Research article

Stability of well-posed stochastic evolution equation

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A B S T R A C T

The stability of a non-classical stochastic evolution equation with impulsive and nonlocal initial conditions is examined from a general perspective. We investigate the effect of the nonlocal conditions on the well-posedness and the stability of the solution. We show that the concept of Ulam-type stability holds for this class of equation under the given condition $(1 - M L_{\eta\xi}) < 1$. An example is also provided to support our claim.

1. Introduction

The progression in time of many physical problems is mostly defined through a system of differential equations which can also be rewritten as a Cauchy problem. The interest in studying well-posedness of these problems together with the properties of their solution is on the increase. One of the major analytical difficulties in the theory of classical and quantum stochastic differential equations arise whenever the coefficients driving the equation also consist of unbounded operators, a requirement that is largely unavoidable when dealing with differential equations. Ulam-Hyers (or Ulam-Hyers-Rissias) stability has been used extensively to study stability and has found applications in real life problems such as in economics, biology, population dynamics, etc. that deal with both linear and nonlinear systems. See [1, 2, 3] and the references therein. We consider the following impulsive quantum stochastic differential equation introduced by Bishop et al. [4];

$$d\phi(t) = A(t)\phi(t) + U(t,\phi(t))d\wedge_\pi(t) + V(t,\phi(t))dA_f^+(t)$$
$$+ W(t,\phi(t))dA_g(t) + Z(t,\phi(t))dt$$
$$t \in I = [0,T] \subseteq \mathbb{R}_+, t \neq t_k, k = 1,...,m$$
$$\triangle\phi(t_k) = J_k(\phi(t_k)), t \in t_k$$
$$\phi(0) = \phi_0 - g(\phi), t \in [0,T]. \tag{1.1}$$

Here $J_k \in C(\tilde{\mathcal{B}}, \tilde{\mathcal{B}})$, A is the infinitesimal generator of a family of semi-group defined in [4] while g is a continuous function.

$\phi(t_k^+) = \lim_{\epsilon\to 0^+} \phi(t_k + \epsilon)$ and $\phi(t_k^-) = \lim_{\epsilon\to 0^-} \phi(t_k + \epsilon)$ are the right and left limits of $\phi(t)$ at $t = t_k$ while $\Delta\phi(t_k) = \phi(t_k^+) - \phi(t_k^-)$ is the jump in

the state ϕ at t_k and the coefficients U, V, W, Z are stochastic processes defined in [4, 5, 6]. Associated with equation (1.1) is the following nonclassical stochastic differential equation (NSDE) with impulse effect:

$$\frac{d}{dt}\langle\eta,\phi(t)\xi\rangle = A(t)\phi(t)(\eta,\xi) + P(t,\phi(t))(\eta,\xi), t \in I, t \neq t_k, k = 1,...,m$$
$$\Delta\phi(t_k) = J_k(\phi(t_k^-)), t \in t_k, k = 1,...,m$$
$$\phi(0) = \phi_0 - g(\phi), t \in [0,T], \tag{1.2}$$

where $(t,\phi) \to P(t,\phi)(\eta,\xi)$ is a sesquilinear-form valued stochastic process well defined in [4]. $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ ($\mathbb{D}$ is a pre-Hilbert space and $\mathbb{E}$ is a linear space of exponential vectors). For details of these equations and how they are connected, we refer the reader to [4] and the references therein. Existence of solution of (1.1) using the equivalent nonclassical ordinary differential equation (1.2) with initial condition was first considered by Ogundiran and Payne [7]. Later Bishop and Oguntunde [6] considered a weaker form of [7], where the evolution operator $A = 0$. In [4], the existence of solution of (1.1) was established with nonlocal conditions that are completely continuous. Several results on the qualitative and topological properties of solutions of (1.1) with and without the impulse effect have been studied [5, 6, 8, 9, 10]. In the case of ordinary differential equations (ODEs), functional ordinary differential equations (FODEs), etc. with impulse effect, local initial and nonlocal initial conditions, so much have been done on the existence of solutions and stability of solutions of these types of equations, see [3, 7, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21] and the references therein.

In [7, 21], the authors established Ulam stability of impulsive differential equations with local and nonlocal initial conditions respectively. Ulam-Hyers-Rassias stability was also considered by Liao and Wang

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[15]. Several other studies in literature show the use of this concept of stability [3, 7, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. Recently Ali et al. [25] studied Hyers–Ulam type stability of solutions of a coupled system of implicit type impulsive boundary value problems of fractional ODEs under some strict conditions. Within the context of (1.1), no study has been done on Ulam's type of stability.

We study Ulam, Ulam-Hyers, Ulam-Hyers-Rassias types of stability for (1.1). Here the impulsive conditions are combinations of the nonlocal problem and the short term perturbations. We rely on the structure of the topological space and additional properties to establish the main result.

2. Preliminaries

We state the following definitions and notations of some spaces.

1. $\tilde{B}$ is a topological vector space, where $\text{clos}(\tilde{B})$ denotes its nonempty closed subset.
2. $sesq(\mathbb{D}\otimes\mathbb{E})$ is a complex space of sesquilinear-form valued stochastic processes.
3. $\text{PC}(I,\tilde{B})$, $PC^1(I,\tilde{B})$, $\text{PC}(I,sesq(\mathbb{D}\otimes\mathbb{E}))$ are Banach spaces with the usual supremum norm defined in [2, 12].

Let $I_0 = [0,t_1], I_1 = (t_1, t_2], ..., I_m = (t_m, T]$, and $I_k = (t_k, t_{k+1}]$, where $k = 1, ..., m, t_0 = 0$.

Definition 2.1. A stochastic process $z \in PC(I,\tilde{B})$ is said to be a solution of (1.1) if it satisfies

$$z(0) = \phi_0 - g(z)$$

$$\Delta z(t_k) = J_k(z(t_k^-))$$

$$z(t) = S(t)[z_0 - g(z)] + \int_0^t S(t-s)(U(s,z(s))d\wedge_\pi(s)$$

$$+ V(s,z(s))dA_f^+(s) + W(s,z(s))dA_g(s) + Z(s,z(s))ds), \ t \in [0,t_1]$$

$$+ \sum_{0<t_k<t} S(t-s)J_k(z(t_k^-)), \ k = 1, ..., m. \tag{2.1}$$

Notation 2.1. Denote Ulam, Ulam-Hyers and Ulam-Hyers-Rassias by U, U-H and U-H-R respectively.

Next, we introduce the concept of Ulam stability within the context of this paper. We state the following useful inequalities. Let $\delta > 0, \Psi \geq 0$ and $\varphi \in PC(I,\tilde{A})$ be a non-decreasing function. Then

$$|\frac{d}{dt}\langle\eta,\phi(t)\xi\rangle - A(t)\phi(t)(\eta,\xi) - P(t,\phi(t)(\eta,\xi)| \leq \delta, t \neq t_k$$

$$|\triangle\phi(t_k)(\eta,\xi) - J_k(\phi(t_k^-))(\eta,\xi) + g(t)| \leq \delta, \ k = 1, ..., m \tag{2.2}$$

$$|\frac{d}{dt}\langle\eta,\phi(t)\xi\rangle - A(t)\phi(t)(\eta,\xi) - P(t,\phi(t)(\eta,\xi)| \leq \varphi_{\eta\xi}(t)$$

$$|\triangle\phi(t_k)(\eta,\xi) - J_k(\phi(t_k^-))(\eta,\xi) + g(t)| \leq \Psi_{\eta\xi} \tag{2.3}$$

$$|\frac{d}{dt}\langle\eta,\phi(t)\xi\rangle - A(t)\phi(t)(\eta,\xi) - P(t,\phi(t)(\eta,\xi)| \leq \delta\varphi_{\eta\xi}(t)$$

$$|\triangle\phi(t_k)(\eta,\xi) - J_k(\phi(t_k^-))(\eta,\xi) + g(t)| \leq \delta\Psi_{\eta\xi}. \tag{2.4}$$

Subsequently, $t \in I$, $\eta, \xi \in (\mathbb{D}\otimes\mathbb{E}))$ and $k = 1, ..., m$ except otherwise stated.

Definition 2.2.

(i) (1.1) is U-H stable if $\exists$ a real number $c_{p,l_k,L_{\eta\xi},m,M} > 0$ such that for each $\delta > 0$ and for each solution $\phi \in PC^1(I,\tilde{B})$ of (2.2) there exists a solution $y \in PC^1(I,\tilde{B})$ of (1.1) with

$$||\phi(t) - y(t)||_{\eta\xi} \leq c_{p,l_k,L_{\eta\xi},m,M}\delta.$$

(ii) (1.1) is generalized U-H stable if $\exists$ $\theta_{p,l_k,L_{\eta\xi},m,M} \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\theta_{p_{l_k},L_{\eta\xi},M}(0) = 0$, so that for a solution $\phi \in PC^1(I,\tilde{B})$ of (2.2) we obtain

$$||\phi(t) - y(t)||_{\eta\xi} \leq \theta_{p,l_k,L_{\eta\xi},m,M}(\delta).$$

(iii) (1.1) is U-H-R stable with respect to $(\varphi_{\eta\xi}(t), \Psi_{\eta\xi})$ if we can find $c_{l_k,L_{\eta\xi},m,M,\varphi_{\eta\xi}} > 0$ and a $\delta > 0$ such that for each solution $y \in PC^1(I,\tilde{B})$ of (2.4) a solution $\phi \in PC^1(I,\tilde{B})$ of (1.2) exists with

$$||\phi(t) - y(t)||_{\eta\xi} \leq c_{l_k,L_{\eta\xi},m,M}\delta(\varphi_{\eta\xi})(\varphi_{\eta\xi} + \Psi_{\eta\xi}).$$

(iv) (1.1) is generalized U-H-R stable with respect to $(\varphi_{\eta\xi}, \Psi_{\eta\xi})$ if $c_{P,l_k,L_{\eta\xi},m,M,\varphi_{\eta\xi}} > 0$ exists such that for each solution $\phi \in PC^1(I,\tilde{B})$ of (2.3) we find a solution $y \in PC^1(I,\tilde{B})$ of (1.1) with

$$||\phi(t) - y(t)||_{\eta\xi} \leq c_{p,l_k,L_{\eta\xi},m,M,\varphi_{\eta\xi}}(\varphi_{\eta\xi}(t) + \Psi_{\eta\xi}).$$

Definition 2.3. Let $q \in PC(I,\tilde{A})$ and $q_k, k = 1, ..., \infty$, be a sequence which also depends on η, ξ such that the following holds:

(i) $|q(t)| \leq M\delta, \quad t \in I$, and $|q_k| < \delta$;
(ii) $d\phi(t) = A(t)\phi(t) + U(t,\phi(t))d\wedge_\pi(t) + V(t,\phi(t))dA_f^+(t) + W(t,\phi(t))dA_g(t) + Z(t,\phi(t))dt + q(t), t \in I', I' = I \setminus [t,t_m]$;
(iii) $\Delta\phi(t_k) = J_k(\phi(t_k^-)) + q_k$.
Then $\phi \in PC^1(I,\tilde{B})$ is a solution of (2.2).

Similar definitions can be obtained for (2.3) and (2.4).

By Definition 2.1, if $\phi \in PC^1(I,\tilde{B})$ is a solution of inequality (2.2), then ϕ is a solution of the following:

$$||\phi(t)||_{\eta\xi} - ||S(t-s)[\phi_0 - g(\phi)]$$

$$+ \int_0^t S(t-s)(P(s,\phi(s))ds + \sum_{i=j}^k J_i(\phi(t_j^-)))||_{\eta\xi}$$

$$\leq M(m+t)\delta. \tag{2.5}$$

So that by Definition 2.3 (ii)-(iii), we obtain

$$||\phi(t)||_{\eta\xi} = ||S(t-s)[\phi_0 - g(\phi)] + \int_0^t S(t-s)(P(s,\phi(s))ds + \int_0^t q(s)ds$$

$$+ \sum_{i=j}^k J_i(\phi(t_j^-)) + \sum_{j=1}^k q_j)||_{\eta\xi}, \quad t \in (t_k, t_{k+1}],$$

from which we get

$$||\phi(t) - S(t-s)[\phi_0 - g(\phi)] + \int_0^t S(t-s)P(s,\phi(s))ds + \sum_{j=1}^k J_j(\phi(t_j))||_{\eta\xi}$$

$$\leq M\sum_{j=1}^m |q_j| + M\int_0^t |q(s)|ds$$

$$\leq Mm\delta + Mt\delta = M(m+t)\delta.$$

Repeating the process for (2.3) and (2.4), we obtain similar results with respect to $\varphi_{\eta\xi}(t)$ and $\Psi_{\eta\xi}$.

3. Main results

The following assumptions will be used to establish the main results: Let $K_{\eta\xi}$, l_k, M, and $L_{\eta\xi}$ be positive constants.

A_1. If $(t,\phi) \to P(t,x)(\eta,\xi)$ is continuous, for $\phi, y \in \tilde{B}$ we get

$$||P(t,\phi) - P(t,y)||_{\eta\xi} \leq K_{\eta\xi}^P ||\phi - y||_{\eta\xi}.$$

A_2. For $J_k : \tilde{\mathcal{B}} \to \tilde{\mathcal{B}}$, we have

$$||J_k(\phi) - J_k(y)||_{\eta\xi} \le l_k ||\phi - y||_{\eta\xi}, \phi \in \tilde{\mathcal{B}}.$$

A_3. Let $t \ge 0$, then $||S(t)||_{\eta\xi} \le M$.

A_4. For $g : \tilde{\mathcal{B}} \to PC(I, sesq(\mathbb{D} \underline{\otimes} \mathbb{E}))$ we obtain

$$||g(\phi) - g(y)||_{\eta\xi} \le L_{\eta\xi} ||\phi - y||_{\eta\xi}, \quad \phi, y \in \tilde{\mathcal{B}}.$$

A_5. For each $t \in I$, let the function $\varphi_{\eta\xi} \in C(I, \mathbb{R}_+)$ be non decreasing. Then

$$\int_0^t \varphi_{\eta\xi}(s) ds \le c_\varphi \varphi(t), \, c_\varphi > 0.$$

Theorem 3.1. *Let the conditions $A_1 - A_5$, and H_4 in [4] be satisfied. Then (1.2) respectively (1.1) is generalized U-H-R stable with respect to $(\varphi_{\eta\xi}(t), \Psi_{\eta\xi})$ provided $(1 - M L_{\eta\xi}) < 1$.*

Proof. Let $\phi \in PC^1(I, \tilde{\mathcal{B}})$ be a solution of (2.3) and y a unique solution of (1.1) where $\phi(0) = y(0)$ and $\phi_0 = y_0$. Then we obtain

$$\phi(t) = S(t-s)[y(0) + g(\phi)] + \int_0^t S(t-s)[U(s, \phi(s))d\wedge_\pi(s)$$
$$+ V(s, \phi(s)dA_f^+(s) + W(s, \phi(s))dA_g(s) + Z(s, \phi(s))ds], \quad t \in (0, t_1],$$

$$= S(t-s)([y(0) + g(\phi)] + J_1(\phi(t_1^-))) + \int_0^t S(t-s)[U(s, \phi(s))d\wedge_\pi(s)$$
$$+ V(s, \phi(s)dA_f^+(s) + W(s, \phi(s))dA_g(s) + Z(s, \phi(s))ds], \quad t \in (t_1, t_2],$$

$$= S(t-s)([(y(0) + g(\phi)] + J_1(\phi(t_1^-)) + J_2(\phi(t_2^-)))$$
$$+ \int_0^t S(t-s)[U(s, \phi(s))d\wedge_\pi(s) + V(s, \phi(s))dA_f^+(s)$$
$$+ W(s, \phi(s))dA_g(s) + Z(s, \phi(s))ds], \quad t \in (t_2, t_3],$$

$$\vdots$$

$$= S(t-s)([y(0) + g(\phi)) + \sum_{k=1}^m J_k(\phi(t_k)))$$
$$+ \int_0^t S(t-s)[U(s, \phi(s))d\wedge_\pi(s) + V(s, \phi(s))dA_f^+(s)$$
$$+ W(s, \phi(s))dA_g(s) + Z(s, \phi(s))ds], \quad t \in (t_m, T].$$

Similarly, by definition (2.1), we obtain for each $t \in (t_k, t_{k+1}]$,

$$||y(t)||_{\eta\xi} - ||S(t-s)[y(0) + g(\phi)] - \sum_{j=1}^k S(t-s)J_j(y(t_j^-))||_{\eta\xi}$$

$$- ||\int_0^t S(t-s)P(s, y(s))ds||_{\eta\xi}$$
$$\le M(m + c_\varphi)(\varphi_{\eta\xi}(t) + \Psi_{\eta\xi}), \quad t \in I.$$

Therefore, for each $t \in (t_k, t_{k+1}]$, we get the following:

$$||y(t) - \phi(t)||_{\eta\xi} \le ||y(t) - S(t-s)[(y(0) - \sum_{j=1}^k J_j(y(t_j)) - \int_0^t P(s, \phi(s))ds||_{\eta\xi}$$

$$+ \sum_{j=1}^k S(t-s)||J_j(\phi(t_j)) - J_j(y(t_j))||_{\eta\xi}$$
$$+ S(t-s)[||g(\phi) - g(y)||_{\eta\xi}$$

$$+ \int_0^t ||P(s, \phi(s)) - P(s, y(s))||_{\eta\xi} ds]$$

$$\le M(m + c_\varphi)(\varphi_{\eta\xi}(t) + \Psi_{\eta\xi}) + M \sum_{j=1}^k l_j ||\phi(t_j) - y(t_j)||_{\eta\xi}$$

$$+ M L_{\eta\xi} ||\phi(t) - y(t)||_{\eta\xi} + M \int_0^t K_{\eta\xi} ||\phi(s) - y(s)||_{\eta\xi} ds.$$

$$(3.1)$$

Case 1: When $K_{\eta\xi}^P(t)$ is a function.

Since the map $s \to ||\phi_1 - \phi_0||_{\eta\xi}$ is continuous for $\phi \in \tilde{\mathcal{B}}$, $t \in [t_0, T]$, we let

$$R_{\eta\xi} = \sup_{t \in [0,T]} ||\phi_1 - \phi_0||_{\eta\xi}$$

and

$$N_{\eta\xi}(t) = \int_0^t K_{\eta\xi}^P(s) ds.$$

Hence, from (3.1), we obtain

$$||y(t) - \phi(t)||_{\eta\xi} \le M(m + c_\varphi)(\varphi_{\eta\xi}(t) + \Psi_{\eta\xi})$$

$$+ M \sum_{j=1}^k l_j ||\phi(t_j) - y(t_j)||_{\eta\xi}$$

$$+ M L_{\eta\xi} R_{\eta\xi} + M R_{\eta\xi} N_{\eta\xi}(t)$$

$$\le M(m + c_\varphi)(\varphi_{\eta\xi}(t) + \Psi_{\eta\xi})$$

$$+ M c_k R_{\eta\xi} + M R_{\eta\xi}[L_{\eta\xi} + N_{\eta\xi}(t)]$$

$$\le M(m + c_\varphi)(\varphi_{\eta\xi}(t) + \Psi_{\eta\xi}) + M \sum_{j=1}^k l_j ||\phi(t_j) - y(t_j)||_{\eta\xi}$$

$$+ M R_{\eta\xi}[L_{\eta\xi} + N_{\eta\xi}(t)].$$

Obviously, Lemma 2.1 in [23] cannot be applied directly in this case. For more on the Lipschitz function $K_{\eta\xi}^P(t)$, see the references [4, 5, 6, 7, 8].

Case 2: When $K_{\eta\xi}^P > 0$ is a constant independent of t.

Again, subtracting $M L_{\eta\xi} || \phi(t) - y(t) ||_{\eta\xi}$ from both sides of (3.1) yields:

$$(1 - M L_{\eta\xi}) || \phi(t) - y(t) ||_{\eta\xi} \le M(m + c_\varphi)(\varphi_{\eta\xi}(t) + \Psi_{\eta\xi})$$

$$+ M K_{\eta\xi} \int_0^t || \phi(s) - y(s) ||_{\eta\xi} ds$$

$$+ M \sum_{j=1}^k l_j || \phi(t_j) - y(t_j) ||_{\eta\xi}.$$

By applying Lemma 2.1 in [23], we obtain

$$|| \phi(t) - y(t) ||_{\eta\xi} \le M(m + c_\varphi)(\varphi_{\eta\xi}(t) + \Psi_{\eta\xi}) \left(\prod_{0 < t_k < t} (1 + l_k) e^{K_{\eta\xi}^P t} \right)$$

$$\le c_{p, l_k, L_{\eta\xi}, m, M, \varphi_{\eta\xi}} (\varphi_{\eta\xi}(t) + \Psi_{\eta\xi}), \quad t \in I,$$

where $c_{p, l_k, L_{\eta\xi}, m, M, \varphi_{\eta\xi}} := M(m + c_\varphi) \prod_{k=1}^m (1 + l_k) e^{K_{\eta\xi}^P T} > 0$ (3.2)

and $(1 - M L_{\eta\xi}) < 1$. This completes the proof.

4. Example

Let

$$P(t, \phi(t))(\eta, \xi) = (-\tfrac{1}{2}\phi(t))(\eta, \xi), t \in I = [0, 1], \tag{4.1}$$

and

$$A\phi(t)(\eta, \xi) = \phi(t)(\eta, \xi), t \in [0, 1]. \tag{4.2}$$

Also Let

$$\|S(t)\|_{\eta\xi} \le 1 = M, t \ge 0, \tag{4.3}$$

$$\phi(0) + g(\phi) = \phi(0) + \sum_{i=1}^{n} \kappa_i \phi(t_i) = 1, \tag{4.4}$$

where $0 < t_1 < t_2 < ... < t_n < 1, \kappa_i > 0, i = 1, ..., n > 0, m = 1, t_1 = 1/2$ and set $\sum_{i=1}^{n} \kappa_i \le 1/3$.

Set

$$\Delta\phi(t_1) = J_1(\phi(t_i)) = c\phi(\tfrac{1}{2}^{-}),$$

and

$$\langle \eta, \phi(0)\xi \rangle = \langle \eta, \xi \rangle = e^{\langle \eta, \xi \rangle} = 1.$$

Then we can have the following nonclassical evolution problem:

$$\frac{d}{dt}\langle \eta, \phi(t)\xi \rangle = \phi(t)(\eta, \xi) + (-\tfrac{1}{2}\phi(t))(\eta, \phi\xi), t \in [0, 1], t \ne \tfrac{1}{2},$$

$$\Delta\phi(t_k) = c\phi(\tfrac{1}{2}^{-}), t \in [0, 1]$$

$$\phi_0 = \phi(0) + \sum_{i=1}^{n} \kappa_i \phi(t_i) = 1. \tag{4.5}$$

We state that hypothesis H_4 in [4] guarantees existence of solution of (4.5). Obviously, the map P defined by (4.1) is continuous and hence, A_1 holds. By setting $c_\varphi = 1/2$, A_2 holds with $l_k = 1/2$. Also A_3 holds by (4.3) and A_4 holds by (4.4) so that $L_{\eta\xi} = 1/3$.

For A_5, assume that $\varphi_{\eta\xi}(t) = e^{2t}$ and $\Psi_{\eta\xi} = 1$. Then, if $c_\varphi = \tfrac{1}{2}$, we get the required result, where $K_{\eta\xi}^{P} = \tfrac{1}{2}$. Thus A_5 holds. Also the claim that $(1 - ML_{\eta\xi}) < 1$, holds. That is $(1 - ML_{\eta\xi}) = \tfrac{2}{3} < 1$.

Lastly, we obtain $\|\phi(t) - y(t)\|_{\eta\xi} \le 9/4(e^{2t} + 1)e^{\frac{1}{2}t}$ by (3.2) and by setting $\langle \eta, \xi \rangle = e^{\langle \eta, \xi \rangle} = 1$. Thus, from Theorem 3.1, it follows that the problem (4.5) is generalized U-H-R stable with respect to $(e^{2t}, 1)$.

5. Conclusion

The Ulam-Hyers stability considered here involves a function say $y(t)$ which can closely solve (1.2) provided one can find an exact solution $\phi(t)$ of the given equation which is close to $y(t)$. Hence, by Definition 2.2 (i), (1.2) is U-H stable if it has an exact solution and if there is a $\delta > 0$ such that if $\phi(t)a$ is an approximation for the solution of (1.2) then there is an exact solution $\phi(t)$ of (1.2) which is close to ϕ. This means that in application, if one is studying stability problem of this type, one does not have to attain the exact solution (which is quite difficult in this case). What is required is to obtain a function which satisfies Definition 2.2 and U-H stability guarantees that there is a solution close to the exact solution. This is quite useful in many applications, e.g. numerical analysis, optimization, biology, economics, etc., where finding the exact solution is quite difficult. It also helps, if the stochastic effects are small, to use deterministic model to approximate a stochastic one. It is pertinent to note that U-H stability considered here is independent of the conversant Lyapunov stability which states that (1.2) is Lyapunov stable if both $\phi(t)$ and $y(t)$ are exact solutions of (1.2), see [1, 2, 3] and the references therein. This explanation can be repeated for U-H-R stability and the result follows for the Generalized U-H-R stability by using (ii)-(iii) in Definition 2.2 and inequality (2.3). This implies that (1.2) is generalized U-H-R stable with respect to $(\varphi_{\eta\xi}, \Psi_{\eta\xi})$ provided $(1 - ML_{\eta\xi}) < 1$ and $K_{\eta\xi}^{P}$ (Lipschitz function) is a constant.

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