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A Note on Analytical Roots of the Navier-Stokes Equation

G. O. Akinlabi[1*] , S. O. Edeki[1*], and J.A. Braimah[2]

[1]Department of Mathematics, Covenant University, Ota, Nigeria
[2]Department of Physical and Computer Sciences, McPherson University PMB 2094 Abeokuta, Nigeria

Contact E-mails: grace.akinlabi@covenantuniversity.edu.ng; soedeki@yahoo.com

Abstract: A new approach called the Generalised Picard Iteration Sch eme (GPIS) is used to solve the Navier-Stokes equations in this paper. The solutions are organized in a series with components that are readily computed. Because it delivers the exact solution to the solved issue with minimal computing effort while retaining a high degree of accuracy, this method appears to be extremely adaptable, efficient, effective, and dependable. It is not necessary to identify Lagrange multipliers. As a result, the presented method is recommended for dealing with higher-order linear and non-linear models.

Keywords: PIM, Iterative Method, Navier-Stokes model, Analytical solutions

1. Introduction

Navier-Stokes equations (NSEs) are important equations in the sketch of motion of viscous fluid substances in the physical sciences, such as engineering, mathematics, computational fluid dynamics, and other fields. NSEs relate the reaction of a fluid flow to pressure and external forces operating on it [1]. In their most basic form, the NSEs are:

$$\frac{\partial w}{\partial t} - P = v\left(\frac{\partial^2 w}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial w}{\partial \eta}\right), \quad w(\eta, 0) = g(\eta), \tag{1.1}$$

where v is the kinematics viscosity, $P = -\frac{\partial P}{\rho \partial z}$, w is the flow velocity, P is the pressure, t is the time.

Some semi-analytical approaches have been used to solve NSEs [2-6]. Analytic and numerical approaches have been developed and used to provide solutions (numerical or exact) to differential equations [7-29]. Our goal in this paper is to present analytical solutions to NSEs using the generalised Picard Iteration Scheme.

2 The Generalised Picard Iteration Scheme

2.1 Lipschitzian Continuity Condition

Let $f(t, y)$ be given function, so $f(t, y)$ satisfies a Lipchitz condition with respect to y in a certain region referred to as D in the XY-plane, if there exists a non-negative constant ζ , such that

$$\left| f(t,y_a) - f(t,y_b) \right| \leq \zeta \left| y_a D\!\!- y_b \right| \quad \zeta$$

whenever (t, y_a) and (t, y_b) are in , and is called the Lipchitz constant.

2.2 Overview of the Successive Iteration Method

Suppose a first-order non-linear ordinary differential equation (ODE) is given as follows with an initial condition:

$$\begin{cases} \dfrac{dy}{dt} = g(t,y), \\ y(t_0) = y_0 \end{cases} \tag{2.1}$$

Suppose $\int_{t_0}^{t} (\cdot)\, dy = I_{t_0}^{t}(\cdot)$ denotes a one-fold integral operator w.r.t. a concerned variable; thus, by direct integration of both sides (2.1) over $(t_0,\ t)$, we have:

$$\begin{cases} I_{t_0}^{t}(dy) = I_{t_0}^{t}\left(g(s,y)\right) \\ y(t_0) = y_0 \end{cases} \tag{2.2}$$

This implies that:

$$\begin{cases} y(t) = y_0 + I_{t_0}^{t}\left(g(s,y)\right), \\ y = y(t) \end{cases} \tag{2.3}$$

By iteration, we substitute $y(t) = y_{n+1}(t) = y_{n+1}$ and $y_n(s) = y_n$

Therefore, (2.3) becomes:

$$\begin{cases} y_{n+1} = y_0 + I_{t_0}^{t}\left(g(s,y_n)\right), \\ y = y(t) \end{cases} \tag{2.4}$$

3. The Method Applied

Here, the Generalised Picard Iteration Scheme (GPIS) is applied to the NSE:

Case 1:

$$\begin{cases} \dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial \eta^2} + \dfrac{1}{\eta}\dfrac{\partial w}{\partial \eta}, \\ w(\eta, 0) = \eta. \end{cases} \tag{3.1}$$

By the GPIM, we have:

Equation (3.1) is rewritten as follows:

$$\begin{cases} w_t = w_{\eta\eta} + \dfrac{1}{\eta} w_\eta \ , w = w(\eta, t) \\ w(\eta, 0) = \eta. \end{cases} \tag{3.2}$$

Applying the SIM to (3.2), give the following relation:

$$\left\{ w_{j+1} = w_0 + I_0^t \left(\left(w_j\right)_{\eta\eta} + \frac{1}{\eta}\left(w_j\right)_\eta \right), \ j \geq 0. \right. \tag{3.3}$$

Thus, the following are obtained: $w_1,\ w_2,\ w_3,\ \cdots$, using $w_0 = \eta$:

$$\begin{cases} \eta = m \\ w_0 = \eta, \\ w_1 = m + \dfrac{t}{m}, \\ w_2 = \dfrac{t^2}{2m^3} + \dfrac{t}{m}, \\ w_3 = m + \dfrac{3t^3}{2m^5} + \dfrac{t^2}{2m^3} + \dfrac{t}{m} + m, \\ w_4 = \dfrac{75t^4}{8m^7} + \dfrac{3t^3}{2m^5} + \dfrac{t^2}{2m^3} + \dfrac{t}{m} + m, \\ w_5 = \dfrac{735t^5}{8m^9} + \dfrac{75t^4}{8m^7} + \dfrac{3t^3}{2m^5} + \dfrac{t^2}{2m^3} + \dfrac{t}{m} + m, \\ w_6 = \dfrac{19845t^6}{16m^{11}} + \dfrac{735t^5}{8m^9} + \dfrac{75t^4}{8m^7} + \dfrac{3t^3}{2m^5} + \dfrac{t^2}{2m^3} + \dfrac{t}{m} + m, \\ \cdots \end{cases}$$

$$\begin{aligned} \therefore \quad w(\eta, t) &= \eta + \frac{t}{\eta} + \frac{1}{2}\frac{t^2}{\eta^3} + \frac{3}{2}\frac{t^3}{\eta^5} + \frac{75}{8}\frac{t^4}{\eta^7} + \frac{735}{8}\frac{t^5}{\eta^9} + \frac{19845}{16}\frac{t^6}{\eta^{11}} + \frac{343035}{16}\frac{t^7}{\eta^{13}} + \cdots \\ &= \eta + \sum_{\iota=1}^{\infty} \frac{1^1 \times 3^2 \times 5^2 \times \cdots \times (2\iota - 3)^2}{\eta^{2\iota-1}} \frac{t^\iota}{\iota!} \ . \end{aligned} \tag{3.4}$$

Note: we use $\eta \in [0.1,\ 5]$ and $t \in [0,\ 10]$ for the purpose graphical solution, Figure 1 and Figure 2 below represent the 3D plots of the solution for terms up to power seven and power five (in terms of the time variable t), respectively.

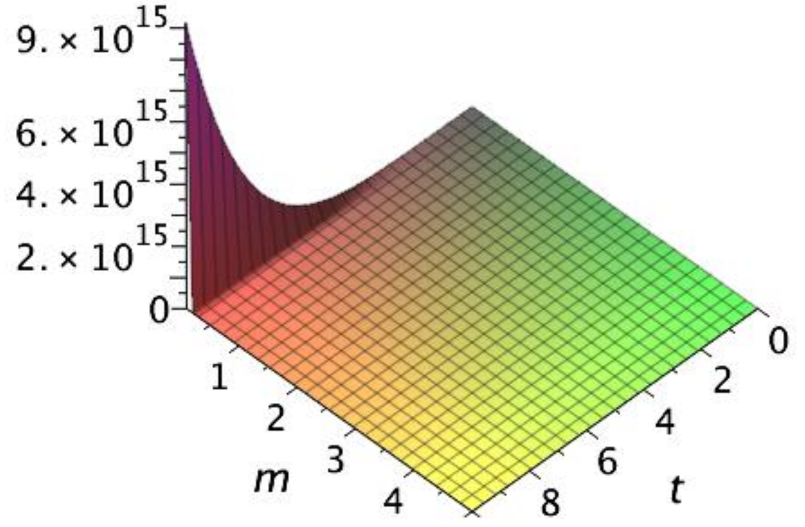

Figure 1: GPIM Solution up to t^5 term.to term

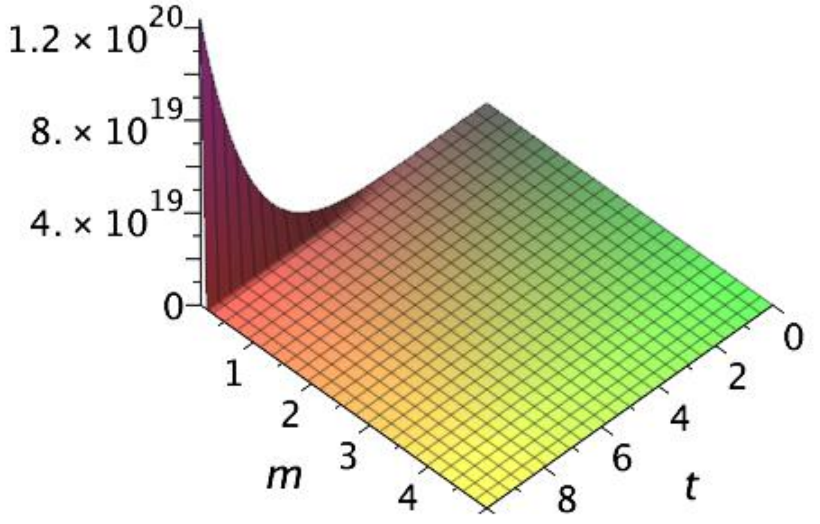

Figure 2: GPIM Solution up to t^6 term.

4. Conclusion

The generalized Picard Iteration Scheme has been used to the Navier-Stokes model solutions in this article. The solutions were in a series format, with components that were readily computed. Because the method gives the exact solution of the solved problem with minimal computing effort while keeping a high degree of accuracy, this suggested technique seems to be extremely adaptable, effective, efficient, and dependable. It is not necessary to identify Lagrange multipliers. As a result, the suggested approach is essential for handling higher-order linear and non-linear models. Maple 18 was used for the numerical calculations and graphics in this research.

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REFERENCES

[1] D. Kumar, J. Singh, S Kumar, "A fractional model of Navier–Stokes equation arising in unsteady flow of a viscous fluid", *Journal of the Association of Arab Universities for Basic and Applied Sciences* **17**, (2015): 14–19.
[2] S.O. Edeki, G.O. Akinlabi, and M.E. Adeosun, "Analytical solutions of the Navier-Stokes model by He's polynomials", Proceedings of the World Congress on Engineering 2016, Vol I WCE 2016, June 29 - July 1, 2016, London, U.K.
[3] H.A. Wahab, A. Jamal, S. Bhatti, M. Naeem, M. Shahzad, S. Hussain, Application of homotopy perturbation method to the Navier-Stokes equations in cylindrical coordinates, *Computational Ecology and Software*, **5**(2), (2015): 139-151.
[4] K. Haldar, "Application of adomian approximations to the Navier-Stokes equation in cylindrical coordinates", *Applied Mathematics Letters*, **9**(4), (1995):109-113.
[5] N.A Khan, A. Ara, S.A. Ali, A. Mahmood, "Analytical study of Navier–Stokes equation with fractional orders using He's homotopy perturbation and variational iteration methods" *Int J Nonlinear Sci Numer Simul*, **10**(9), (2011):1127–34.
[6] S.Momani, Z. Odibat, Analytical solution of a time-fractional Navier–Stokes equation by Adomian decomposition method, *Appl Math Comput*, **177,** (2006): 488–94.
[7] A.M. Wazwaz, M.S. Mehanna, 'The combined Laplace–Adomian method for handling singular integral equation of heat transfer", *Int J Nonlinear Sci.* **10** (2010): 248-52.
[8] G.O. Akinlabi and S. O. Edeki, The solution of initial-value wave-like models via perturbation iteration transform method, International MultiConference of Engineers and Computer Scientists, Hong Kong, 15-17 March 2017, (Accepted-2017).
[9] S.O. Edeki, G.O. Akinlabi, A.O. Akeju, "A Handy Approximation Technique for Closed-form and Approximate Solutions of Time-Fractional Heat and Heat-Like Equations with Variable Coefficients", Proceedings of the World Congress on Engineering, 2016, Vol II, WCE 2016, June 29 – July 1, London, UK.
[10] G.O. Akinlabi and S. O. Edeki, "On Approximate and Closed-form Solution Method for Initial-value Wave-like Models", *International Journal of Pure and Applied Mathematics,* 107 (2), (2016): 449-456.
[11] M.M. Rashidi, "The modified differential transform method for solving MHD boundary-layer equations', *Comput Phys Commun*, 180, (2009):2210–7.
[12] S.O. Edeki, G.O. Akinlabi, S.A. Adeosun, "On a modified transformation method for exact and approximate solutions of linear Schrödinger equations", *AIP Conference proceedings* 1705, 020048 (2016); doi: 10. 1063/1.4940296.
[13] S.O. Edeki, O.O. Ugbebor, E.A. Owoloko, "Analytical Solutions of the Black–Scholes Pricing Model for European Option Valuation via a Projected Differential Transformation Method", *Entropy,* **17 (**11), (2015): 7510-7521.
[14] J.H. He, "Homotopy perturbation technique", *Computer Methods in Applied Mechanics and Engineering*, **178** (3/4): (1999): 257-262.
[15] J.H. He, "A coupling method of homotopy techniques and perturbation technique for non-linear problems", *International Journal of Non-Linear Mechanics*, **35**(1) (2000): 37-43.
[16] M. Shakil, T. Khan, H.A. Wahab, S. Bhatti, "A Comparison of Adomian Decomposition Method (ADM) and Homotopy Perturbation Method (HPM) for Non-linear Problems", *International Journal of Research in Applied, Natural and Social Sciences*, **1**(3), (2013): 37-48.

[17] S.O. Edeki, I. Adinya, O.O. Ugbebor, “The Effect of Stochastic Capital Reserve on Actuarial Risk Analysis via an Integro-differential equation”, *IAENG International Journal of Applied Mathematics,* **44** (2), (2014): 83-90.
[18] J. Singh, D. Kumar and S. Rathore, “Application of Homotopy Perturbation Transform Method for Solving Linear and Nonlinear Klein-Gordon Equations”, *Journal of Information and Computing Science,* **7** (2), (2012): 131-139.
[19] N.H. Sweilam, M.M. Khader, “Exact solutions of some coupled non-linear partial differential equations using the homotopy perturbation method’, *Computers & Mathematics with Applications,* **58**: (2009): 2134–2141.
[20] J. Saberi-Nadjafi, A. Ghorbani, “He’s homotopy perturbation method: an effective tool for solving non-linear integral and integro-differential equations”, *Computers & Mathematics with Applications*, **58**, (2009):1345–1351.
[21] S. O. Edeki, G. O. Akinlabi, and S. A. Adeosun, “On a modified transformation method for exact and approximate solutions of linear Schrödinger equations”, AIP Conference Proceedings **1705**, 020048 (2016); doi: 10.1063/1.4940296.
[22] A. Ghorbani and J. S. Nadjfi, “He’s homotopy perturbation method for calculating Adomian’s polynomials”*, Int. J. Nonlin. Sci. Num. Simul*. **8** (2) (2007): 229-332.
[23] A. Ghorbani, “Beyond Adomian’s polynomials: He polynomials”, *Chaos, Solitons & Fractals,* (2007), in press.
[24] H. K. Mishra, A Comparative Study of Variational Iteration Method and He-Laplace Method, *Applied Mathematics*, 2012, 3, 1193-12**01.**
[25] S.O. Edeki, O.O. Ugbebor, and E.A. Owoloko, “He’s polynomials for analytical solutions of the Black-Scholes pricing model for stock option valuation”, Proceedings of the World Congress on Engineering 2016, Vol II, WCE 2016, June 29 - July 1, 2016, London, U.K.
[26] J.H. He, “Homotopy perturbation method: A new non-linear analytical technique”, *Appl. Math. Comput.* **135** (2003):73-79.
[27] S.O. Edeki, P.O. Ogunniyi, O. F. Imaga, Coupled method for the solution of a one-dimensional heat equation with axial symmetry, 2021 Journal of Physics: Conference Series 1734(1),012046.
[28] J. G. Oghonyon , S. A. Okunuga, S. A., Bishop A 5-step block predictor and 4-step corrector methods for solving general second order ordinary differential equations, Global Journal of Pure and Applied Mathematics11 (5), 2015, 3847-386.
[29] J. G. Oghonyon, N. A. Omoregbe, S.A. Bishop Implementing an order six implicit block multistep method for third order ODEs using variable step size approach, Global Journal of Pure and Applied Mathematics12 (2), 2016, 1635-1646.