL \ : \ Lx = N_\lambda x, \ \lambda \in (0, 1]\}. \tag{3.6}$$

For $x \in V_1$, $x \notin ker\ L$. Hence $N_\lambda x \in Im\ L = ker\ Q$. Therefore $Q\ N_\lambda x = 0$ and by (H_2) there exists $t_0 \in [0, \infty)$ such that

$$\left|x^{(n-2)}(t_0)\right| \le M_n. \tag{3.7}$$

$$\begin{aligned}\left|x^{(n-2)}(0)\right| &= \left|x^{(n-2)}(t_0) - \int_0^{t_0} x^{(n-1)}(s)ds\right| \\ &\le M_n + \|x^{(n-1)}\|_1.\end{aligned} \tag{3.8}$$

For $x \in V_1$, $(I - P)\ x \in dom\ L \cap kerP$. Therefore from (2.24) and (2.29) we obtain

$$\|(I - P)\ x\| = \|R(x, \lambda)\| \le L_n. \tag{3.9}$$

From the definition of the projection P we derive

$$(Px)^{(i)}(t) = \frac{x^{(n-2)}(0)}{(n-2-i)!}t^{n-2-i}, \ i = 0, 1, 2, \cdots, n-2, \ t \in (0, \infty). \tag{3.10}$$

$$\begin{aligned}\|Px\| &= \max_{0\le i\le n-2}\frac{x^{(n-1)}(0)}{(n-2-i)!}sup_{t\in[0,\infty)}e^{-t}t^{n-2-i} \\ &\le \left|x^{(n-2)}(0)\right|\max_{0\le i\le n-2}sup_{t\in[0,\infty)}e^{-t}t^{n-2-i} \\ &\le A_n\left|x^{(n-2)}(0)\right| \\ &\le A_n\left[M_n + \|x^{(n-1)}\|_1\right] \\ &= A_n\|x^{(n-1)}\|_1 + A_nM_n.\end{aligned} \tag{3.11}$$

$$\begin{aligned}\|x\| &= \|Px + (I - P)\ x\| \le \|Px\| + \|(I-P)x\| \\ &\le A_n\|x^{(n-1)}\|_1 + A_nM_n + L_n = A_n\|x^{(n-1)}\|_1 + B_n,\end{aligned} \tag{3.12}$$

where $B_n = A_nM_n + L_n$.

If $p \le 2$ then from 2.6), 2.16) and (3.3) we obtain

$$\begin{aligned}&\|x^{(n-1)}\|_1 \\ &= \int_0^\infty\left|\phi_q\left(\frac{1}{\omega(s)}\right)\phi_q\left(\int_0^s \lambda h(\tau, x(\tau), x'(\tau), \cdots, x^{(n-1)}](\tau)d\tau\right)\right|ds \\ &\le \left\|\phi_q\left(\frac{1}{\omega}\right)\right\|_1\phi_q\left[\|a_0\|_1 + \sum_{i=1}^n\|a_i\|_1\phi_p(\|x\|)\right] \\ &\le \left\|\phi_q\left(\frac{1}{\omega}\right)\right\|_1 2^{q-2}\left[\|a_0\|_1^{q-1} + \left(\sum_{i=1}^n\|a_i\|_1\right)^{q-1}(\|x\|)\right].\end{aligned}$$

From (3.12) we get

$$\|x^{(n-1)}\|_1 \le \left\|\phi_q\left(\frac{1}{\omega}\right)\right\|_1 2^{q-2}\left[\|a_0\|_1^{q-1} + \left(\sum_{i=1}^n\|a_i\|_1\right)^{q-1}(A_n\|x^{(n-1)}\|_1 + B_n)\right]$$

or

$$\begin{aligned}&\left(1 - 2^{q-2}\left\|\phi_q\left(\frac{1}{\omega}\right)\right\|_1\left(\sum_{i=1}^n\|a_i\|_1\right)^{q-1}A_n\right)\|x^{(n-1)}\|_1 \\ &\le \left\|\phi_q\left(\frac{1}{\omega}\right)\right\|_1 2^{q-2}\left[\|a_0\|_1^{q-1} + B_n\left(\sum_{i=1}^n\|a_i\|_1\right)^{q-1}\right].\end{aligned}$$

$$\|x^{(n-1)}\|_1 \le \frac{2^{q-2}\left\|\phi_q\left(\frac{1}{\phi}\right)\right\|_1\left[\left\|a_0\right\|^{q-1} + B_n\left(\sum_{i=1}^n\|a_i\|_1)^{q-1}\right]}{1 - 2^{q-2}\left\|\phi_q\left(\frac{1}{\omega}\right)\right\|_1\left(\left(\sum_{i=1}^n\|a_i\|_1)^{q-1}\right)A_n}. \tag{3.13}$$

Therefore using (3(3.12) and (3.1)(3.1) we conclude that there exists $D_n > 0$ such that

$$\|x\| < D_n. \tag{3.14}$$

Similarly if $p > 2$

$$\|x^{(n-1)}\|_1 \le \frac{\left\|\phi_q\left(\frac{1}{\omega}\right)\right\|_1\left[\left\|a_0\right\|^{q-1} + B_n\left(\sum_{i=1}^n\|a_i\|_1)^{q-1}\right]}{1 - \left\|\phi_q\left(\frac{1}{\omega}\right)\right\|_1\left(\sum_{i=1}^n\|a_i\|_1)^{q-1}A_n}. \tag{3.15}$$

Again from (3(3.12) and (3.2)(3.2) we obtain $||x|| < D_n^*$ for some $D_n^* > 0$.

Therefore V_1 is bounded.

Let $V_2 = \{x \in kerL : N_\lambda x \in ImL\}$.

For $x \in V_2$, $x(t) = ct^{n-2}$, $c \in R$, $t \in (0, \infty)$

$$N_\lambda x \in ImL \text{ implies } N_\lambda x \in \ker Q \text{ and hence } QN_\lambda x = 0. \tag{3.16}$$

From (H_4) we obtain

$$|c| < \frac{M_n^*}{(n-2)!}. \tag{3.17}$$

Therefore for $x \in V_2$,

$$||x|| = |c| \max_{0 \le i \le n-2} \left(sup_{t\in[0,\infty)} e^{-t}\left(t^{(n-2)(i)}\right) \le M_n^* A_n. \tag{3.18}$$

Thus V_2 is bounded.

We choose M_0 large enough such that

$$V = \{x \in V : ||x|| < M_0\} \supset \overline{V_1} \cup \overline{V_2}$$

Therefore from the above computations

$$Lx \ne N_\lambda x \text{ for } x \in \partial V \cap domL.$$

Thus the first part of theorem 2.1 is verified.

$$\text{Let } H(x, \lambda) = \lambda x + (1-\lambda)\, JQN\, x,\ \lambda \in [0, 1]. \tag{3.19}$$

Where $J : ImQ \to kerL$ is the homeomorphism defined by

$$J(\rho c) = ct^{n-2}. \tag{3.20}$$

Then for $x \in \partial V \cap kerL$, $x = ct^{n-2} \ne 0 H(x, 1) = \lambda c\, t^{n-2} \ne 0$ and $H(x, 0) = JQNx \ne 0$, since $Nx \notin Im\, L$.

Therefore for $\lambda = 0$ or $\lambda = 1$,$H(x, \lambda) \ne 0$. Let $0 < \lambda < 1$. Suppose $H(x, \lambda) = 0$ then from (3.19) and (3.4) we have

$$-c^2 = c\frac{(1-\lambda)}{\lambda} QN\left(ct^{n-2}\right) > 0,$$

which is a contradiction. Similarly using

$H(x, \lambda) = -\lambda x + (1-\lambda)\, JQNx = 0$ and (3.5) we obtain the contradiction

$$c^2 = \frac{1-\lambda}{\lambda}\, c\, QN\left(ct^{n-2}\right) < 0.$$

Thus $H(x, \lambda) \ne 0$ for $x \in \partial V \cap kerL$, $\lambda \in [0, 1]$. Therefore by the invariance of the degree under a homotopy we derive

$$deg(JQN,\ V \cap \ker L,\ 0) = deg\ (H(.,\ 1),\ V \cap kerL,\ 0)$$

$$= deg\ (H(.,\ 0),\ V \cap kerL,\ 0)$$

$$= deg\ (\pm 1,\ V \cap kerL,\ 0) \ne 0.$$

Therefore we conclude from theorem 2.1 that 1.1), 1.2) has at least one solution.

4. Example

We consider the following boundary value problem

$$\left(w(t)\phi_p(x'''(t))\right)' = 3 + e^{-t(p-1)}\left[\frac{|x(t)|^2}{4(1+t)^2} + \frac{|x'(t)|^2}{8(1+t)^3} + \frac{|x''(t)|^2}{16(1+t)^4} + |\cos t||x'''(t)|^2\right] \tag{4.1}$$

$$x''(0) = 2x(1),\ x'''(0) = x'(0) = x(0) = 0,\ \ x''(\infty) = \int_0^1 x''(s)ds\ . \tag{4.2}$$

Here

$$\omega(t) = e^t,\ t \in [0, \infty),\ p = \frac{3}{2},\ q = 3,\ \eta = 1,\ A(s) = s,\ n = 4.$$

$$h(t, x, x', x'', x''') = 3 + e^{-\frac{t}{2}}\left[\frac{|x(t)|^2}{4(1+t)^2} + \frac{|x'(t)|^2}{8(1+t)^3} + \frac{|x''(t)|^2}{16(1+t)^4} + |\cos t|\ |x'''(t)|^2\right]$$

$$a_1(t) = \frac{1}{4(1+t)^2},\ a_2(t) = \frac{1}{8(1+t)^3},\ a_3(t) = \frac{1}{16(1+t)^4}.$$

$$||a_1||_1 = \frac{1}{4},\ ||a_2||_1 = \frac{1}{16},\ ||a_3||_1 = \frac{1}{48}.$$

$$\begin{aligned} A_n &= \max_{0\le i\le n-1}\left(sup_{t\in[0,\infty)} e^{-t} t^{n-2-i}\right) \\ &= \max_{0\le i\le n-1}\left(sup_{t\in[0,\infty)} e^{-t} t^{2-i}\right. \\ &= max_{t\in[0,\infty)}\left[t^2 e^{-t},\ t\, e^{-t},\ e^{-t}\right] = 1. \end{aligned}$$

$$\sum_{i=1}^{n-1} ||a_i||_1 = \sum_{i=1}^{3} ||a_i||_1 = \left(\frac{1}{4} + \frac{1}{16} + \frac{1}{48}\right) = \frac{1}{3}.$$

$$\left|\phi_q\left(\frac{1}{\omega}\right)|\right|_1 = \int_0^\infty e^{-2t}dt = \frac{1}{2}.$$

Thus,

$$2^{q-2}A_n\left|\phi_q\left(\frac{1}{\omega}\right)|\right|_1\left(\sum_{i=1}^{n-1}||a_i||_1\right)^{q-1} = 2\ A_n\left|\phi_q\left(\frac{1}{\omega}\right)|\right|_1\left(\sum_{i=1}^{3}||a_i||_1\right)^2$$

$$= 2 \times 1 \times \frac{1}{2} \times \left(\frac{1}{3}\right)^2 = \frac{1}{9} < 1.$$

and

$$|h(t, x, x', x'', x'''| \le 3 + e^{-\frac{t}{2}}\left[\frac{|x(t)|}{4(1+t)^2} + \frac{|x'(t)|}{8(1+t)^3} + \frac{|x''(t)|}{16(1+t)^4} + |x'''(t)|\right].$$

This verifies H_1.

$$\begin{aligned} r &= \int_0^\infty w_q\left(\frac{1}{w(s)}\right)\phi_q\left(\int_0^s e^{-\tau}d\tau\right)ds - \int_0^\eta\int_0^t \phi_q\left(\frac{1}{w(s)}\right)\phi_q\left(\int_0^s e^{-\tau}d\tau\right)dsdt \\ &= \int_0^1\int_0^\infty \phi_q\left(\frac{1}{w(s)}\right)\phi_q\left(\int_0^s e^{-\tau}d\tau\right)dsdt - \int_0^1\int_0^t \phi_q\left(\frac{1}{w(s)}\right)\phi_q\left(\int_0^s e^{-\tau}d\tau\right)dsdt \\ &= \int_0^1\int_t^\infty \phi_q\left(\frac{1}{w(s)}\right)\phi_q\left(\int_0^s e^{-\tau}d\tau\right)dsdt > 0. \end{aligned}$$

$$\begin{aligned} QN_\lambda x &= \rho\left[\int_0^\infty \phi_q\left(\frac{1}{w(s)}\right)\phi_q\left(\int_0^s N_\lambda(\tau)d\tau\right)ds\right. \\ &- \int_0^1\int_0^t \phi_q\left(\frac{1}{w(s)}\right)\phi_q\left(\int_0^s N_\lambda(\tau)d\tau\right)dsdt \\ &= \rho\int_0^1\int_t^\infty \phi_q\left(\frac{1}{w(s)}\right)\phi_q\left(\int_0^s N_\lambda(\tau)d\tau\right)dsdt \end{aligned}$$

Since $h(t, x, y, z, \omega) > 0$ for all $(t, x, y, z, \omega) \in [0, \infty) \times \mathbb{R}^4$ and $\rho(t) = re^{-t} > 0$ then $QN_\lambda x(t) \neq 0$ on $[0, \infty)$. This verifies H_2 and H_3.To verify H_4,

Let $x(t) = ct^{n-2} = c\,t^2 \in ker\, L$. Then,

$$c\,QN_\lambda x(t) = c\int_0^1 \int_t^\infty e^{-2s}\left[\int_0^s \left(3 + e^{-\frac{\tau}{2}}\left(\frac{|c\tau|^2}{4(1+\tau)^2} + \frac{|2c\tau|^2}{8(1+\tau)^3} + \frac{|2c\tau|^2}{16(1+\tau)^4}\right)d\tau\, ds\, dt\right]^2.$$

If $c > 1$ then,

$$cQN_\lambda x \geq c\int_0^1 \int_t^\infty 9\, s^2 e^{-2s}\, ds\, dt > 0,$$

and if $c < -1$ then,

$$cQN_\lambda x < c\int_0^1 \int_t^\infty 9\, s^2 e^{-2s}\, ds\, dt < 0.$$

Therefore assumptions (3.4) or (3.5) are satisfied respectively if

$$|c| > \frac{M_n^*}{(n-2)!} = \frac{2}{2!} = 1.$$

Thus all the assumptions of theorem 3.1 are satisfied. Hence example 4.1–4.2 has at least one solution.

Declarations

Author contribution statement

S. A. Iyase: Conceived and designed the experiments; Analyzed and interpreted the data; Wrote the paper..

K. S. Eke: Analyzed and interpreted the data.

Funding statement

This work was supported by Covenant University.

Competing interest statement

The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

Acknowledgements

The authors are grateful to Covenant University for the financial support in carrying out this research.

References

[1] R.P. Agarwal, D.O. O'Regan, Singular problems on the infinite interval modelling phenomena in draining flows, IMA J. Appl. Math. 66 (2001) 621–635.

[2] R.P. Agarwal, Boundary Value Problem for Higher Order Differential Equations, World Scientific, Singapore, 1986.

[3] K.S. Eke, H. Akewe, S.A. Bishop, On random fixed point theorem with applications to integral equations,, Heliyon (2019).

[4] A. Frioui, A. Guesane-Lakoud, R. Khaldi, Higher order boundary value problems at resonance on an unbounded interval, Eletr. J. Diff. Equ. 29 (2016) 1–10.

[5] Y. Gao, M. Pei, Solvability for two classes of higher order multipoint boundary value problems at resonance, Bound. Value Probl. 2008 (2008) 1–14. Article ID 723828.

[6] W. Ge, J. Ren, An extension of Mawhin's continuation theorem and its application to boundary value problems with a p- Laplacian, Nonlinear Anal. 58 (2004) 477–488.

[7] S.A. Iyase, J.O. Adeleke, On some higher order boundary value problems at resonance with integral boundary conditions, Arab J. Math. Sci. 24 (2) (2018) 225–234.

[8] W. Jang, N. Kosmatov, Resonance P- Laplacian problems with functional boundary conditions, Bound. Value Probl. 2018 (2018) 72.

[9] X. Lin, Q. Zhang, Existence of solutions for a p-Laplacian multipoint boundary value problem at resonance, Qual. Theory Dynamic. Syst. 17 (2018) 143–154.

[10] Y. Liu, W. Ge, Solvability of multipoint boundary value problems for higher order differential equations, Electron. J. Differ. Equ. 120 (2003) 1–19.

[11] J. Mawhin, Topological degree methods in onolinear boundary value problems. NSF-CBMS Regional Conference Series in Math, Am. Math. Soc. Provid. 40 (1979).

[12] X. Ni, W. Ge, Multipoint boundary value problems for the p- Laplacian at resonance, Electron J. Diff. Equ. (2003) 1–7.

[13] H. Pang, N. Ge, M. Tian, Solvability of nonlocal boundary value proplem for ordinary differential equation of higher order with a p-Laplacian, Comput. Math. Appl. 56 (2008) 127–142.

[14] H.I. Royden, Real Analysis, third ed., Prentice Hall, Englewood cliffs, 1988.

[15] Y. Wang, G. Zhang, Multipoint boundary value problems for one dimensional p-Laplacian at resonance, J. Appl. Math. Comput. 22 (2006) 361–372.

[16] A. Yang, G. Miao, W. Ge, Solvability for second order nonlocal boundary value problems with a p-Laplacian at resonance on half-line, Electron. J. Qual. Theor. Differ. Equ. 19 (2009) 1–15.