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Numerical treatment of first order delay differential equations using extended block backward differentiation formulae

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Abstract. In this research, we developed and implemented extended backward differentiation methods (formulae) in block forms for step numbers k = 2, 3 and 4 to evaluate numerical solutions for certain first-order differential equations of delay type, generally referred to as delay differential equations (DDEs), without the use of interpolation methods for estimating the delay term. The matrix inversion approach was applied to formulate the continuous composition of these block methods through linear multistep collocation method. The discrete schemes were established through the continuous composition for each step number, which evaluated the error constants, order, consistency, convergent and area of absolute equilibrium of these discrete schemes. The study of the absolute error results revealed that, as opposed to the exact solutions, the lower step number implemented with super futures points work better than the higher step numbers implemented with super future points.

Keywords: First order differential equations, Delay ODE, block method, backward differentiation.

1. Introduction

For several years, there have been much research activities in the area of numerical solution for delay differential equations which are of great interest to researchers; scientists and engineers. The results of these researches are methods which can be applied to many real life problems in quantum mechanics, nuclear and theoretical physics, astrophysics, molecular dynamics and economic dynamics and control an so on. In natural sciences, [1], delay differential equations was used in the modeling of El Nino temperature oscillations in the Equatorial Pacific to determine the model single-species population growth. Many physical processes can be converted into mathematical language using the principle of mathematical models in research, medicine , and engineering. The mathematical model describes these physical processes as differential equations in the form of delays. Delay Differential equations are one of mathematical modeling 's commonly used and important methods, and can be solved analytically or using computational approaches. In the case where the empirical solution is difficult to find, the value of

computational methods will then be put into play. Numerical approaches are critical techniques for solving the differential delay equation which cannot be overlooked
Delays are used in electrical systems, so it takes time for a signal to pass across a transmission wire. These DDEs differ from ordinary differential equations from a mathematical point of view, since the evolution of DDEs includes a time series of historical values of dependent variables and derivatives, while the evolution of ODEs depends only on the present values of these quantities.

Our interest in this research work is to determine the discrete solutions of the first order DDEs of the following form:

$$\begin{cases} y'(t) - f(t, y(t), y(t-\tau)) = 0, \ t > t_0, \tau > 0 \\ y(t) - \phi(t) = 0, \ t \le t_0 \end{cases} \tag{1}$$

where the initial function is given as $\phi(t)$, τ is known as the delay term, $(t-\tau)$ is called the delay argument and $y(t-\tau)$ is the solution of the delay argument. Researches on solution approaches DDEs, stiff DEs can be seen in [2-19].

2. Development of Multistep Collocation Procedure

The *k*-step multistep collocation procedure with *m* collocation points was derived [2, 3] as

$$y(x) = \sum_{v=0}^{c-1} \alpha_v(x) y_{u+v} + g \sum_{v=0}^{z-1} \beta_v(x) f_{u+v}(x, y(x)) \tag{2}$$

where $\alpha_v(x)$ and $\beta_v(x)$ are continuous coefficients of the technique defined as

$$\alpha_v(x) = \sum_{m=0}^{c+z-1} \alpha_{v,m+1} x^m \text{ for } v = \{0,1,...,c-1\} \tag{3}$$

$$g\beta_v(x) = \sum_{m=0}^{c+z-1} g\beta_{v,m+1} x^m \text{ for } v = \{0,1,...,z-1\} \tag{4}$$

where $x_0, \ldots, x_{z-1}$ are the z collocation points, $x_{u+v}, v = 0,1,2,...,c-1$ are the p arbitrarily chosen interpolation points and m is the constant step size.
To get $\alpha_v(x)$ and $\beta_v(x)$, [2,3] formulated a matrix equation of the form

$$LM = \mathrm{I} \tag{5}$$

where I is the square matrix of dimension $(c+z)\times(c+z)$ while M and L are matrices defined as

$$M = \begin{bmatrix} \alpha_{0,1} & \alpha_{1,1} & \cdots & \alpha_{c-1,1} & m\beta_{0,1} & \cdots & m\beta_{z-1,1} \\ \alpha_{0,2} & \alpha_{1,2} & \cdots & \alpha_{c-1,2} & m\beta_{0,2} & \cdots & m\beta_{z-1,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{0,c+z} & \alpha_{1,c+z} & \cdots & \alpha_{c-1,c+z} & m\beta_{0,c+z} & \cdots & m\beta_{z-1,c+z} \end{bmatrix} \tag{6}$$

$$L=\begin{bmatrix} 1 & x_u & x_u^2 & \cdots & x_u^{c+z-1} \\ 1 & x_{u+1} & x_{u+1}^2 & \cdots & x_{u+1}^{c+z-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{u+c-1} & x_{u+c-1}^2 & \cdots & x_{u+c-1}^{c+z-1} \\ 0 & 1 & 2x_0 & \cdots & (c+z-1)x_0^{c+z-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2x_{z-1} & \cdots & (c+z-1)x_{z-1}^{c+z-2} \end{bmatrix} \quad (7)$$

From the matrix equation (5), the columns of $M=L^{-1}$ give the continuous coefficients of the continuous scheme of (2).

2.1 Development of EBBDF Method Incorporating One Super Future Point for $k=2$

Here, we incorporate one super future point at $x=x_{u+3}$ as a collocation point, thus the interpolation points, $c=2$ and the collocation points $z=2$ are considered, therefore, (2) becomes:

$$y(x)=\alpha_0(x)y_n+\alpha_1(x)y_{n+1}+h[\beta_2(x)f_{n+2}+\beta_3(x)f_{n+3}] \quad (8)$$

The matrix L in (5) becomes

$$L=\begin{pmatrix} 1 & x_u & x_u^2 & x_u^3 \\ 1 & x_u+g & (x_u+g)^2 & (x_u+g)^3 \\ 0 & 1 & 2x_u+2g & 3(x_u+2g)^2 \\ 0 & 1 & 2x_u+3g & 3(x_u+3g)^2 \end{pmatrix}$$

The inverse of the matrix $M=L^{-1}$ is examined using Maple 18 from which the continuous scheme is obtained using (2), evaluating and simplifying it at $x=x_{u+2}$, $x=x_{u+3}$ and its derivative at $x=x_{u+1}$,the following discrete schemes are obtained;

$$y_{u+1}=\frac{23}{12}gf_{u+1}-\frac{4}{3}gf_{u+2}+\frac{5}{12}gf_{u+3}+y_u$$
$$y_{u+2}=-\frac{5}{23}y_u+\frac{28}{23}y_{u+1}+\frac{22}{23}gf_{u+2}-\frac{4}{23}gf_{u+3}$$
$$y_{u+3}=-\frac{4}{23}y_u+\frac{27}{23}y_{u+1}+\frac{36}{23}gf_{u+2}+\frac{6}{23}gf_{u+3} \quad (9)$$

2.2 Development of EBBDF Method Incorporating Two Super Future Point for $k=2$

In this case, we incorporate two super future points at $x=x_{u+3}, x=x_{u+4}$ as collocation points, thus the interpolation points, $c=2$ and the collocation points $z=3$ are considered, therefore, (2) becomes:

$$y(x)=\alpha_0(x)y_u+\alpha_1(x)y_{u+1}+h[\beta_2(x)f_{u+2}+\beta_3(x)f_{u+3}+\beta_4(x)f_{u+4}] \quad (10)$$

The matrix L in (5) becomes

$$L=\begin{pmatrix}1 & x_u & x_u^2 & x_u^3 & x_z^4\\ 1 & x_u+g & (x_u+g)^2 & (x_u+g)^3 & (x_u+g)^4\\ 0 & 1 & 2x_u+2g & 3(x_u+2g)^2 & 4(x_u+2g)^3\\ 0 & 1 & 2x_u+3g & 3(x_u+3g)^2 & 4(x_u+3g)^3\\ 0 & 1 & 2x_u+4g & 3(x_u+4g)^2 & 4(x_u+4g)^3\end{pmatrix} \quad (11)$$

The inverse of the matrix $M=L^{-1}$ is examined using Maple 18 from which the continuous scheme is obtained using (2), evaluating and simplifying it at $x=x_{u+2,}x=x_{u+3,}, x=x_{u+4}$ and its derivative at $x=x_{u+1}$,the following discrete schemes are obtained;

$$y_{u+1}=\frac{55}{24}gf_{u+1}-\frac{59}{24}gf_{u+2}+\frac{37}{24}gf_{u+3}-\frac{3}{8}gf_{u+4}+y_u$$
$$y_{u+2}=-\frac{9}{55}y_u+\frac{64}{55}y_{u+1}+\frac{197}{165}gf_{u+2}-\frac{76}{165}gf_{u+3}+\frac{17}{165}gf_{u+4}$$
$$y_{u+3}=-\frac{8}{55}y_u+\frac{63}{55}y_{u+1}+\frac{93}{55}gf_{u+2}+\frac{6}{55}gf_{u+3}+\frac{3}{55}gf_{u+4}$$
$$y_{u+4}=-\frac{9}{55}y_u+\frac{64}{55}y_{u+1}+\frac{84}{55}gf_{u+2}+\frac{48}{55}gf_{u+3}+\frac{24}{55}gf_{u+4} \quad (12)$$

2.3 Development of EBBDF Method Incorporating One Super Future Point for $k=3$

Here, we incorporate one super future point at $x=x_{u+4}$ as a collocation point, thus the interpolation points, $c=3$ and the collocation points $z=2$ are considered, therefore, (2) becomes:

$$y(x)=\alpha_0(x)y_u+\alpha_1(x)y_{u+1}+h[\beta_2(x)f_{u+2}+\beta_3(x)f_{u+3}+\beta_4(x)f_{u+4}] \quad (13)$$

The matrix L in (5) becomes

$$L=\begin{pmatrix}1 & x_u & x_u^2 & x_u^3 & x_z^4\\ 1 & x_u+g & (x_u+g)^2 & (x_u+g)^3 & (x_u+g)^4\\ 1 & x_u+2g & (x_u+2g)^2 & (x_u+2g)^3 & (x_u+2g)^4\\ 0 & 1 & 2x_u+6g & 3(x_u+3g)^2 & 4(x_u+3g)^3\\ 0 & 1 & 2x_u+8g & 3(x_u+4g)^2 & 4(x_u+4g)^3\end{pmatrix} \quad (14)$$

The inverse of the matrix $M=L^{-1}$ is examined using Maple 18 from which the continuous scheme is obtained using (2), evaluating and simplifying it at $x=x_{u+3}, x=x_{u+4}$ and its derivative at $x=x_{u+1}$ and $x=x_{u+2}$,the following discrete schemes are obtained;

$$y_{u+1}=-\frac{197}{120}gf_{u+1}-\frac{17}{40}gf_{u+3}+\frac{7}{60}gf_{u+4}-\frac{19}{40}y_u+\frac{59}{40}y_{u+2}$$
$$y_{u+2}=\frac{197}{165}gf_{u+2}-\frac{76}{165}gf_{u+3}+\frac{17}{165}gf_{u+4}-\frac{9}{55}y_u+\frac{64}{55}y_{u+1}$$

$$y_{u+3}=\frac{17}{197}y_u-\frac{99}{197}y_{u+1}+\frac{279}{197}y_{u+2}+\frac{150}{197}gf_{u+3}-\frac{18}{197}gf_{u+4}$$
$$y_{u+4}=\frac{9}{197}y_u-\frac{64}{197}y_{u+1}+\frac{252}{197}y_{u+2}+\frac{288}{197}gf_{u+3}+\frac{60}{197}gf_{u+4} \quad (15)$$

2.4 Development of EBBDF Method Incorporating Two Super Future Point for $k=3$

Here, we incorporate one super future point at $x=x_{u+4}, x=x_{u+5}$ as collocation points, thus the interpolation points, $c=3$ and the collocation points $z=3$ are considered, therefore, (2) becomes:

$$y(x)=\alpha_0(x)y_u+\alpha_1(x)y_{u+1}+h[\beta_2(x)f_{u+2}+\beta_3(x)f_{u+3}+\beta_4(x)f_{u+4}+\beta_5(x)f_{u+5}] \quad (16)$$

The matrix L in (5) becomes

$$L=\begin{pmatrix} 1 & x_u & x_u^2 & x_u^3 & x_u^4 & x_u^5 \\ 1 & x_u+g & (x_u+g)^2 & (x_u+g)^3 & (x_u+g)^4 & (x_u+g)^5 \\ 1 & x_u+2g & (x_u+2g)^2 & (x_u+2g)^3 & (x_u+2g)^4 & (x_u+2g)^5 \\ 0 & 1 & 2x_u+6g & 3(x_u+3g)^2 & 4(x_u+3g)^3 & 5(x_u+3g)^4 \\ 0 & 1 & 2x_u+8g & 3(x_u+4g)^2 & 4(x_u+4g)^3 & 5(x_u+4g)^4 \\ 0 & 1 & 2x_u+10g & 3(x_u+5g)^2 & 4(x_u+5g)^3 & 5(x_u+5g)^4 \end{pmatrix} \quad (17)$$

The inverse of the matrix $M=L^{-1}$ is examined using Maple 18 from which the continuous scheme is obtained using (2), evaluating and simplifying it at $x=x_{u+3}$, $x=x_{u+4}, x=x_{u+5}$ and its derivative at $x=x_{u+1}$ and $x=x_{u+2}$,the following discrete schemes are obtained;

$$y_{u+1}=-\frac{211}{168}gf_{u+1}-\frac{5}{8}gf_{u+3}+\frac{29}{84}gf_{u+4}-\frac{1}{14}gf_{u+5}-\frac{17}{56}y_u+\frac{73}{56}y_{u+2}$$
$$y_{u+2}=\frac{8018}{5703}gf_{u+2}-\frac{1609}{1901}gf_{u+3}+\frac{724}{1901}gf_{u+4}-\frac{413}{5703}gf_{u+5}-\frac{251}{1901}y_u+\frac{2152}{1901}y_{u+1}$$
$$y_{u+3}=\frac{413}{8018}y_u-\frac{1467}{4009}y_{u+1}+\frac{10539}{8018}y_{u+2}+\frac{7503}{8018}gf_{u+3}-\frac{963}{4009}gf_{u+4}+\frac{333}{8018}gf_{u+5}$$
$$y_{u+4}=\frac{153}{4009}y_u-\frac{1184}{4009}y_{u+1}+\frac{5040}{4009}y_{u+2}+\frac{6012}{4009}gf_{u+3}+\frac{1092}{4009}gf_{u+4}+\frac{36}{4009}gf_{u+5}$$
$$y_{u+5}=\frac{234}{4009}y_u-\frac{1575}{4009}y_{u+1}+\frac{5350}{4009}y_{u+2}+\frac{4950}{4009}gf_{u+3}+\frac{4500}{4009}gf_{u+4}+\frac{1470}{4009}gf_{u+5} \quad (18)$$

2.5 Development of EBBDF Method Incorporating One Super Future Point for $k=4$

Here, we incorporate one super future point at $x=x_{u+5}$ as a collocation point, thus the interpolation points, $c=3$ and the collocation points $z=3$ are considered, therefore, (2) becomes:

$$y(x)=\alpha_0(x)y_u+\alpha_1(x)y_{u+1}+h[\beta_2(x)f_{u+2}+\beta_3(x)f_{u+3}+\beta_4(x)f_{u+4}+\beta_5(x)f_{u+5}] \quad (19)$$

The matrix L in (5) becomes

$$L=\begin{pmatrix}1 & x_u & x_u^2 & x_u^3 & x_u^4 & x_u^5\\ 1 & x_u+g & (x_u+g)^2 & (x_u+g)^3 & (x_u+g)^4 & (x_u+g)^5\\ 1 & x_u+2g & (x_u+2g)^2 & (x_u+2g)^3 & (x_u+2g)^4 & (x_u+2g)^5\\ 1 & x_u+3g & (x_u+3g)^2 & (x_u+3g)^3 & (x_u+3g)^4 & (x_u+3g)^5\\ 0 & 1 & 2x_u+8g & 3(x_u+4g)^2 & 4(x_u+4g)^3 & 5(x_u+4g)^4\\ 0 & 1 & 2x_u+10g & 3(x_u+5g)^2 & 4(x_u+5g)^3 & 5(x_u+5g)^4\end{pmatrix} \quad (20)$$

The inverse of the matrix $M=L^{-1}$ is examined using Maple 18 from which the continuous scheme is obtained using (2), evaluating and simplifying it at $x=x_{u+4}, x=x_{u+5}$ and its derivative at $x=x_{u+1}, x=x_{u+2}$ and $x=x_{u+3}$,the following discrete schemes are obtained;

$$\begin{aligned}
y_{u+1}&=-\frac{2501}{2478}gf_{u+1}+\frac{184}{1239}gf_{u+4}-\frac{29}{826}gf_{u+5}-\frac{268}{1239}y_u+\frac{724}{413}y_{u+2}-\frac{95}{177}y_{u+3}\\
y_{u+2}&=-\frac{2501}{336}gf_{u+2}-\frac{97}{112}gf_{u+4}+\frac{31}{168}gf_{u+5}+\frac{19}{42}y_u-\frac{475}{112}y_{u+1}+\frac{1609}{336}y_{u+3}\\
y_{u+3}&=\frac{7503}{8018}f_{u+3}-\frac{963}{4009}f_{u+4}+\frac{333}{8018}gf_{u+5}+\frac{413}{8018}y_u-\frac{1467}{4009}y_{u+1}+\frac{10539}{8018}y_{u+2}\\
y_{u+4}&=-\frac{111}{2501}y_u+\frac{728}{2501}y_{u+1}-\frac{2124}{2501}y_{u+2}+\frac{4008}{2501}y_{u+3}+\frac{1644}{2501}gf_{u+4}-\frac{144}{2501}gf_{u+5}\\
y_{u+5}&=-\frac{24}{2501}y_u+\frac{225}{2501}y_{u+1}-\frac{1000}{2501}y_{u+2}+\frac{3300}{2501}y_{u+3}+\frac{3600}{2501}gf_{u+4}+\frac{780}{2501}gf_{u+5}
\end{aligned} \quad (21)$$

2.6 Development of EBBDF Method Incorporating Two Super Future Point for $k=4$

Here, we incorporate two super future point at $x=x_{u+5}, x=x_{u+6}$ as collocation points, thus the interpolation points, $c=4$ and the collocation points $z=3$ are considered, therefore, (2) becomes:

$$y(x)=\alpha_0(x)y_u+\alpha_1(x)y_{u+1}+h[\beta_2(x)f_{u+2}+\beta_3(x)f_{u+3}+\beta_4(x)f_{u+4}+\beta_5(x)f_{u+5}+\beta_6(x)f_{u+6}] \quad (22)$$

The matrix L in (5) becomes

$$L=\begin{pmatrix}1 & x_u & x_u^2 & x_u^3 & x_u^4 & x_u^5 & x_u^6\\ 1 & x_u+g & (x_u+g)^2 & (x_u+g)^3 & (x_u+g)^4 & (x_u+g)^5 & (x_u+g)^6\\ 1 & x_u+2g & (x_u+2g)^2 & (x_u+2g)^3 & (x_u+2g)^4 & (x_u+2g)^5 & (x_u+2g)^6\\ 1 & x_u+3g & (x_u+3g)^2 & (x_u+3g)^3 & (x_u+3g)^4 & (x_u+3g)^5 & (x_u+3g)^6\\ 0 & 1 & 2x_u+8g & 3(x_u+4g)^2 & 4(x_u+4g)^3 & 5(x_u+4g)^4 & 6(x_u+4g)^5\\ 0 & 1 & 2x_u+10g & 3(x_u+5g)^2 & 4(x_u+5g)^3 & 5(x_u+5g)^4 & 6(x_u+5g)^5\\ 0 & 1 & 2x_u+12g & 3(x_u+6g)^2 & 4(x_u+6g)^3 & 5(x_u+6g)^4 & 6(x_u+6g)^5\end{pmatrix} \quad (23)$$

The inverse of the matrix $M=L^{-1}$ is examined using Maple 18 from which the continuous scheme is obtained using (2), and are evaluated and simplified at $x=x_{u+4}, x=x_{u+5}, x=x_{u+6}$.

nd its derivative at $x = x_{u+1}, x = x_{u+2}$ and $x = x_{u+3}$,the following discrete schemes are obtained;

$$
\begin{aligned}
y_{n+1} &= -\frac{10673}{12390} f_{n+1} h + \frac{368}{1239} hf_{n+4} - \frac{117}{826} hf_{n+5} + \frac{8}{315} hf_{n+6} - \frac{1756}{11151} y_n \\
&+ \frac{740}{413} y_{n+2} - \frac{7073}{11151} y_{n+3} \\
&\vdots \\
y_{n+6} &= -\frac{7100}{224133} y_n + \frac{2448}{10673} y_{n+1} - \frac{7875}{10673} y_{n+2} + \frac{345200}{224133} y_{n+3} + \frac{77400}{74711} hf_{n+4} \\
&+ \frac{7200}{5747} hf_{n+5} + \frac{25180}{74711} hf_{n+6}
\end{aligned} \tag{24}
$$

3. Convergence analysis

Here, the investigations of order, zero stability, consistency, error constant, and region of the absolute stability of (9), (12), (15) , (18) , (21) and (24) are worked-out .

3.1 Order and Error Constant

The order and error constants for (9) are obtained as follows:

$$c_0 = c_1 = c_2 = c_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \; c_4 = \begin{pmatrix} -\frac{3}{8} \\ \frac{17}{138} \\ \frac{3}{46} \end{pmatrix}$$

Therefore, (9) has order $p = 3$ and error constants, $\left(-\frac{3}{8}, \frac{17}{138}, \frac{3}{46} \right)^T$

With the same approach, (12) and can be presented as:

$$c_0 = c_1 = c_2 = c_3 = c_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, (12) has order $p = 4$ and error constants, $\left(\frac{251}{720}, -\frac{413}{4950}, -\frac{17}{275}, -\frac{26}{275} \right)^T$

Applying the same approach, (15) and can be obtained as:

$$c_0 = c_1 = c_2 = c_3 = c_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, (15) has order $p=4$ and error constants, $\left(-\frac{19}{150},-\frac{413}{4950},\frac{111}{1970},\frac{12}{985}\right)^T$

Applying the same approach, (18) and can be obtained as:

$$c_0 = c_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, (18) has order $p=1$ and error constants, $\left(-\frac{5}{14},-\frac{2065}{5703},\frac{1665}{8018},\frac{180}{4009},\frac{85525}{8018}\right)^T$

Applying the same approach, (21) and can be obtained as:

$$c_0 = c_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, (21) has order $p=1$ and error constants, $\left(-\frac{5}{14},-\frac{2065}{5703},\frac{1665}{8018},\frac{180}{4009},\frac{85525}{8018}\right)^T$

Also applying the same approach, (24) and can be obtained as:

$$c_0 = c_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, (24) has order $p=1$ and error constant,

$$\left(-\frac{6887}{12390},\frac{2071}{1743},\frac{383985}{616669},-\frac{6672}{10673},-\frac{232625}{21346},-\frac{103674}{10673}\right)^T$$

3.2 Consistency

Since the schemes in (9), (12) , (15) , (18) , (21) and (24) satisfy the condition for consistency of order $a \geq 1$, then they are consistent.

3.3 Convergence

Equations (9), (12) , (15) , (18) , (21) and (24) are consistent and zero stable, as such they are convergent.

3.4 Overview of Region of Absolute Stability (RAS)

Here, the RAS of the numerical approaches (methods) linked with DDEs are considered. We considered finding the P - and Q -stability by applying (9), (12) , (15) , (18) , (21) and (24) to the DDEs of this form:

$$\begin{cases} f^{'}(t) = \xi f(\mathrm{t}) + \upsilon f(t-\tau), & t \geq t_0 \\ f(t) = \eta(t), & t \leq t_0 \end{cases} \tag{25}$$

where $\eta(t)$ denotes the initial function, while ξ, υ are complex coefficients, $\tau = ug, u \in \mathbb{Z}^{+}$, m is the step size and $u = \dfrac{\tau}{g}, u$ is a positive integer. Let $G_1 = m\xi$ and $G_2 = m\upsilon$, then

Making use of Maple 18 and MATLAB, the region of P - and Q -stability for (9), (12) , (15) , (18) , (21) and (24) are shown in Fig.1 to 12.

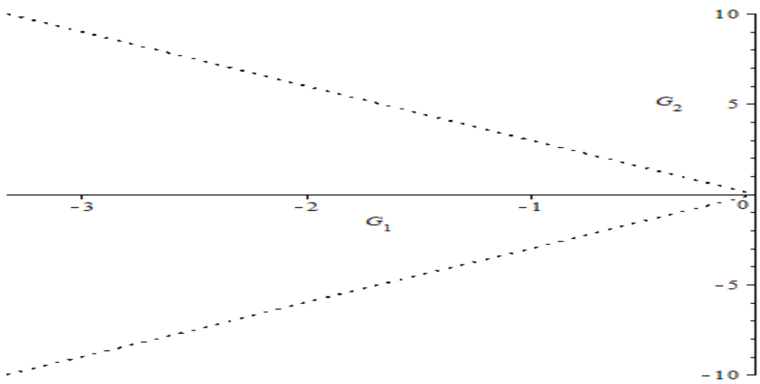

Fig.1.Region of P -stability (EBBDFM) in (9)

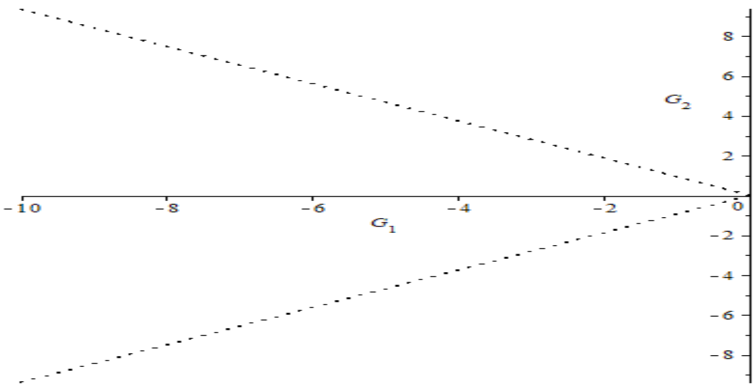

Fig.2.Region of P -stability (EBBDFM) in (12)

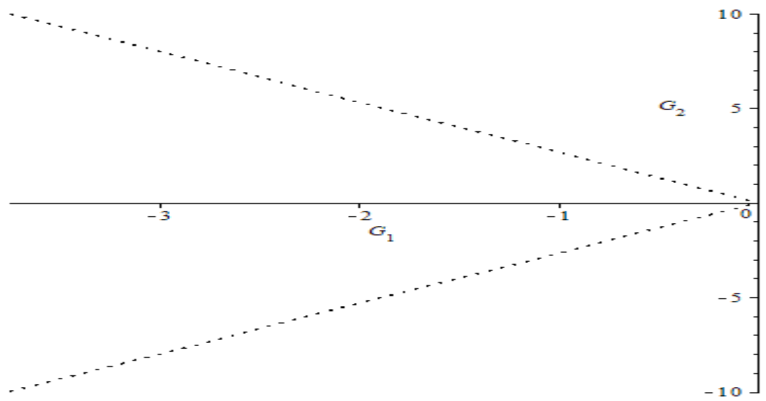

Fig.3.Region of P -stability (EBBDFM) in (15)

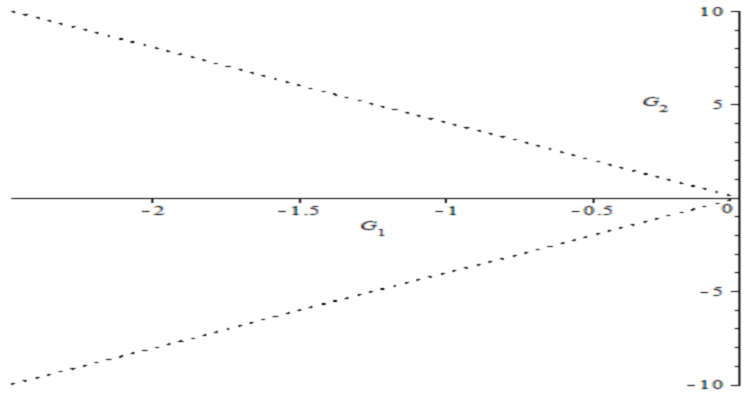

Fig.4.Region of P -stability (EBBDFM) in (18)

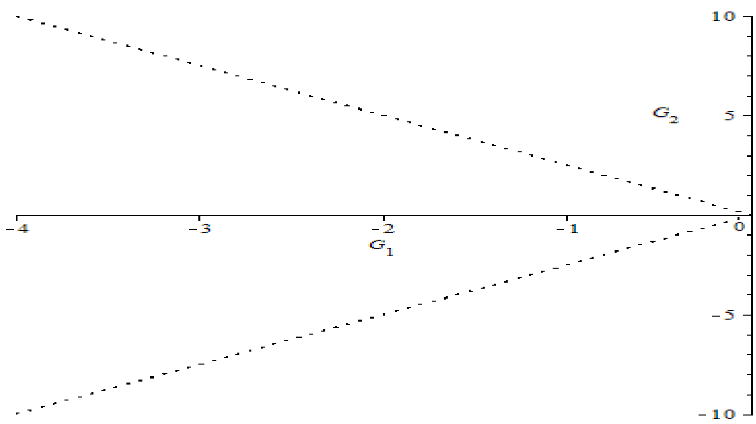

Fig.5.Region of P -stability (EBBDFM) in (21)

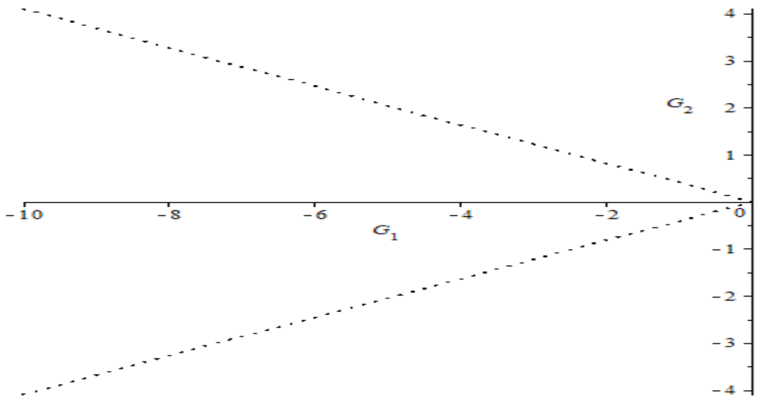

Fig.6.Region of P-stability (EBBDFM) in (24)

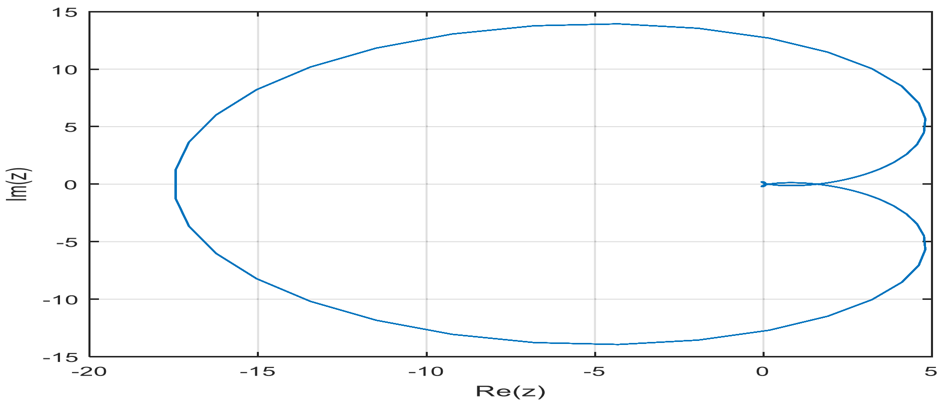

Fig.7. Region of Q-stability (EBBDFM) in (9)

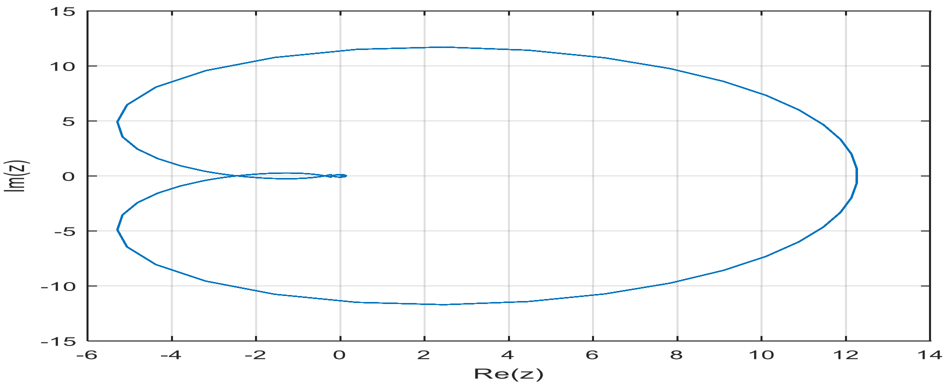

Fig.8. Region of Q-stability (EBBDFM) in (12)

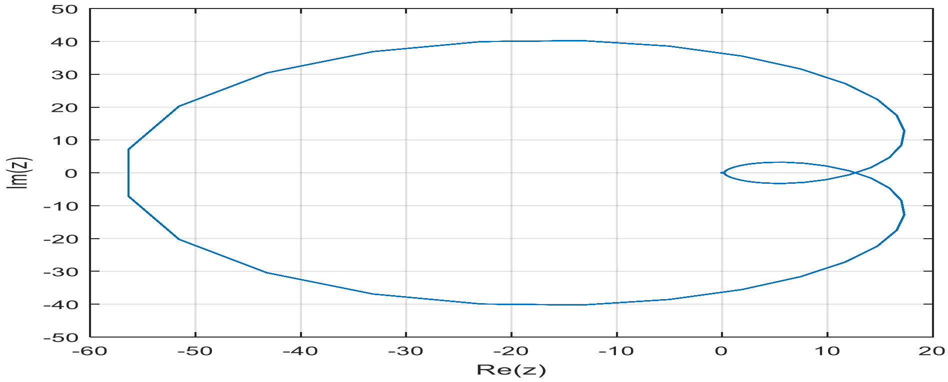

Fig.9. Region of Q-stability (EBBDFM) in (15)

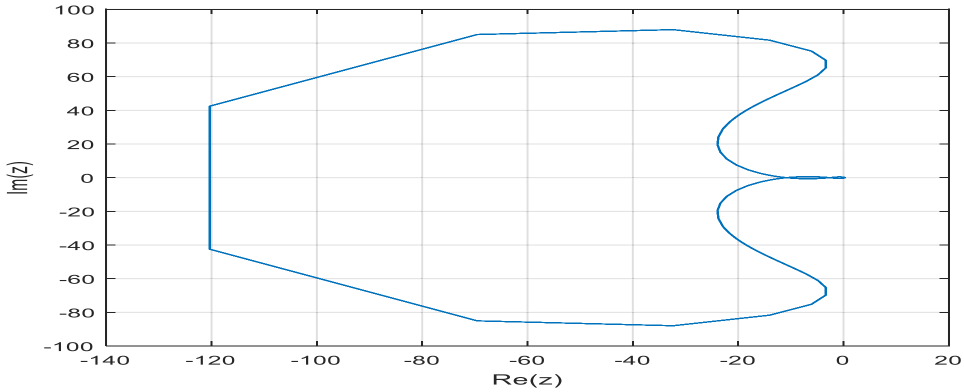

Fig.10. Region of Q-stability (EBBDFM) in (18)

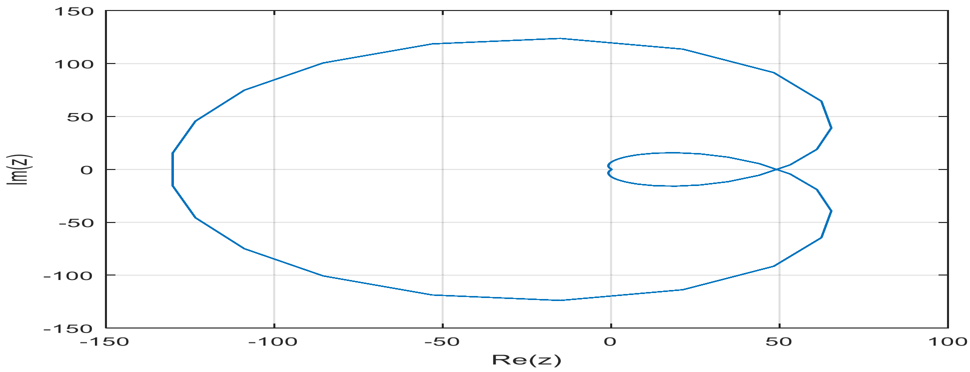

Fig.11. Region of Q-stability (EBBDFM) in (21)

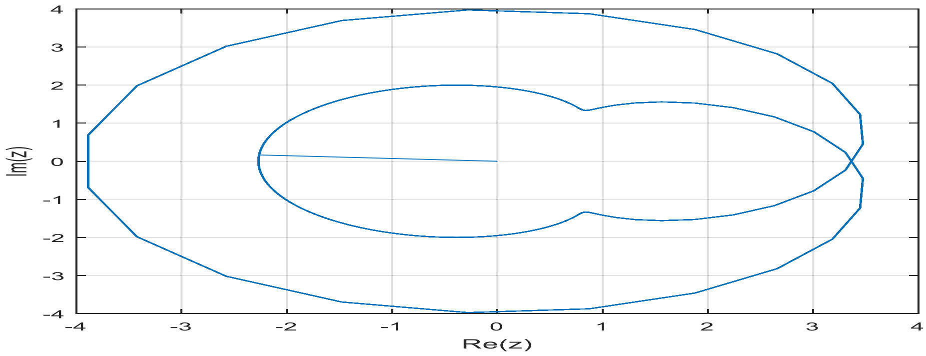

Fig.12. Region of Q-stability (EBBDFM) in (24)

The P-stability regions in Figs 1 to 6 lie inside the open-ended region while the Q-stability regions in Figs 7 to 12 lie inside the enclosed region.

4. Numerical Computations

Here, some first-order DDEs are solved using (9), (12),(15) , (18),(21) and (24) of the discrete schemes been established. The delay argument are evaluated using the notion of sequence developed by [2, 3].

Problem 1

$$y'(t) = -1000y(t) + y(t - (\ln(1000-1))),\ 0 \le t \le 3$$

$$y(t) = e^{-t}, t \le 0$$

Exact solution $y(t) = e^{-t}$

Table 1: Numerical Solution of One Super Future Point for Problem 1

Time (t)	Exact Solution	Numerical Solution (k = 2)	Numerical Solution (k = 3)	Numerical Solution (k = 4)
0.01	0.990049834	0.990049834	0.990049835	0.990049835
0.02	0.980198673	0.980198674	0.980198674	0.980198674
0.03	0.970445534	0.970445536	0.970445534	0.970445534
0.04	0.960789439	0.96078944	0.960789439	0.96078944
0.05	0.951229425	0.951229425	0.951229425	0.951229426
0.06	0.941764534	0.941764532	0.941764534	0.941764534
0.07	0.93239382	0.932393821	0.93239382	0.93239382
0.08	0.923116346	0.923116347	0.923116348	0.923116347
0.09	0.913931185	0.913931183	0.913931186	0.913931185
0.1	0.904837418	0.904837419	0.904837418	0.90483742
0.11	0.895834135	0.895834135	0.895834136	0.895834136
0.12	0.886920437	0.886920434	0.886920437	0.886920437
0.13	0.878095431	0.878095431	0.878095432	0.878095431
0.14	0.869358235	0.869358236	0.869358235	0.869358236

0.15	0.860707976	0.860707976	0.860707977	0.860707977
0.16	0.852143789	0.85214379	0.852143789	0.852143789
0.17	0.843664817	0.843664817	0.843664817	0.843664817
0.18	0.835270211	0.835270211	0.835270211	0.835270212
0.19	0.826959134	0.826959135	0.826959134	0.826959134
0.2	0.818730753	0.818730754	0.818730754	0.818730753
0.21	0.810584246	0.810584245	0.810584247	0.810584246
0.22	0.802518798	0.802518798	0.802518798	0.802518798
0.23	0.794533603	0.794533603	0.794533603	0.794533603
0.24	0.786627861	0.786627862	0.786627861	0.786627861
0.25	0.778800783	0.778800784	0.778800783	0.778800783
0.26	0.771051586	0.771051586	0.771051586	0.771051586
0.27	0.763379494	0.763379493	0.763379495	0.763379495
0.28	0.755783741	0.755783742	0.755783741	0.755783742
0.29	0.748263568	0.748263568	0.748263569	0.748263568
0.3	0.740818221	0.740818219	0.740818221	0.74081822

Table 2: Numerical Solution of Two Super Future Point for Problem 1

Time (t)	Exact Solution	Numerical Solution (k = 2)	Numerical Solution (k = 3)	Numerical Solution (k = 4)
0.01	0.990049834	0.990049834	0.990049834	0.990049835
0.02	0.980198673	0.980198672	0.980198674	0.980198674
0.03	0.970445534	0.970445533	0.970445534	0.970445534
0.04	0.960789439	0.960789447	0.960789439	0.960789439
0.05	0.951229425	0.951229424	0.951229428	0.951229423
0.06	0.941764534	0.941764535	0.941764536	0.941764539
0.07	0.93239382	0.932393822	0.932393819	0.932393821
0.08	0.923116346	0.923116343	0.923116346	0.923116347
0.09	0.913931185	0.913931185	0.913931187	0.913931185
0.1	0.904837418	0.904837418	0.904837414	0.904837418
0.11	0.895834135	0.895834136	0.895834137	0.895834134
0.12	0.886920437	0.886920436	0.886920436	0.88692044
0.13	0.878095431	0.87809543	0.878095431	0.878095431
0.14	0.869358235	0.869358235	0.869358237	0.869358236
0.15	0.860707976	0.860707977	0.860707977	0.860707976
0.16	0.852143789	0.852143789	0.852143789	0.852143789
0.17	0.843664817	0.843664817	0.843664816	0.843664816
0.18	0.835270211	0.835270211	0.835270212	0.83527021
0.19	0.826959134	0.826959135	0.826959135	0.826959134

0.2	0.818730753	0.818730757	0.818730751	0.818730754
0.21	0.810584246	0.810584246	0.810584247	0.810584246
0.22	0.802518798	0.802518798	0.802518798	0.802518798
0.23	0.794533603	0.794533603	0.794533601	0.794533602
0.24	0.786627861	0.786627862	0.786627862	0.786627865
0.25	0.778800783	0.778800783	0.778800787	0.778800783
0.26	0.771051586	0.771051585	0.771051588	0.771051587
0.27	0.763379494	0.763379493	0.763379494	0.763379494
0.28	0.755783741	0.755783744	0.755783742	0.755783742
0.29	0.748263568	0.748263567	0.748263569	0.748263568
0.3	0.740818221	0.74081822	0.740818219	0.740818225

4.1 Results Analysis and Discussion

The solutions of the schemes referred to in (9), (12), (15), (18), (21) and (24) are explored here, in order to solve the problem referred to above by calculating their absolute errors.
The results analysis is accomplished by evaluating the absolute differences between the exact and the obtained numerical solutions. The details are seen in Tables 3 and 4.

Table 3: Absolute Error of One Super Future Point for Problem 1

t	Exact Solution	k = 2 Error	k = 3 Error	k = 4 Error
0.01	0.990049834	3.50832E-10	8.50832E-10	1.45083E-09
0.02	0.980198673	2.93245E-10	5.93245E-10	2.93245E-10
0.03	0.970445534	2.05149E-09	4.51492E-10	6.51492E-10
0.04	0.960789439	1.04768E-09	1.47677E-10	5.47677E-10
0.05	0.951229425	1.99286E-10	4.99286E-10	1.59929E-09
0.06	0.941764534	1.38425E-09	8.42487E-11	4.15751E-10
0.07	0.93239382	1.39405E-09	2.94052E-10	3.94052E-10
0.08	0.923116346	3.13364E-10	1.61336E-09	1.13364E-10
0.09	0.913931185	2.57123E-09	4.28772E-10	7.12281E-11
0.1	0.904837418	4.6404E-10	1.6404E-10	1.96404E-09
0.11	0.895834135	1.03472E-10	2.03472E-10	5.03472E-10
0.12	0.886920437	2.71716E-09	1.17158E-10	2.82843E-10
0.13	0.878095431	3.20561E-10	6.79439E-10	7.94387E-11
0.14	0.869358235	1.01194E-10	3.98806E-10	2.01194E-10
0.15	0.860707976	5.25058E-10	7.49422E-11	1.74942E-10
0.16	0.852143789	1.13379E-09	1.66211E-10	4.33789E-10
0.17	0.843664817	6.03616E-10	6.03616E-10	3.03616E-10
0.18	0.835270211	2.11272E-10	3.11272E-10	1.88728E-10
0.19	0.826959134	1.15664E-09	3.56638E-10	1.56638E-10

0.2	0.818730753	4.22018E-10	1.02202E-09	4.77982E-10
0.21	0.810584246	9.70187E-10	1.02981E-09	3.29813E-10
0.22	0.802518798	2.37522E-10	2.62478E-10	1.37522E-10
0.23	0.794533603	3.96666E-10	9.6666E-11	1.96666E-10
0.24	0.786627861	5.33447E-10	2.66553E-10	3.34466E-11
0.25	0.778800783	6.28595E-10	7.14049E-11	2.85951E-11
0.26	0.771051586	3.56626E-12	2.03566E-10	5.96434E-10
0.27	0.763379494	1.03685E-09	2.63147E-10	2.63147E-10
0.28	0.755783741	6.44275E-10	3.55725E-10	2.44274E-10
0.29	0.748263568	7.85653E-11	9.21435E-10	1.21435E-10
0.3	0.740818221	1.88172E-09	8.17179E-11	2.81718E-10

Table 4 Absolute Error of Two Super Future Point for Problem 1

t	Exact Solution	k = 2 Error	k = 3 Error	k = 4 Error
0.01	0.990049834	5.50832E-10	2.50832E-10	7.50832E-10
0.02	0.980198673	1.40676E-09	8.93245E-10	8.93245E-10
0.03	0.970445534	8.48508E-10	1.51492E-10	4.51492E-10
0.04	0.960789439	7.54768E-09	1.47677E-10	2.47677E-10
0.05	0.951229425	7.00714E-10	2.99929E-09	1.50071E-09
0.06	0.941764534	1.01575E-09	2.01575E-09	5.41575E-09
0.07	0.93239382	2.09405E-09	6.05948E-10	5.94052E-10
0.08	0.923116346	3.78664E-09	1.33642E-11	3.13364E-10
0.09	0.913931185	4.71228E-10	2.12877E-09	2.71228E-10
0.1	0.904837418	6.40404E-11	4.33596E-09	2.35959E-10
0.11	0.895834135	1.10347E-09	1.30347E-09	1.39653E-09
0.12	0.886920437	6.17157E-10	3.17158E-10	3.48284E-09
0.13	0.878095431	1.02056E-09	1.20561E-10	1.79439E-10
0.14	0.869358235	4.98806E-10	1.20119E-09	6.01194E-10
0.15	0.860707976	6.74942E-10	8.74942E-10	4.25058E-10
0.16	0.852143789	6.62114E-11	4.66211E-10	3.33789E-10
0.17	0.843664817	1.03616E-10	3.96384E-10	1.96384E-10
0.18	0.835270211	4.11272E-10	2.88728E-10	1.51127E-09
0.19	0.826959134	6.56638E-10	9.56638E-10	4.56638E-10
0.2	0.818730753	4.02202E-09	1.67798E-09	5.22018E-10
0.21	0.810584246	7.01871E-11	7.29813E-10	7.01871E-11
0.22	0.802518798	1.37522E-10	6.24785E-11	3.75215E-11
0.23	0.794533603	8.96666E-10	1.10333E-09	6.03334E-10
0.24	0.786627861	5.33447E-10	7.33447E-10	3.83345E-09
0.25	0.778800783	3.28595E-10	3.8286E-09	3.28595E-10

0.26	0.771051586	9.03566E-10	1.79643E-09	9.96434E-10
0.27	0.763379494	1.33685E-09	3.68532E-11	3.68532E-11
0.28	0.755783741	2.74427E-09	4.42746E-11	4.42746E-11
0.29	0.748263568	5.78565E-10	1.02143E-09	2.14347E-11
0.3	0.740818221	8.81718E-10	1.78172E-09	3.91828E-09

5. Conclusions

In conclusion, the discrete schemes (9), (12),(15),(18),(21), and (24) were generated from their various continuous formulations and were found to be converging, P-and Q-stable. It was also noted in Tables 3 and 4 that the lower step number of this numerical approach integrated with super future points is greater than the higher step numbers implemented with super future points relative to the same solutions. It is recommended that the EBBDFM Step Number and 4 schemes are appropriate for DDE solution. Further studies for step numbers on the development of discrete schemes of EBBDFM for DDE solutions should be carried out without the use of interpolation techniques to estimate their delay arguments.

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