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Frobenius method for the solution of Klein-Gordon-Fock equation with equal scalar and vector oscillator plus potential

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Frobenius method for the solution of Klein-Gordon-Fock equation with equal scalar and vector oscillator plus potential

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Abstract. In this paper, the solution of Klein Gordon Equation is sought. Frobenius method was used to solve the Klein Gordon (KG) equation with equal scalar and vector harmonic oscillator plus inverse quadratic potential for s - waves. A corresponding un-normalized wavefunction was obtained for the Frobenius equation in the form of a power series.

Keywords: Klein-Gordon equation, hypergeometric equation, Harmonic oscillator potential, inverse quadratic potential, Frobenius method

1. Introduction

Klein-Gordon-Fock Equation (KGFE) also referred to as Klein-Gordon- Equation is a second order in space-and-time differential model whose solutions contain a quantum scalar. The KGFE has a wide range of application when modelling various problems in relation to quantum mechanics, condensed matter physics and so on [1-5]. Solutions of KGFEs and similar differential equations in pure and applied sciences have been considered using some analytical, numerical and approximate methods [6-14]. In this paper, the solutions of the Klein-Gordon-Fock equation will be considered via Frobenius method of solution. The KGFE to be considered takes the following form with the harmonic oscillator plus inverse quadratic (HO+IQ) potential:

$$V(z) = kz^2 + \frac{g}{z^2} \tag{1}$$

where z represents spherical coordinate, k is arbitrary constant and g is the inverse quadratic potential strength. Dong and Lozada-Cassou [15] have used algebraic method to solve the Schrodinger equation in three dimensions with the potential in equation (1) and obtained eigenfunctions and eigenvalues of the Schrodinger equation. Also, Ikhdair and Sever [16] solved the D-dimensional radial Schrodinger equation with some molecular potential and obtained the solution for $(HO+IQ)$ potential as a special case of pseudo harmonic oscillator for $l>0$ waves.

In this paper, we shall obtain a power series solution for the Klein-Gordon equation with $(HO+IQ)$ potential by Frobenius method.

2. Klein Gordon Equation

The Klein - Gordon (KG) equation with equal scalar potential $S(z)$ and vector potential $V(z)$ in natural units $(h = c = 1)$ is given as

$$\frac{d^2U(z)}{dz^2} + [(E^2 - M^2 - 2(E+M)V(z))]U(z) = 0 \tag{2}$$

where M is the rest mass and E is the relativistic energy. Using an appropriate transformation, $z = r^2$, (2) is reduced to the ordinary differential equation of the form:

$$\frac{d^2U(r)}{dr^2} + \frac{1}{2r}\frac{dU(r)}{dr} + \frac{1}{4r^2}(-Dr^2 + Cr - F)U(r) = 0 \tag{3}$$

where the radial wave function is $U(r)$; C, D and F are potential parameters given by

$$C = E^2 - M^2, \quad D = 2(E+M)k, \quad F = -2(E+M)g \ , \tag{4}$$

where k is an arbitrary constant and g is the inverse quadratic potential strength (as earlier stated in (1).

For simplicity sake, we introduce the notation, N1 as follows:

$$N1: \quad \begin{cases} P(r) = \dfrac{1}{2}, \\ Q(r) = \dfrac{-Dr^2 + Cr - F}{4} \end{cases}$$

into (3), to have:

$$\frac{d^2U(r)}{dr^2} + \frac{P(r)}{r}\frac{dU(r)}{dr} + \frac{Q(r)}{r^2}U(r) = 0 \tag{5}$$

In this form, Frobenius method can be applied to solve (5). This demands the expansion of the solution around regular singular points, at $r = 0$ and $r = \infty$, of the differential equation. In what follows, we shall only consider the regular point for which the solution is physically meaningful namely; at $r = 0$. Therefore, the radial wave function is represented by the generalized power series:

$$U(r) = r^\beta \sum_{i=0}^{\infty} a_i r^i \tag{6}$$

where $a_0 \neq 0$. Substituting Eqn. (6} in Eqn. (5) yields:

$$\sum_{i=0}^{\infty}(i+\beta)(i+\beta-1)a_i r^{i+\beta-2} + \frac{P(r)}{r}\sum_{i=0}^{\infty}(i+\beta)a_i r^{i+\beta-1} + \frac{Q(r)}{r^2}\sum_{i=0}^{\infty}a_i r^{i+\beta} = 0 \tag{7}$$

That is,

$$\sum_{i=0}^{\infty}\left[(i+\beta)(i+\beta-1) + P(r)(i+\beta) + Q(r)\right]a_i r^{i+\beta-2} = 0 \ . \tag{8}$$

By isolating the first term of the sum starting from $i=0$, we obtained:

$$\left(\beta(\beta-1) + P(r)\beta + Q(r)\right)a_0 r^{\beta-2} = 0 \ . \tag{9}$$

Now, since each coefficient goes to zero from the linear independence of powers of r and noting that $a_0 \neq 0$ we obtained the indicial equation as follows

$$\begin{cases} \beta(\beta-1)+P(r)\beta+Q(r)=0 \\ \Rightarrow \beta^2+\left(P(r)-1\right)\beta+Q(r)=0 \end{cases} \tag{10}$$

which is solved for β, after replacing $P(r)$ and $Q(r)$ (see notation N1 for the expressions). Thus, we have:

$$\begin{cases} \beta=\lim_{r\to 0}\frac{1}{4}\left\{1\pm\sqrt{1+4\left(Dr^2-cr+F\right)}\right\}=\frac{1}{4}\left\{1\pm\sqrt{1+4F}\right\}, \\ \Rightarrow \beta_1=\frac{1}{4}\left\{1+\sqrt{1+4F}\right\} \text{ and } \beta_2=\frac{1}{4}\left\{1-\sqrt{1+4F}\right\}. \end{cases}. \tag{11}$$

By Fuchs's theorem [16], the generalized series (6) converges and the Klein-Gordon equation has two linearly independent solutions obtained as generalized series.

Now at $i=1$, after replacing $P(r)$ and $Q(r)$ in (8), we obtain:

$$\frac{C}{4}a_0+\left[(1+\beta)\beta+\frac{1}{2}(1+\beta)+\frac{F}{4}\right]a_1=0 \tag{12}$$

which can only be true if $C=0$ and $a_1=0$. Similarly at $i>1$ we have:

$$\left[(i+\beta)(i+\beta-1)+\frac{1}{2}(i+\beta)+\frac{F}{4}\right]a_i+\frac{C}{4}a_{i-1}-\frac{D}{4}a_{i-2}=0. \tag{13}$$

We re-write (13) and apply little algebra as follows:

$$\begin{aligned} &\left[4(i+\beta)(i+\beta-1)+2(i+\beta)+F\right]a_i=Da_{i-2}-Ca_{i-1}. \\ &\Rightarrow a_i=\frac{Da_{i-2}-Ca_{i-1}}{\left[4(i+\beta)(i+\beta-1)+2(i+\beta)+F\right]} \\ &\qquad =\frac{Da_{i-2}-Ca_{i-1}}{4(i+\beta)(i+\beta-\frac{1}{2})+F},\ i>1. \end{aligned} \tag{14}$$

Equation (14) denotes the recurrence relation for the coefficients of the series (6) as

Putting the values of C, D and F in (14) yields:

$$a_i=\frac{2\left(E+M\right)ka_{i-2}-\left(E^2-M^2\right)a_{i-1}}{4\left(i+\beta\right)\left(i+\beta-\frac{1}{2}\right)-2\left(E+M\right)g},\quad i>1. \tag{15}$$

Notice that the value of β determines the behaviour of the radial wave function $U(r)$ as $r\to 0$. Clearly, for β to be well behaved, $F>\frac{1}{4}$ thus, the acceptable solution would be the one that contains the series with $\beta=\beta_1$.

Finally, the solution of the Klein-Gordon equation, that is the radial wave function, is obtained in the form:

$$U(r)=r^{\beta_1}\sum_{i=0}^{\infty}a_i r^i \tag{16}$$

Here, a_i is defined by (15) for all i, where $\beta=\beta_1$.

3. Conclusions

In conclusion, we have obtained the corresponding un-normalized wave function (16) using the Frobenius method for the Klein Gordon equation with equal scalar and vector harmonic oscillator plus inverse quadratic potential for S-waves. The energy eigenvalues are obtained as roots of the series (16), after truncating at a suitably high order, say N . This is possible with arbitrary accuracy because the series converges.

Conflict of interest

The authors declare that there exists no conflict of interest regarding the publication of this paper.

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