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Gradient estimates for bounded solutions of semilinear elliptic equations and the Allen-Cahn equation on manifolds

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Abstract. This paper aims at deriving apriori bounds on the gradient of positve solutions to a class of semilinear elliptic equation, with applications focusing on establishing a liouville type property for the bounded solutions of the Allen-Cahn equation on a complete noncompact Riemannian manifold with nonnegative Ricci curvature.

1. Introduction

The Allen–Cahn equation is a mathematical physics reaction-diffusion equation that explains the steps of phase separation in multicomponent alloy systems, encompassing order-disorder transitions. The equation represents the time evolution of a scalar-valued state variable on a specific domain during such time intervals. The Allen-Cahn equation is a semilinear PDE with a singular limit that is intimately related to the theory of minimal hypersurfaces. A powerful form of the multiplicity one hypothesis and the index lower bound conjecture of Marques-Neves in 3-dimensions about min-max constructions of minimum surfaces in the Allen-Cahn context.

Bounded positive solutions to some semilinear elliptic partial differential equations are considered in this paper. This class of equations arises naturally as a model of problems coming from differential geometry and even mathematical physics. Indeed, the model to be considered reads in general as

$$\Delta w - \mathbf{g}(w) = 0 \text{ in } \mathcal{M}, \tag{1.1}$$

where $\mathcal{M}$ is a complete Riemmannian manifold of dimension $m \geq 2$ with Ricci curvature tensor simply bounded from below. Here Δ is the Laplacian on $\mathcal{M}$, which is a second order partial differential operator intrinsically defined with respect to the

Riemannian metric, the function $\mathbf{g}$ is a first order derivative of another function $\mathbf{G}$ satisfying the properties that $\mathbf{G} : \mathbb{R} \to \mathbb{R}$ is a smooth, real, nonnegative function and $\mathbf{G}(s) = 0$ only in a discrete set of $s \in \mathbb{R}$. Typical examples of this class of equations include the cases (i). $\mathbf{g}(w) = w^3 - w$ for which $\mathbf{G}(w)$ is an equal double well function $\mathbf{G}(w) = 1/4(1 - w^2)^2$, (ii). $\mathbf{g}(w) = \pm \sin w$ for which $\mathbf{G}(w) = \mp \cos w$ and a more general case (iii). $\mathbf{g} = w^q - w^p$, $q > p \geq 1$. The mathematicians' interests in any of these cases are on problems concerning local and global apriori estimates on bounded solutions, Liouville type properties of entire bounded solutions, symmetry properties, asymptotic behaviour of solutions and so on. More specifically in the present paper, apriori gradient estimates are obtained on the bounded positive solutions to (1.1) with $\mathbf{g}(w)$ defined as

$$\mathbf{g}(w) = bw^q - aw^p, \quad a, b > 0, \quad q > p \geq 1,$$

and then recover Liouville type properties for these solutions. Particular applications will be discussed for the case

$$\mathbf{g}(w) = w^3 - w,$$

that is, the elliptic Allen-Chan equation having bounded solutions $0 < w(y) \leq 1$ on a complete noncompact Riemannian manifold.

It is worthy mentioning that the entire solution of Allen-Cahn equation and its connection with minimal surfaces theory and Bernstein conjecture [10] motivated the popular De Giorgi conjecture [16], which states that an entire solution w is one dimensional if it is monotone in one direction.

De Giorgi Conjecture(*[16]*): *Let w be an entire solution to $\Delta w(y)+w(y)-w^3(y) = 0$ in $\mathbb{R}^m$ satisfying $|w(y)| \leq 1$, $\frac{\partial w}{\partial y_m} > 0$ for $y = (y', y_m) \in \mathbb{R}^m$. Then the level sets of w, i.e, the sets $\{w = t, \forall\ t \in \mathbb{R}\}$ are hyperplanes provided $m \leq 8$ atleast.*

This conjecture has been proved for $m = 2$ [21], $m = 3$ [8], $4 \leq m \leq 8$ (with additional condition) [30] and counterexamples constructed [17] for the case $m \geq 9$. Allen-Cahn equation (theory and applications) is well studied in literature, see [13, 18, 20, 24] for examples. It originally arose from the theory of phase separation in iron alloys, including order-disorder transitions [7] while its connection with the theory of minimal hypersurfaces has been greatly epxloited by several authors, see [14, 19, 24] and references therein for instance.

The main goal of the present paper is to prove apriori gradient estimates on bounded positive solutions to a specific class of (1.1) and obtain condition(s) under which the gradient estimate will give rise to a Liouville type result in the case $\mathbf{g}(w) = w^3 - w$ with bounded solutions. By the classical Liouville theorem it is known that any bounded positive solution must be a constant. The results of this paper reiterate further the usefulness of gradient estimates in the analysis of partial differential equations. Gradient estimate was pioneered by [33] (see also [15]). See [11, 22, 25, 26, 32, 34] for the development of the concept on manifolds. Other useful references on solution methods, improvements, extensions and applications of gradient estimates include [1, 2, 3, 4, 5, 6, 27, 28, 29].

Recently, the authors in [9] and [23] have respectively considered Harnack and gradient estimates on parabolic and elliptic Allen-Cahn equations and obtained interesting results. The line of approach in this paper is different from those in [9] and [23], though our results can be compared with that of [23]. The adopted methodology towards obtaining the gradient estimate in this paper follows closely the program introduced by Brighton [11] where the classical Bochner formula will be applied to the power of solution in contrast to Yau's idea [33] of using logarithm of positive solution to derive some initial inequality. The classical Bochner formula [31] states that

$$\frac{1}{2}\Delta(|\nabla h|^2) = |\nabla^2 h|^2 + \langle \nabla h, \nabla \Delta h\rangle + Ric(\nabla h, \nabla h) \tag{1.2}$$

for smooth function h on $\mathcal{M}$. The Hessian term in the Bochner formula which is controlled by the Laplacian, (i.e., $|\nabla^2 h|^2 \geq \frac{1}{m}(\Delta h)^2$), gives rise to some nonlinear terms due to the nonlinear source in (2.1), and when squared the cross term must be controlled by choosing a suitable function of the solution, while the two cases must be examined so as to obtain the desired inequality. The obtained inequality is then multiplied by a smooth cut-off function having 'nice' properties, and then subjected to Laplacian comparison theorem and the maximum principle.

2. Statement of Results and a basic Lemma

Here we consider a more general semilinear equation

$$\Delta w + aw^p - bw^q = 0, \quad y \in \mathcal{M}, \tag{2.1}$$

where a, b are nonnegative constants and $q > p \geq 1$. In the sequel, $\mathcal{M}$ represents a m-dimensional complete Riemannian manifold ($m \geq 2$). For a fixed point $\xi \in \mathcal{M}$, $B_\xi(2R)$ denotes a geodesic ball centered at ξ with radius $2R$, $R \geq 1$. The Ricci tensor of $\mathcal{M}$ is denoted by $Ric(\mathcal{M})$ or by $Ric(B_\xi(2R))$ when restricted to $B_\xi(2R)$. Supposing that solutions to (2.1) satisfy $0 < w \leq D$ for some constant D and scaling $w \mapsto \widetilde{w} := w/D$, then $0 < \widetilde{w} \leq 1$ and $\widetilde{w}$ satisfies

$$\Delta\widetilde{w} + A\widetilde{w}^q - B\widetilde{w}^p = 0$$

with $A = aD^{q-1}$ and $B = dD^{p-1}$. Due to this, the assumption that $0 < w \leq 1$ in in order and perfectly fits into the application to Allen Cahn equation ($p = 1, q = 3, A = 1$ and $B = 1$) that we have in mind.

With the notations: $\mathbf{D}_1 := \inf_{B_\xi(2R)} w$ and $\mathbf{D}_2 := \sup\limits_{B_\xi(2R)} w$, the main results of this paper can be stated as follows.

2.1. Main results

Theorem 2.1. *Let $w(y)$ be a bounded positive solution to (2.1) in $B_\xi(2R)$ with $Ric(B_\xi(2R)) \geq -(m-1)\kappa(2R)$ for some $\kappa(2R) \geq 0$. Then the following estimate holds in $B_\xi(2R)$*

Initial conditions of the model **??** *variables are given as follows:*

$$S(0) > 0,\ V(0) \geq 0,\ I(0) \geq 0,\ T(0) \geq 0,\ R(0) \geq 0,\ P(0) > 0. \tag{2.2}$$

$$\begin{aligned} |\nabla w(y)|^2 \leq & c_m(\mathbf{D}_2)^2\Big(\Big[(m-1)\kappa - [b(\beta+q-1)]\mathbf{D}_1^{q-1} \\ & + [a(\beta+p-1)]\mathbf{D}_2^{p-1}\Big] + \Psi(m, R, \kappa, \theta_1, \theta_2)\Big), \end{aligned} \tag{2.3}$$

where

$$\Psi(m, R, \kappa, \theta_1, \theta_2) = \frac{1}{R^2}\Big[(m-1)\sqrt{\theta_1}(1 + R\sqrt{\kappa}) + \theta_2 + (m+2)\theta_1\Big],$$

θ_1 and θ_2 are positive constants and c_m is a positive constant depending only on m.

The following global estimate immediately follows by passing to the limit as R is sent to infinity.

Corollary 2.2. *Let $w(y)$ be a bounded positive solution to (2.1) in noncompact $\mathcal{M}$ with $Ric(\mathcal{M}) \geq -(m-1)\kappa$ for some $\kappa \geq 0$. Then the following estimate holds:*

$$|\nabla w(y)|^2 \leq c_m \mathbf{D}_2^2 \Big(\Big[(m-1)\kappa - [b(\beta+q-1)]\mathbf{D}_1^{q-1} + [a(\beta+p-1)]\mathbf{D}_2^{p-1} \Big] \Big), \tag{2.4}$$

where $\mathbf{D}_1 := \inf_{\mathcal{M}} w$ and $\mathbf{D}_2 := \sup_{\mathcal{M}} w$ and c_m is the same as in Theorem 2.1.

Remark 2.3. If $a = 0 = b$ in (2.1), then the estimate in (2.3) of Theorem 2.1 reads

$$|\nabla w| \leq \sup_{B_\xi(2R)} \{w\} \sqrt{\frac{c_{1,m}}{R^2} + c_{2,m}\kappa}.$$

Note that the last estimate is a replica of the classical Yau gradient estimate [33] for bounded harmonic function on $B_\xi(R)$ with $Ric \geq -m\kappa$, $\kappa \geq 0$.

However, we recover the following apriori gradient estimate on any bounded solution to the Allen-Cahn equation and consequently the desired Liouville property.

Theorem 2.4. *Let $w(y)$ be a bounded solution to*

$$\Delta w(y) + w(y) - w^3(y) = 0 \quad in \quad \mathcal{M}$$

satisfying $0 < w(y) \leq 1$, where $\mathcal{M}$ is a complete noncompact Riemannian manifold of dimension $m \geq 2$ with $Ric(\mathcal{M}) \geq -(m-1)\kappa, \kappa \geq 0$. Then for $\beta \in (0,1)$

$$|\nabla w| \leq c_m \sqrt{(m-1)\kappa + \beta(1 - \inf_{\mathcal{M}} w^2)}. \tag{2.5}$$

Moreover, if $Ric(\mathcal{M}) \geq 0$, then w is a constant and identically equal to 1.

Lastly in this section, a basic lemma that is essential to the proof of Theorem 2.1 is presented.

2.2. Basic Lemma

Define $G = w^\beta$, where $\beta \in (0,1)$ will be chosen later. Note that by direct computation

$$\frac{|\nabla G|^2}{G^2} = \beta^2 \frac{|\nabla w|^2}{w^2}$$

and

$$\Delta G = \frac{\beta - 1}{\beta} \frac{|\nabla G|^2}{G} - \beta a G^{1+\frac{p-1}{\beta}} + \beta b G^{1+\frac{q-1}{\beta}}. \tag{2.6}$$

Using the Bochner formula (1.2) on G together with the inequality $|\nabla^2 G|^2 \geq \frac{1}{m}(\Delta G)^2$ and the Ricci tensor condition $Ric \geq -(m-1)\kappa$, we have

$$\frac{1}{2}\Delta(|\nabla G|^2) \geq \frac{1}{m}(\Delta G)^2 + \langle \nabla G, \nabla \Delta G \rangle - (m-1)\kappa |\nabla G|^2. \tag{2.7}$$

The first two terms in the RHS of (2.7) can be expressed as follows using (2.6):

$$\begin{aligned}\frac{1}{m}(\Delta G)^2 = \frac{(\beta-1)^2}{m\beta^2}\frac{|\nabla G|^4}{G^2} + \frac{2(\beta-1)}{m\beta}\frac{|\nabla G|^2}{G}\left(\beta b G^{1+\frac{q-1}{\beta}} - \beta a G^{1+\frac{p-1}{\beta}}\right)\\ + \frac{1}{m}\left(\beta b G^{1+\frac{q-1}{\beta}} - \beta a G^{1+\frac{p-1}{\beta}}\right)^2\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\langle \nabla G, \nabla \Delta G\rangle = \frac{\beta-1}{\beta}\frac{\nabla G}{G}\nabla(|\nabla G|^2) - \frac{\beta-1}{\beta}\frac{|\nabla G|^4}{G^2}\\ + \left[b(\beta+q-1)G^{\frac{q-1}{\beta}} - a(\beta+p-1)G^{\frac{p-1}{\beta}}\right]|\nabla G|^2.\end{aligned} \tag{2.9}$$

Substituting (2.8) and (2.9) into (2.7) gives

$$\begin{aligned}\frac{1}{2}\Delta(|\nabla G|^2) \geq \frac{2(\beta-1)}{m\beta}\frac{|\nabla G|^2}{G}\left(\beta b G^{1+\frac{q-1}{\beta}} - \beta a G^{1+\frac{p-1}{\beta}}\right) + \left(\frac{(\beta-1)^2}{m\beta^2} - \frac{\beta-1}{\beta}\right)\frac{|\nabla G|^4}{G^2}\\ + \frac{1}{m}\left(\beta b G^{1+\frac{q-1}{\beta}} - \beta a G^{1+\frac{p-1}{\beta}}\right)^2 + \frac{\beta-1}{\beta}\frac{\nabla G}{G}\nabla(|\nabla G|^2)\\ - \left[(m-1)\kappa - b(\beta+q-1)G^{\frac{q-1}{\beta}} + a(\beta+p-1)G^{\frac{p-1}{\beta}}\right]|\nabla G|^2.\end{aligned} \tag{2.10}$$

There is a need to control the first term in the RHS of (2.10). Indeed, two cases arise:
Case 1: If for any fixed point δ in $B_\xi(2R)$ there exists a positive constant λ such that $\beta b G^{1+\frac{q-1}{\beta}} - \beta a G^{1+\frac{p-1}{\beta}} \leq \lambda\frac{|\nabla G|^2}{G}$. Then

$$\frac{2(\beta-1)}{m\beta}\frac{|\nabla G|^2}{G}\left(\beta b G^{1+\frac{q-1}{\beta}} - \beta a G^{1+\frac{p-1}{\beta}}\right) \geq \frac{2(\beta-1)}{m\beta}\frac{|\nabla G|^2}{G}\left(\lambda\frac{|\nabla G|^2}{G}\right)$$

since $\beta \in (0,1)$, and (2.10) will then imply

$$\begin{aligned}\frac{1}{2}\Delta(|\nabla G|^2) \geq \frac{2(\beta-1)}{m\beta}\frac{|\nabla G|^2}{G}\left(\lambda\frac{|\nabla G|^2}{G}\right) + \left(\frac{(\beta-1)^2}{m\beta^2} - \frac{\beta-1}{\beta}\right)\frac{|\nabla G|^4}{G^2}\\ + \frac{1}{m}\left(\beta b G^{1+\frac{q-1}{\beta}} - \beta a G^{1+\frac{p-1}{\beta}}\right)^2 + \frac{\beta-1}{\beta}\frac{\nabla G}{G}\nabla(|\nabla G|^2)\\ - \left[(m-1)\kappa - b(\beta+q-1)G^{\frac{q-1}{\beta}} + a(\beta+p-1)G^{\frac{p-1}{\beta}}\right]|\nabla G|^2.\end{aligned}$$

Case 2: In contrast to **Case 1** if $\beta b G^{1+\frac{q-1}{\beta}} - \beta a G^{1+\frac{p-1}{\beta}} \geq \lambda\frac{|\nabla G|^2}{G}$ at the point δ in $B_\xi(2R)$. Then

$$\frac{2(\beta-1)}{m\beta}\frac{|\nabla G|^2}{G}\left(\beta b G^{1+\frac{q-1}{\beta}} - \beta a G^{1+\frac{p-1}{\beta}}\right) \geq \frac{2(\beta-1)}{m\beta}\frac{1}{\lambda}\left(\beta b G^{1+\frac{q-1}{\beta}} - \beta a G^{1+\frac{p-1}{\beta}}\right)^2$$

and (2.10) implies

$$\begin{aligned}\frac{1}{2}\Delta(|\nabla G|^2) \geq \left(\frac{(\beta-1)^2}{m\beta^2} - \frac{\beta-1}{\beta}\right)\frac{|\nabla G|^4}{G^2} + \frac{\beta-1}{\beta}\frac{\nabla G}{G}\nabla(|\nabla G|^2)\\ + \left(\frac{2(\beta-1)}{m\beta\lambda} + \frac{1}{m}\right)\left(\beta b G^{1+\frac{q-1}{\beta}} - \beta a G^{1+\frac{p-1}{\beta}}\right)^2\\ - \left[(m-1)\kappa - b(\beta+q-1)G^{\frac{q-1}{\beta}} + a(\beta+p-1)G^{\frac{p-1}{\beta}}\right]|\nabla G|^2.\end{aligned}$$

Given that $\frac{2(\beta-1)}{m\beta\lambda}+\frac{1}{m}\geq 0$, then in the two cases described, (2.10) yields the following inequality

$$\begin{aligned}\frac{1}{2}\Delta(|\nabla G|^2) \geq & \Big(\frac{(\beta-1)^2}{m\beta^2}-\frac{\beta-1}{\beta}+\frac{2\lambda(\beta-1)}{m\beta}\Big)\frac{|\nabla G|^4}{G^2}+\frac{\beta-1}{\beta}\frac{\nabla G}{G}\nabla(|\nabla G|^2)\\ & -\Big[(m-1)\kappa-b(\beta+q-1)G^{\frac{q-1}{\beta}}+a(\beta+p-1)G^{\frac{p-1}{\beta}}\Big]|\nabla G|^2.\end{aligned}$$

By this we have summarized the proof of the below given Lemma.

Lemma 2.5. *Suppose $\mathcal{M}$ is a complete Riemannian manifold of dimension $m\geq 2$. Let $w(y)$ be a positive solution to (1.1) in $B_\xi(2R)$ with $Ric(B_\xi(2R))\geq -(m-1)\kappa, \kappa\geq 0$. For a function $G=w^\beta$, $\beta\in(0,1)$, then there exists a positive constant λ satisfying*

$$\frac{1}{m}+\frac{2(\beta-1)}{m\beta\lambda}\geq 0 \tag{2.11}$$

such that

$$\begin{aligned}\frac{1}{2}\Delta(|\nabla G|^2) \geq & \frac{\beta-1}{\beta}\frac{\nabla G}{G}\nabla(|\nabla G|^2)+\Big(\frac{\beta-1}{m\beta^2}-\frac{\beta-1}{\beta}+\frac{2\lambda(\beta-1)}{m\beta}\Big)\frac{|\nabla G|^4}{G^2}\\ & -\Big[(m-1)\kappa-b(\beta+q-1)G^{\frac{q-1}{\beta}}+a(\beta+p-1)G^{\frac{p-1}{\beta}}\Big]|\nabla G|^2\end{aligned} \tag{2.12}$$

holds on $B_\xi(2R)$.

3. Proof of results

Gradient estimates and Liouville type theorem

3.1. Proof of Theorem 2.1

The estimate (2.12) of Lemma 2.5 will be multiplied by a cut-off function, then the maximum principle combined with Laplacian comparison theorem will be applied. First choose λ in (2.12) such that the coefficient of $\frac{|\nabla G|^4}{G^2}$ is positive. Since $0<\beta<1$ and $\lambda>0$, we can choose $\beta=\frac{4}{m+4}$ and letting $\lambda\to\frac{m}{2}$ so that (2.11) holds and (2.12) becomes

$$\frac{1}{2}\Delta(|\nabla G|^2)\geq\frac{m}{16}\frac{|\nabla G|^4}{G^2}-\frac{m}{4}\frac{\nabla G}{G}\nabla(|\nabla G|^2)-\vartheta_\kappa|\nabla G|^2, \tag{3.1}$$

where

$$\vartheta_\kappa := (m-1)\kappa-[b(\beta+q-1)]\inf_{B_\xi(2R)}\{w^{q-1}\}+[a(\beta+p-1)]\sup_{B_\xi(2R)}\{w^{q-1}\}.$$

Now define a smooth cut-off function σ in $[0,+\infty)$ such that $\sigma(s)=1$ for $s\in[0,R]$, $\sigma(s)=0$ for $s\in[2R,+\infty)$ and $\sigma(s)\in[0,1]$ with the properties

$$0\geq\sigma'(s)\sigma^{-\frac{1}{2}}\geq-\sqrt{\theta_1}\quad\text{and}\quad\sigma''(s)\geq-\theta_2$$

for some constants $\theta_1,\theta_2>0$. For a fixed point $\xi\in\mathcal{M}$, denote by $d(\xi,y)$ the distance function between ξ and y in $\mathcal{M}$. Let

$$\eta(y)=\sigma\Big(\frac{d(\xi,y)}{R}\Big).$$

By Calabi's trick [12] we assume that η is smoothly supported in $B_\xi(2R)$ and clearly by the Laplacian comparison theorem [31] we obtain

$$\frac{|\nabla\eta|^2}{\eta} \leq \frac{\theta_1}{R^2} \quad \text{and} \quad \Delta\eta \geq -\frac{(m-1)\sqrt{\theta_1}(1+R\sqrt{\kappa})+\theta_2}{R^2}.$$

Setting $H = \eta|\nabla G|^2$. We suppose G reaches its maximum at $y_0 \in B_\xi(2R)$ with emphasis that y_0 is not in the cut locus of ξ and assumption that $G(y_0) > 0$. Then at y_0, it holds that $\nabla H = 0$ which implies

$$\nabla(|\nabla G|^2) = -\frac{\nabla\eta}{\eta}|\nabla G|^2, \tag{3.2}$$

and $\Delta H \leq 0$. Thus by (3.2)

$$\begin{aligned} 0 \geq \Delta H &= \eta\Delta(|\nabla G|^2) + |\nabla G|^2\Delta\eta + 2\nabla\eta\nabla(|\nabla G|^2) \\ &= \eta\Delta(|\nabla G|^2) + \Big(\Delta\eta - 2\frac{|\nabla\eta|^2}{\eta}\Big)\frac{H}{\eta}. \end{aligned} \tag{3.3}$$

Combinning (3.1) with (3.3) using the definition $H = \eta|\nabla G|^2$ we have

$$0 \geq \frac{m}{8}\frac{H^2}{\eta G^2} - \frac{m}{2}\frac{\nabla G}{G}\frac{\nabla\eta}{\eta}H - 2\vartheta_\kappa H + \Big(\Delta\eta - 2\frac{|\nabla\eta|^2}{\eta}\Big)\frac{H}{\eta}.$$

Multiplying both sides of the last inequality by $\frac{\eta}{H}$ leads to

$$\frac{m}{8}\frac{H}{G^2} \leq -\frac{m}{2}\frac{\nabla G}{G}\nabla\eta + 2\vartheta_\kappa\eta - \Big(\Delta\eta - 2\frac{|\nabla\eta|^2}{\eta}\Big). \tag{3.4}$$

Clearly for $\mu \in (0,1)$, the Cauchy-Schwarz inequality implies

$$\begin{aligned} -\frac{m}{2}\frac{\nabla G}{G}\nabla\eta &\leq \frac{m}{2}\frac{|\nabla G|}{G}|\nabla\eta| \\ &\frac{m}{4\mu}\frac{|\nabla\eta|^2}{\eta} + \frac{m\mu}{4}\frac{|\nabla G|^2}{G^2}\eta = \frac{m}{4\mu}\frac{|\nabla\eta|^2}{\eta} + \frac{m\mu}{4}\frac{H}{G^2}. \end{aligned} \tag{3.5}$$

Putting (3.5) into (3.4) we have

$$\frac{(1-2\mu)m}{8}\frac{H}{G^2} \leq 2\vartheta_\kappa\eta - \Big(\Delta\eta - \Big(\frac{m+8\mu}{4\mu}\Big)\frac{|\nabla\eta|^2}{\eta}\Big). \tag{3.6}$$

In particular we choose $\mu = \frac{1}{4}$ in (3.6) and obtain

$$\frac{m}{16}\frac{H}{G^2} \leq 2\vartheta_\kappa\eta + \Psi, \tag{3.7}$$

where

$$\Psi = (\theta_1, \theta_2, m, k, R) := \frac{(m-1)\sqrt{\theta_1}(1+R\sqrt{\kappa})+\theta_2}{R^2} + \frac{(m+2)\theta_1}{R^2}.$$

Therefore for $y \in B_{y_0}(2R)$, $R \geq 1$, it follows from (3.7) that

$$\frac{m}{16}H(y) \leq \frac{m}{16}H(y_0) \leq G^2(y_0)(2\vartheta_\kappa + \Psi)\eta.$$

Finally, using the definition $G = w^\beta$, $0 < \beta < 1$ and $H = \eta|\nabla G|^2$ we find

$$H = \eta\beta^2 G^2 \frac{|\nabla w|^2}{w^2}$$

so that

$$\frac{|\nabla w(y)|^2}{w^2(y)} \leq c_{m,\beta}(2\vartheta_\kappa + \Psi),$$

where $c_{m,\beta} > 0$ is a constant which depends on β and m. The expected estimate therefore follows at once.

□

3.2. Proof of Corollary 2.2

We note that Ψ in (2.3) vanishes by letting $R \to +\infty$. Therefore we arrive at (2.4) by passing to the limit on noncompact manifold.

□

3.3. Proof of Theorem 2.4

Recall that $w(y)$ is a bounded solution to

$$\begin{aligned} \Delta w(y) + w(y) - w^3(y) = 0 \\ 0 < w(y) \leq 1 \end{aligned} \tag{3.8}$$

on a complete noncompact $\mathcal{M}$. By this one can apply (2.4) of Corollary 2.2. Now, choosing $p = 1$, $q = 3$, $a = b = 1$ and $\inf_y w = \mathbf{D}_1$ and $\sup_y w = \mathbf{D}_2 = 1$, then

$$\begin{aligned} |\nabla w|^2 &\leq c_m \left[(m-1)\kappa - (\beta+2)(\inf w)^2 + \beta\right] \\ &\leq c_m \left[(m-1)\kappa + \beta(1 - (\inf w)^2)\right], \end{aligned} \tag{3.9}$$

where c_m is a positive constant with dependency only on m, and consequently we obtain (2.5).

Moreover, since $\mathcal{M}$ has nonnegative Ricci curvature, $Ric(\mathcal{M}) \geq 0$, and w is a positive solution satisfying $0 < w \leq 1$, then by (3.9) it follows that $|\nabla w| = 0$ meaning that w is a constant and $w \equiv 1$ identically. This concludes the proof.

□

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References

[1] A. Abolarinwa, *Gradient estimates for a weighted nonlinear elliptic equation and Liouville type theorems*, J. Geom Phy. 155 (2020), 103737.
[2] A. Abolarinwa, *Elliptic gradient estimates and Liouville theorems for a weighted nonlinear parabolic equation*, J. Math. Anal. Appl. 473 (2019), 297–312.
[3] A. Abolarinwa, S. O. Salawu, C. A. Onate, *Gradient estimates for a nonlinear elliptic equation on smooth metric measure spaces and applications*, Heliyon 5(2019).
[4] A. Abolarinwa, *Gradient estimates for a nonlinear elliptic equation on complete noncompact Riemannian manifold*, J. Math. Ineq. 12 (2018), no. 2, 391–402.

[5] A. Abolarinwa, *Differential Harnack and logarithmic Sobolev inequalities along Ricci-harmonic map flow*, Pacific J. Math. 278(2)(2015), 257–290.
[6] A. Abolarinwa, *Gradient estimates for a nonlinear parabolic equation with potential under geometric flow*, Electr. J. Diff. Eqn. (2015)(12) (2015), 1–11.
[7] S. M. Allen, J. W. Cahn, *A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening*, Acta Metall. 27 (1979), 1085–1095.
[8] L. Ambrosio, X. Cabré, *Entire solutions of semilinear elliptic equations in R^3 and a conjecture of De Giorgi*, J. Amer. Math. Soc. 13(4) (2000), 725–739.
[9] M. Băileşteanu, *A Harnack inequality for the parabolic Allen-Cahn equation*, Ann. Glob. Anal. Geom. 51(4),(2017), 367–378.
[10] S. Bernstein, *Sur un theoreme de geometrie et son application aux equations aux derivees partielle dy type elliptique*, Comm. Math Kharkow 15(1915), 38–45.
[11] K. Brighton, *A Liouville-type theorem for smooth metric measure spaces*. J. Geom. Anal. 23 (2013), 562–570.
[12] E. Calabi, *An extension of E. Hopf's maximum principle with application to Riemannian geometry*, Duke Math. J. 25 (1958), 45–46.
[13] L. Calatroni, P Colli, Nonl. Anal. 79 (2013), 12–27.
[14] X. Chen, *Generation and propagation of interfaces for reaction-diffusion equations*, J. Diff. Eq., 96(1), (1992), 116–141.
[15] S. Y. Cheng, S. T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. 28 (3)(1975), 333–354.
[16] E. De-Giorgi, *Convergence problems for functionals and operators*, In Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), 131– 188. Pitagora, Bologna, 1979.
[17] M. Del Pino, M. Kowalczyk, J. Wei, *On De-Giorgi's conjecture in dimension $N \geq 9$*, Ann. Math. 174 (2011), 1485–1569.
[18] M. Del Pino, M. Kowalczyk, F. Pacard, J. Wei, *Multiple solutions to the Allen-Cahn equation in $\mathbb{R}^2$*, J. Funct. Anal. 258 (2010), 458–503
[19] L. C. Evans, H. M. Soner, P. E. Souganidis, Comm. Pure Appl. Math 45 (9) (1992), 1097–1123.
[20] Y. Fukao, Y. Morita, H. Ninomiya, *Some entire solutions of the Allen-Cahn equation*, Taiwan J. Math. 8(2004), 15–32.
[21] N. Ghoussoub, C. Gui, *On a conjecture of De Giorgi and some related problems*, Math. Ann., 311(3) (1998), 481–491.
[22] R. Hamilton, *A matrix Harnack estimate for the heat equation*, Comm. Anal. Geom. 1(1993), 113–126.
[23] S. Hou, *Gradient estimates for the Allen-Cahn equation on Riemannian manifolds*, Proc. Amer. Math. Soc. 147 (2019), 619–628.
[24] T. Ilmanen, *Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature*, J. Diff. Geom 38(2) (1993), 417–461.
[25] J. Li, *Gradient estimates and Harnack inequalities for nonlinear parabolic and nonlinear elliptic equations on Riemannian manifolds*, J. Funct. Anal. 100 (1991) 233–256
[26] P. Li, S-T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. 156 (1986), 153–201.
[27] G.O. Akinlabi, R.B. Adeniyi, E.A. Owoloko, *The solution of boundary value problems with mixed boundary conditions via boundary value methods*, International Journal of Circuits, Systems and Signal Processing, 12, (2018), 1-6.
[28] G.O. Akinlabi, R.B. Adeniyi, *Sixth-order and fourth-order hybrid boundary value methods for systems of boundary value problems*, WSEAS Transactions on Mathematics. 17 (2018), 258-264.
[29] B. Qian, *Yau's gradient estimates for a nonlinear elliptic equation*, Arch. Math. (2016)
[30] O. Savin, *Regularity of flat level sets in phase transitions*, Ann. of Math., 169 (1) (2009), 41–78.
[31] R. Schoen, S.-T. Yau, *Lectures on differential geometry*, International Press, Cambridge, MA, 1994.
[32] P. Souplet, Q. S. Zhang, *Sharp gradient estimate and Yau's Liouville theorem for the heat equation on noncompact manifolds*, Bull. London Math. Soc. 38 (6)(2006), 1045–1053.
[33] S-T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. 28 (1975), 201–228.
[34] S-T. Yau, *Some function-theoretic properties of complete Riemannian manifold and their applications to geometry*, Indiana Univ. Math 25 (1976), 659–670.