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# Elzaki Adomian decomposition method applied to Logistic differential model

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**Abstract.** This study applies Elzaki Adomian Decomposition Method (EADM) to solve the Logistic Differential Model (LDM) of different forms and coefficients. Illustrative examples are considered, and the obtained results are in good agreement compared to those already in the literature. This study, therefore, recommends the proposed method (EADM) for application in other aspects of applied mathematics for real-life problems.

**Keywords:** Logistic differential model, non-linear model**,** approximate solutions; Transform method

## 1. Introduction

A logistic differential equation is a conventional mathematical expression whose solution is a logistic function. Exponential functions fail to consider limitations that prevent infinite resources, whereas logistic functions do [1-3]. Many other areas, such as machine learning, chess ratings, cancer therapies (such as the modeling of tumor development), economics, and language adoption studies, rely on these types of models [4, 5]. This model is unrealistic since the environment constrains population expansion.

$$\left.\begin{array}{l} \dfrac{dP}{dt}=rP\left(1-\dfrac{P}{K}\right),\ P=P(x,t) \\ P_0=P(x,0) \end{array}\right\} \tag{1.1}$$

where $P=P(x,t)$ is the population size of the species at time $t$, $r$ denotes the rate of growth in the absence of limited resources, and $K$ denotes the carrying capacity or the maximum population that the ecosystem can support indefinitely.
The goal of this study is to apply the EADM to find a solution for the Logistic Differential Model (LDM) [6-10]. The aim is to present a simple and practical method for obtaining a better approximation to find the exact solution to the LDM. Thus, the objectives are to: apply EADM to the logistic differential equation and compare the results obtained via applying the EADM and exact solutions of the Logistic differential model. Although, numerical approaches have been arbitrated more efficient and reliable in solving the dynamical models (equations) and other differential models in this regard [11-20].

Different solution experts have recently discussed numerous methods for finding an exact or numerical solution to ordinary or partial differential models [21-26]. In this work, a novel approach termed Successive Approximation Method (SAM) is applied to some non-linear evolution models.

## 2. Note on Elzaki Adomian Decomposition Method (EADM)

The EADM will be discussed here in relation to the Logistic Differential Equation.

*2.1 Adomian Decomposition Method (ADM)*
Let us examine the differential equation of the following form:

$$Dw + Rw + Nw = g\left(x,t\right),\ w = w\left(x,t\right) \tag{2.1}$$

where the linear operator (differential) is $D$, the differential operator has a remaining part $R$ and a non-linear part, $N$, while $g = g\left(x,t\right)$ is a source term. Generally, we choose $D = \frac{d^n}{dx^n}\left(\cdot\right)$, to be the nth-order differential operator, has its inverse $D^{-1}$ follows as the nth-order integration operator. Therefore, the inverse linear $D^{-1}$ used on (2.1), we have

$$D^{-1}\left[Dw + Rw + Nw\right] = D^{-1}g\left(x,t\right) \tag{2.2}$$

where,

$$D^{-1}Dw = y - \alpha \tag{2.3}$$

and $\alpha$ signifies the initial value.
Thus, (2.3) becomes:

$$y - \alpha + D^{-1}\left[Rw + Nw\right] = D^{-1}g \tag{2.4}$$

$$y = D^{-1}g + \alpha - D^{-1}\left[Rw + Nw\right] \tag{2.5}$$

$$y = \beta\left(y\right) - D^{-1}\left[Rw + Nw\right] \tag{2.6}$$

where,

$$\beta\left(y\right) = D^{-1}g + \alpha \tag{2.7}$$

which signifies a function obtained by integrating the source term with respect to the initial condition(s). The ADM expresses the solution $y\left(t\right)$ in the series form:

$$y = \sum_{n=0}^{\infty} y_n \tag{2.8}$$

Also, the non-linear component can be stated as:

$$Nw\left(x,t\right) = \sum_{n=o}^{\infty} A_m \tag{2.9}$$

$$A_m = \frac{1}{\text{n!}}\frac{d^n}{d\lambda^n}\left(f\left(t, \sum_{k=o}^{\infty}\lambda^k y_k\right)\right)_{t=0}, n \geq 0 \tag{2.10}$$

$$\sum_{n=0}^{\infty} y_n = \beta\left(y\right) - D^{-1}\left[R\left(\sum_{n=0}^{\infty} y_n\right) + \sum_{n=0}^{\infty} A_m\right] \tag{2.11}$$

By a recursive equation, we have:

$$y_0\left(x\right) = \beta\left(x\right) \tag{2.12}$$

$$y_{n+1}\left(x\right) = -D^{-1}\left[Ry_n + A_m\right], n \geq 0. \tag{2.13}$$

Thus, the solution is:

$$y(x) = \lim_{n\to\infty}\left(\sum_{n=0}^{\infty} y_n\right). \tag{2.14}$$

*2.2 Elzaki Transform Method (ETM)*

The Elzaki transform method helps in solving differential equations (ODE) and partial (PDE) in the time domain. It is also used as an effective tool in response to initial data analyze the fundamental properties of a linear system governed by the differential equation.

*2.3. Definition of Elzaki Transform*

Let C a function such that

$$C = \left\{ H(t) : \left|H(t)\right| < Me^{|t|k_j},\ for\ \mathrm{M}, k_1, k_2 > 0 \right\}, \tag{2.15}$$

Thus, the Elzaki transform of $H(t)$ is defined and denoted as:

$$E\left[H(t)\right] = H(v) = v\int_0^{\infty} H(t)\mathrm{e}^{-\frac{t}{v}}\,dt. \tag{2.16}$$

*2.4 Properties of Elzaki Transform*

The main properties of Elzaki Transform are:

PE1: $E[1] = v^2$

PE2: $E[t] = v^3$

PE3: $E\left[\mathrm{e}^{at}\right] = \dfrac{v^2}{1-av}$

PE4: $E\left[t^n\right] = n!v^{n+2} \Rightarrow \dfrac{1}{n!}E\left[t^n\right] = v^{n+2}.$

*2.5 Elzaki Adomian Decomposition Method (EADM)*

The EADM consists of a mix of both the Elzaki transform method and the Adomian decomposition approach. The problem can either be linear or non-linear.

Let us consider the general differential equation of the form:

$$Dw + Rw + Nw = g(x,t),\ w = w(x,t) \tag{2.17}$$

where $D$, $N$, $R$, and $g$ are as defined earlier.

Suppose,

$$\begin{cases} g(x,0) = g_1^*, \\ w = w(x,t) \end{cases}$$

then the Elzaki transform of (2.17) is as follows:

$$E[Dw] + E[Rw] + E[Nw] = E[g]$$

$$E[Dw] = E[g] - E[Rw] - E[Nw]$$

$$\frac{1}{v}T(x,v) - vw = E[g] - E[Rw + Nw]$$

$$\left.\begin{aligned} T(x,v) &= v^2 w + vE[g] - vE[Rw + Nw] \\ \therefore T(x,v) &= G(x,t) - vE[Rw + Nw] \end{aligned}\right\} \tag{2.18}$$

where $G(x,t)$ is the resulting term from the source and initial condition terms when used.
Based on the inverse Elzaki transform of (2.18), we have:

$$\left.\begin{aligned} E^{-1}T(x,v) &= E^{-1}\left[G(x,t)\right] - E^{-1}\left\{vE\left[Rw + Nw\right]\right\} \\ h &= E^{-1}\left[G(x,t)\right] - E^{-1}\left\{vE\left[Rw + Nw\right]\right\} \end{aligned}\right\}. \tag{2.19}$$

Using ADM, the series solution is defined as

$$w = \sum_{n=0}^{\infty} w_n \tag{2.20}$$

And the non-linear term as:

$$\begin{cases} Nw = \sum_{n=0}^{\infty} A_n \\ A_n, \text{ as Adomian polynomials.} \end{cases} \tag{2.21}$$

Hence, (2.21) becomes

$$\left.\begin{aligned} w_1 &= g_1^{*} \\ w_{n+1} &= -E^{-1}\left[v\left(R(w_n) + (A_n)\right)\right] \end{aligned}\right\}. \tag{2.22}$$

## 3. Method and Model Discussed

This part discusses the proposed method and the Logistic model, as formulated based on some assumptions. Case examples are also considered via the EADM.
*CASE I:* Consider the following version of the LDE:

$$\begin{cases} \dfrac{dP}{dt} = \dfrac{1}{4}P(1-P) \\ P(0) = \dfrac{1}{3} \end{cases} \tag{3.1}$$

whose exact solution is:

$$P(t) = \frac{e^{0.25t}}{2 + e^{0.25t}} \tag{3.2}$$

By the EADM, we have:

$$E[P_t] = E\left[\frac{P}{4} - \frac{P^2}{4}\right] \tag{3.3}$$

$$\Rightarrow \frac{1}{v}T(x,v) - vP(x,0) = E\left[\frac{P}{4} - \frac{P^2}{4}\right] \tag{3.4}$$

$$T(x,v) = v^2 P(x,0) + vE\left[\frac{P}{4} - \frac{P^2}{4}\right] \tag{3.5}$$

Thus,

$$T(x,v) = \frac{v^2}{3} + vE\left[\frac{P}{4} - \frac{P^2}{4}\right]. \tag{3.6}$$

Taking the Elzaki inverse of (3.6) gives:

$$E^{-1}\left[T(x,v)\right] = E^{-1}\left[\frac{v^2}{3}\right] + E^{-1}\left[vE\left[\frac{P}{4} - \frac{P^2}{4}\right]\right]$$

$$\left.\begin{aligned} \Rightarrow P(x,t) &= \frac{1}{3}E^{-1}\left[v^2\right] + E^{-1}\left[vE\left[\frac{P}{4} - \frac{P^2}{4}\right]\right] \\ &= \frac{1}{3} + E^{-1}\left[vE\left[\frac{P}{4} - \frac{P^2}{4}\right]\right] \end{aligned}\right\} \tag{3.7}$$

Next, we apply the Adomian approach to (3.7), where

$$P(x,t) = \sum_{n=0}^{\infty} P_n \ .$$

Hence,

$$\sum_{n=0}^{\infty} P_n = \frac{1}{3} + \frac{1}{4}\left\{E^{-1}\left[vE\left(\sum_{n=0}^{\infty} P_n - \sum_{n=0}^{\infty} A_n\right)\right]\right\}. \tag{3.8}$$

The recursive relation is:

$$\left.\begin{aligned} P_0 &= \frac{1}{3} \\ P_{n+1} &= E^{-1}\left[vE\left(\frac{P_n}{4} - \frac{A_n}{4}\right)\right],\ n \geq 0 \end{aligned}\right\} \tag{3.9}$$

Thus, for $n = 0,1,2,3,4,5,\cdots$, the following are respectively obtained:

$$\left.P_1 = \frac{t}{18},\ P_2 = \frac{t^2}{432},\ P_3 = -\frac{t^3}{5184},\cdots\right\}$$

$$\left.\begin{aligned} P(t) &= P_0 + P_1 + P_2 + P_3 + \cdots \\ &= \frac{1}{3} + \frac{t}{18} + \frac{t^2}{432} - \frac{t^3}{5184} + \cdots \end{aligned}\right\} \tag{3.10}$$

Exact solution:

$$P^{Exact}(t) = \frac{e^{0.25t}}{2 + e^{0.25t}}.$$

For numerical results, Table 1 and Table 2 are referred.

*Table 1: Numerical results for Case II*

| $t$ | Exact Solution | $EADM - Solution : P(t)$ | Error |
|---|---|---|---|
| 0.0 | 0.3333333333 | 0.3333333333 | 0.00 |
| 0.3 | 0.3502029635 | 0.3502031250 | $1.615 \times 10^{-7}$ |
| 0.6 | 0.3674557720 | 0.3674583333 | $2.6133 \times 10^{-6}$ |
| 0.9 | 0.3850548748 | 0.3850677083 | $1.28335 \times 10^{-5}$ |

*CASE II:* Consider the following version of the LDE:

$$\begin{cases} \dfrac{dP}{dt} = \dfrac{1}{2} P(1-P) \\ p(0) = \dfrac{1}{2} \end{cases} \tag{3.11}$$

whose exact solution is:

$$P(t) = \frac{\mathrm{e}^{0.5t}}{1+\mathrm{e}^{0.5t}}.$$

By the EADM, we have:

$$\left. \begin{aligned} & E[P_t] = E\left[\frac{P}{2}\right] - E\left[\frac{P^2}{2}\right] \\ & \Rightarrow T(x,v) = v^2 P(x,0) + vE\left[\frac{P}{2} - \frac{P^2}{2}\right]. \end{aligned} \right\} \tag{3.12}$$

Thus,

$$T(x,v) = \frac{v^2}{2} + vE\left[\frac{P}{2} - \frac{P^2}{2}\right]. \tag{3.13}$$

Taking the Elzaki inverse of (3.13) gives:

$$P(x,t) = \frac{1}{2} + E^{-1}\left[vE\left[\frac{P}{2} - \frac{P^2}{2}\right]\right]. \tag{3.14}$$

Next, we apply the Adomian approach to (3.14), where

$$P(x,t) = \sum_{n=0}^{\infty} P_n .$$

Hence,

$$\sum_{n=0}^{\infty} P_n = \frac{1}{2} + \frac{1}{2}\left\{E^{-1}\left[vE\left(\sum_{n=0}^{\infty} P_n - \sum_{n=0}^{\infty} P_n^2\right)\right]\right\}.$$

Using Adomian Polynomials gives the following recursive relation:

$$\left. \begin{aligned} & P_0 = \frac{1}{2} \\ & P_{n+1} = E^{-1}\left[vE\left[\frac{P_n}{2} - \frac{A_n}{2}\right]\right] \end{aligned} \right\}.$$

Thus, for $n = 0,1,2,3,4,5,\cdots$, the following are respectively obtained:

$$\left. P_1 = \frac{t}{8}, P_2 = 0, P_3 = -\frac{t^3}{384}, P_4 = 0, \cdots \right\}$$

$$\left. \begin{aligned} P(t) &= P_0 + P_1 + P_2 + P_3 + \cdots \\ &= \frac{1}{2} + \frac{t}{8} - \frac{t^3}{384} + \cdots \end{aligned} \right\} \tag{3.15}$$

$$P^{Exact}(t) = \frac{e^{0.5t}}{1+e^{0.5t}}.$$

*Table 2: Numerical results for Case II*

| $t$ | Exact Solution | $EADM - Solution: P(t)$ | Error |
|---|---|---|---|
| 0.0 | 0.5 | 0.5 | 0 |
| 0.3 | 0.5374298453 | 0.5374296875 | $1.578\times10^{-7}$ |
| 0.6 | 0.5744425217 | 0.5744375000 | $5.0217\times10^{-6}$ |
| 0.9 | 0.6106392339 | 0.6106015625 | $3.7671\times10^{-5}$ |

## 4. Conclusions

In this work, the Elzaki Adomian Decomposition Method (EADM) was applied to the non-linear differential equation known as the Logistic Differential Model. The EADM has an advantage in its applicability, speed of convergence, and accuracy, unlike other numerical methods. Applying the EADM yields a series solution. The EADM is a very effective tool in the solution of the Logistic Differential Model. It can also be applied to several other more complex ordinary differential equations (both linear and non-linear). The results have shown distinctive characteristics of the method in terms of effectiveness and speed of accuracy. The EADM does not require linearization and initial guess points.

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