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Resonant mixed fractional-order p-Laplacian boundary value problem on the half-line

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https://doi.org/10.1515/msds-2020-0141Received August 3, 2021; accepted November 15, 2021Abstract:This study aims at establishing the solvability of a fractional-orderp-Laplacian boundary valueproblem involving both the left Caputo and right Riemann-Liouville fractional derivatives on the half-line. Inorder to overcome the nonlinearity of the fractional dierential operator, we apply the Ge and Ren coincidencedegree theorem to obtain existence results for the boundary value problem at resonance. An example is givento demonstrate the established results.Keywords:Coincidence degree, half-line, fractional derivative, p-Laplacian, resonance.MSC:70K30; 34B10; 34B15; 34A08

1 Introduction

In this paper, we obtain existence results for the following fractional-order p-Laplacian boundary value prob-lem at resonance on the half-line with nonlocal boundary conditions$D^{a}_{-}\varphi_p(D^{b}_{0+}u(t)) = e^{-t}w(t,u(t),D^{b}_{0+}u(t)), t\in(0,\infty)$,(1)$I^{1-b}_{0+}u(0) = 0, \varphi_p(D^{b}_{0+}u(\infty)) = \varphi_p(D^{b}_{0+}u(0))$,(2)where$D^{a}_{-}$is the left Caputo fractional derivative on the half line and$D^{b}_{0+}$the right Riemann-Louville fractionalderivative on the half-line,$0<a,b\le1, 1<a+b\le2, \varphi_p(r) = |r|^{p-2}, p\ne2$, with$\varphi_q=\varphi^{-1}_{p}and1/q+1/p= 1$.$w: [0,+\infty)\times\mathbb{R}^2\to\mathbb{R}$is a continuous function.Recently, fractional dierential equations have become the focus of many researches mainly due to theprogress made in the development of the theory of fractional calculus and due to its widespread applica-tion in engineering and science like in signal processing, viscoelasticity, bioengineering, uid dynamics [16].Fractional order derivatives have useful tools for solving integral and dierential equations and can handlemodels more accurately than integer order derivatives [4]. Fractional order derivatives contain a memory termwhich enables it to describe the memory and hereditary properties of various processes and materials [18].The presence of thep-Laplacian operator on a boundary value problem causes the fractional dierentialopertator to become nonlinear. Boundary value problems involveingp-Laplacian operator occur in combus-tion theory, nonlinear elasticity, population biology, glaciology, non-Newtonian mechancis, plasma physicsand the study of drain ows; see [4, 13]. When$p= 2$the fractional dierential operator becomes linear.Existence results have been been obtained by dierent researchers for fractional order boundary valueproblems using dierent methods. When the corresponding homogenous problem of the fractional dieren-tial operator has a trivial solution, xed point methods are applied; see [1, 6, 7, 11, 12, 17]. For the case where

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the corresponding homogenous fractional dierential equation has a non-tricial solution and the fractionaldierential operator is nonlinear as a result of the presence of the p-Laplacian operator, the Ge and Ren co-incidence degree theorem [5] is used to establish existence results; see [9, 19–21]The associated homogeneous problem of boundary value problem (1)-(2):$D^{a}_{-}\varphi_p(D^{b}_{0+}u(t)) = 0, t\in(0,\infty), I^{1-b}_{0+}u(0) = 0, \varphi_p(D^{b}_{0+}u(\infty)) = \varphi_p(D^{b}_{0+}u(0))$,has s non-trivial solution$u(t) = dt^{b}, d\in\mathbb{R}$, hence (1)-(2) is a resonance problem.In [4], the authors considered the following fractionalp-Laplacian boundary value problem at resonancewhen$p= 2$$D^{\beta}_{0+}\varphi_p(D^{\alpha}_{0+}x(t)) = f(t,x(t),D^{\alpha}_{0+}x(t)), t\in[0,1]$subject to the boundary conditions$D^{\alpha}_{0+}x(0) = D^{\alpha}_{0+}x(1) = 0$,where$0<\alpha,\beta\le1, 1<\alpha+\beta\le2, \varphi_p(s) = |s|^{p-2}s, p>1$,$D^{\alpha}_{0+}$is a Caputo fractional derivative, and$f: [0,1]\times\mathbb{R}^2\to\mathbb{R}$is a continuous function.Bai and Zhang [3] obtained existence of solution for the following three point boundary value problemof fractional dierential equation with nonlinear growth at resonance by applying Coincidence degree$D^{\alpha}_{0+}u(t) = f(t,u(t),D^{\alpha-1}_{0+}u(t)), 0<t<1, u(0) = 0, u(1) = \sigma u(\eta)$,where$1<\alpha\le2$,$D^{\alpha}_{0+}$is the Riemann-Liouville derivative,$f: [0,1]\times\mathbb{R}^2\to\mathbb{R}$is continuous,$\sigma\in(0,\infty), \eta\in(0,1)$and the resonance condition is$\sigma\eta^{\alpha-1}$.In [21], the authors studied the following fractional order boundary value problem$\{D^{a}_{0+}u(t) = f(t,u(t),D^{a-1}_{0+}u(t)), t\in(0,\infty), u(0) = 0, \lim_{t\to\infty}D_{0+}u(t) = \beta u(\eta)$,where$0<a,b\le1, 1<a+b\le2, p= 2$,$f$is a continuous function,$D^{a}_{0+}$and$D^{b}_{0+}$are Caputo fractionalderivatives.In [10], Imaga and Iyase considered a second-order p-Laplacian boundary value problem. at resonance:$(\varphi_p(u'(t)))'+g(t,u(t),u'(t)) = 0, t\in(0,+\infty), \varphi_p(u'(0)) = \int_0^{+\infty}v(t)\varphi_p(u'(t))dt, \varphi_p(u'(+\infty)) = \sum_{j=1}^{m}\beta_j\int_0^{\eta_j}\varphi_p(u'(t))dt$,where$g: [0,+\infty)\times\mathbb{R}^2\to\mathbb{R}$is an$L_1$-Caratheodory function,$0<\eta_1<\eta_2<\cdots\le\eta_m<+\infty, \beta_j\in\mathbb{R}, j= 1,2,\ldots,m, v\in L_1[0,\infty), v(t)>0and\varphi_p(s) = |s|^{p-2}, p\ge2$. They used the Ge and Ren continuationtheory to obtain existence results.Also, in [20], the authors used the Ge and Ren continuation theorem to obtain existence result for thefollowing fractionalp-Laplacian problem at resonance$(\varphi_p(D^{\alpha}_{0+}x(t)))'+f(t,x(t),D^{\alpha}_{0+}x(t)) = 0, 0<t<+\infty$,(3)$x(0) = x'(0) = 0, \varphi_p(D^{\alpha}_{0+}x(+\infty)) = \sum_{i=1}^{n}\alpha_i\varphi_p(D^{\alpha}_{0+}x(\xi_i))$,(4)where$1<\alpha\le2$,$D^{\alpha}_{0+}$is the standard Riemann-Liouville fractional derivative,$0<\xi_1<\xi_2<\ldots\xi_n<+\infty, \alpha_i>0, \sum_{i=1}^{n}\alpha_i= 1, \varphi^{-1}_{p}=\varphi_q$with$1/p+ 1/q= 1$.Motivated by the above results, we study the solvability for a resonant fractional-orderp-Laplacianboundary value problem at resonance on the half-line. Though, some researchers have considered mixed

fractional-order boundary value problem; see [2, 8], to the best of our knowledge this is the rst work toconsiderp-Laplacian mixed fractional boundary value problems on the half-line. In Section 2 of this work,required lemmas, theorem and denitions will be presented, Section 3 contains conditions for existence ofsolutions. An example will be given in Section 4 to illustrate the results obtained.

2 Preliminaries

In this section, we will give denitions, lemmas and theorems that will be used in this work.Denition 2.1.([5]) Let $(U,\|\cdot\|_U)$ and $(Z,\|\cdot\|_Z)$ be any two Banach spaces and$M:U\cap\mathrm{dom}M\to Z$a continuous operator. If$M(X\cap\mathrm{dom}M)$is a closed subset of$Z$and$\ker M=\{u\in U\cap\mathrm{dom}M:Mu= 0\}$is linearly homeomorphic to$\mathbb{R}^n, n<\infty$, then,$M$is quasi-linear.Let$U_1= \ker M$andU_2be the complement ofU_1in$U$, such that$U=U_1\oplus U_2$. Also, let$Z_1\subset Z$andZ_2, thecomplement ofZ_1in$Z$such that$Z=Z_1\oplus Z_2$. Similarly, let$E:U\to U, F:Z\to Z$be continuous projectorsand$\Omega\subset U$be open and bounded with the origin$\theta\in\Omega$.Denition 2.2.([5]) Let$N_k:\Omega\to Z, k\in[0,1]$be a continuous operator and let$\nabla_k=\{u\in\Omega:Mu=N_ku\}$.We denote$N_1byN$.$N_k$is said to be$M$-compact in$\Omega$if there exists a vector subspace$Z_1$of$Z$with$\dim Z_1=\dim U_1$and a continuous and compact operator,$P:\Omega\times[0,1]\to U_2$such that, for$k\in[0,1]$$(\rho_1)(I-F)N_k(\Omega)\subset \mathrm{Im}M\subset(I-F)Z$;$(\rho_2)FN_ku=\theta, k\in(0,1)\Leftrightarrow FNu=\theta$;$(\rho_3)P(\cdot,k)$is the zero operator and$P(\cdot,k)|_{\nabla_k}= (I-E)|_{\nabla_k}$;$(\rho_4)M[E+P(\cdot,k)] =$

$(I-F)N_k$.Theorem 2.1.([5]). Let$(U,\|\cdot\|_U)$and$(Z,\|\cdot\|_Z)$be any two Banach spaces and$\Omega\subset U$be bounded, openand nonempty. Assume that$M:U\cap domM\to Z$is a quasi-linear operator and$N_k:\Omega\to Z,k\in[0,1]isM$-compact on$\Omega$. Suppose the following hold:$(\tau_1)Mu\neq N_k u,\forall(u,k)\in\partial\Omega\times(0,1)$,$(\tau_2)FNu\neq 0,\forall u\in kerM\cap\partial\Omega$,$(\tau_3)deg(JFN,kerM\cap\Omega,0)\neq 0$, where$F:Z\to ImF$is a projector and$J:ImF\to kerM$is a homeomorphismwith$J(\theta)=\theta$.Then at least one solution exists for the abstract equation$Mu=Nu$in $domM\cap\Omega$where$N=N_1$.Denition 2.3.[4]. The set$X\subset U$dened by$X=\{u\in C[0,\infty),\lim_{t\to\infty}u(t)exist\}$is relatively compact if$X_1=\{u(t)1+t_{a-1}:u\in X\},X_2=\{D_{a0+}u(t):u\in X\}$are uniformly bounded; equicontinuous on any compact subinterval of$[0,\infty)$and equiconvergent at∞.Denition 2.4.. Let$\alpha>0$, the left-sided Caputo and right-sided Riemann-Liouville fractional integral of afunction$x:(0,\infty)\to\mathbb{R}$is dened by$I_{\alpha-}x(t)=1\Gamma(\alpha)\infty\int t x(r)(r-t)_{1-\alpha}dr,t\in[0,\infty)andI_{\alpha 0+}x(t)=1\Gamma(\alpha)t\int 0x(r)(t-r)_{1-\alpha}dr,t\in[0,\infty)$respectively.

Denition 2.5.Let$\alpha>0$, the left-sided Caputo and right-sided Riemann-Liouville fractional derivative of afunction$x:(0,\infty)\to\mathbb{R}$is dened by$D_{\alpha-}x(t)=(-1)_n\Gamma(n-\alpha)d_ndt_n\infty\int t x(r)(r-t)_{\alpha-n+1}dr,t\in[0,\infty)$,and$D_{\alpha 0+}x(t)=1\Gamma(n-\alpha)d_ndt_nt\int 0x(r)(t-r)_{\alpha-n+1}dr,t\in[0,\infty)$respectively where$n=[a]+1$.Denition 2.6.Let $a>0$, then the Euler Gamma function is given as$\Gamma(a)=\infty\int 0r_{a-1}e_{-r}dr$.Lemma 2.1.[14]. Let$a\in(0,\infty)$. The general solution of the Riemman-Liouville fractional dierential equa-tion$D_{a0+}w(t)=0$is$w(t)=b_1t_{a-1}+b_2t_{a-2}+\cdots+b_nt_{a-n}$, where$b_j\in\mathbb{R},j=1,2...,n$while, the general solution of the Caputofractional dierential equation$D_{a0+}w(t)=0$is$w(t)=d_0+d_1t+\cdots+d_nt_n$, where$d_i\in\mathbb{R},i=0,1,...,nandn=[a]+1$is the smallest integer greaterthan or equal to$a$.Lemma 2.2.[14