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## Closed form root of a linear Klein–Gordon equation

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**Abstract.** In this paper, solution of the linear version of Klein-Gordon equation is considered via the application of natural transform combined with decomposition method. Hereafter, referred to as natural decomposition method (NDM). This proposed method shows viable improvement and reliability in usage compared to the classical natural transform. Illustrative example(s) are considered, and the solution (root) is shown to follow a closed form. Therefore, the NDM is recommended for highly nonlinear differential models both in pure and applied sciences.

**Keywords:** Analytical solutions; Decomposition Method; Klein–Gordon model, Closed form solutions

### 1. Introduction

In applied mathematics, Klein-Gordon Equation (KGE) is one of the most vital of partial differential equations (PDE). The Klein-Gordon equation plays a significant role in mathematical physics in relation to the study of solitons, condensed matter physics, and so on. Its theoretical relevance is similar to that of the Dirac equation [1-5]. The modeling pattern of the KGE can result to linear and nonlinear models whose solutions, if they exist are not easy to obtain. Hence, the need for effective and reliable semi-approximate methods of solution such as decomposition, numerical, and integral transform methods [6-17]. The general form of the KGE to be considered in this work is of the form:

$$\begin{cases} \eta_{tt} + a\eta_{xx} = -F(\eta), & \eta = \eta(x, t) \\ \eta(x, 0) = z_1(x), \\ \eta_t(x, 0) = z_2(x), \end{cases} \quad (1)$$

where  $F(\eta)$  denotes a known analytic function, and  $a \in (-\infty, \infty)$ .

In recent times, the KGE (1) has drawn the attention of so many authors with respect to solution methods [18-24]. This present work aims at combining decomposition method with natural transform method for the solutions of the linear KGE.



## 2. Natural Transform and Natural Decomposition Method

The preliminaries of Natural Transform (NT), and its basic properties are given in this section [25-27]. Let  $H$  be a class of functions such that:

$$H = \left\{ g(t) : \exists c, k_1, k_2 > 0 \ni |g(t)| < ce^{k|k_i|} \right\}, \quad (2)$$

then, the natural transform of  $g(t)$  is defined and denoted as:

$$N[g(t)] = Q(s, \varphi) = \int_0^{\infty} g(\varphi t) e^{-st} dt, \quad t \in [0, \infty) \quad (3)$$

provided the integral in the right hand side exists. As a consequence, the Inverse Natural Transform (INT) associated with (3) is defined and denoted as:

$$N^{-1}\{N[g(t)]\} = N^{-1}\{Q(s, \varphi)\} = g(t). \quad (4)$$

In general, it is remarked that:

$$N[t^n] = \frac{n! \varphi^n}{s^{n+1}}, \quad n \geq 0. \quad (5)$$

### 2.1 Natural Transform of Derivatives

For a continuous function,  $g(x, t)$  in  $A$  as defined earlier, we have the following:

$$D1: \begin{cases} N[g_t] = \frac{s}{\varphi} Q(s, \varphi) - \frac{g(x, 0)}{\varphi} \\ N\left[\frac{\partial^n g}{\partial t^n}\right] = \frac{s^n}{\varphi^n} Q(s, \varphi) - \sum_{j=0}^{n-1} \frac{s^{n-j-1}}{\varphi^{n-j}} g^{(j)}(x, 0). \end{cases}$$

$$D2: \quad N\left[\frac{\partial^n g}{\partial x^n}\right] = \frac{\partial^n}{\partial x^n} [Q(s, \varphi)].$$

### 2.2 Natural Decomposition Method

Let a general nonlinear nonhomogeneous partial differential equation be defined as:

$$\begin{cases} D\phi(x, t) + R\phi(x, t) + \Pi\phi(x, t) = m(x, t) \\ \phi(x, 0) = h(x) \end{cases} \quad (6)$$

where  $D$  is an  $n^{\text{th}}$  order differential operator in  $t$ ,  $R$  is the remaining part of the linear differential operator,  $\Pi$  and  $m(x, t)$  are nonlinear differential operator and source term respectively.

So taking the natural transform of (6) gives:

$$N[D\phi(x, t)] = N[m(x, t)] - N[(R\phi(x, t) + \Pi\phi(x, t))] \quad (7)$$

$$\Rightarrow \frac{s^n}{\varphi^n} Q(s, \varphi) - \sum_{j=0}^{n-1} \frac{s^{n-j-1}}{\varphi^{n-j}} g^{(j)}(x, 0) = N[m(x, t)] - N[(R\phi(x, t) + \Pi\phi(x, t))] \quad (8)$$

Showing that:

$$Q(s, \varphi) = \frac{\varphi^n}{s^n} \left\{ \sum_{j=0}^{n-1} \frac{s^{n-j-1}}{\varphi^{n-j}} g^{(j)}(x, 0) + N[m(x, t)] - N[(R\phi(x, t) + \Pi\phi(x, t))] \right\}. \quad (9)$$

So, taking the INT  $L_t^{-1}(\cdot)$  on both sides of (9) gives:

$$\begin{cases} \phi(x, t) = \lambda(x, t) - N^{-1} \left\{ \frac{\varphi^n}{s^n} N[(R\phi(x, t) + \Pi\phi(x, t))] \right\}, \\ \lambda(x, t) = N^{-1} \left\{ \frac{\varphi^n}{s^n} \left( \sum_{j=0}^{n-1} \frac{s^{n-j-1}}{\varphi^{n-j}} g^{(j)}(x, 0) + N[m(x, t)] \right) \right\}. \end{cases} \quad (10)$$

According to Adomian and its polynomials, the solution and the non-linear part are:

$$\begin{cases} \phi(x, t) = \sum_{i=0}^{\infty} \phi_i(x, t), \\ \Pi\phi(x, t) = \sum_{i=0}^{\infty} A_i. \end{cases} \quad (11)$$

and  $A_i$  defined as:

$$A_j = \frac{1}{j!} \frac{\partial^j}{\partial \lambda^j} \left[ \Pi \left( \sum_{i=0}^j \lambda^i \phi_i \right) \right]_{\lambda=0}. \quad (12)$$

Thus, (10) becomes:

$$\sum_{i=0}^{\infty} \phi_i(x, t) = \lambda(x, t) - N^{-1} \left\{ \frac{\varphi^n}{s^n} N \left[ \left( R \left( \sum_{i=0}^{\infty} \phi_i(x, t) \right) + \left( \sum_{n=0}^{\infty} A_i \right) \right) \right] \right\}. \quad (13)$$

Therefore, the solution,  $\phi(x, t)$  is obtained using (14) as follows:

$$\begin{cases} \phi_0 = N^{-1} \left\{ \frac{\varphi^n}{s^n} \left( \sum_{j=0}^{n-1} \frac{s^{n-j-1}}{\varphi^{n-j}} g^{(j)}(x, 0) + N[m(x, t)] \right) \right\} \\ \phi_{k+1} = -N^{-1} \left\{ \frac{\varphi^n}{s^n} N[(R(\phi_k) + A_k)] \right\}, k \geq 0. \end{cases} \quad (14)$$

Whence,  $\phi(x, t)$  is finalized as:

$$\phi(x, t) = \lim_{N \rightarrow \infty} \sum_{k=0}^N \phi_k. \quad (15)$$

### 3. Applications

**Case I:** Linear KGE of the following form is considered [1, 24]:

$$\begin{cases} \eta_{tt} - \eta_{xx} = \eta, \\ \eta(x, 0) = 1 + \sin x, \\ \eta_t(x, 0) = 0. \end{cases} \quad (16)$$

The exact solution of (16) is:

$$\eta(x, t) = \sin x + \cosh t. \quad (17)$$

**Procedure w.r.t Case 1:**

By applying the N-transform to (16), we have:

$$N[\eta_t] = N[\eta_{xx} + \eta]. \quad (18)$$

$$\Rightarrow \frac{s^2}{\omega^2} Q(s, \omega) - \frac{s\eta(s, 0)}{\omega^2} - \frac{\eta_t(s, 0)}{\omega} = \bar{N}[\eta_{xx} + \eta]. \quad (19)$$

$$\begin{aligned} \therefore Q(s, \omega) &= \frac{\omega^2}{s^2} \left\{ \frac{s}{\omega^2} \eta(s, 0) + \frac{1}{\omega} \eta_t(s, 0) + N[\eta_{xx} + \eta] \right\} \\ &= \left\{ \frac{1}{s} (1 + \sin x) + \frac{\omega^2}{s^2} N[\eta_{xx} + \eta] \right\}. \end{aligned} \quad (20)$$

Applying the N-inverse,  $N^{-1}[\cdot]$  and the initial condition to (20) gives:

$$\eta(x, t) = (1 + \sin x) N^{-1} \left\{ \frac{1}{s} \right\} + N^{-1} \left\{ \frac{\omega^2}{s^2} N[\eta_{xx} + \eta] \right\}. \quad (21)$$

Hence, the solution in Adomian series form is expressed as:

$$\eta(x, t) = \eta = \sum_{n=0}^{\infty} \eta_n. \quad (22)$$

So, the recursive relation is:

$$\begin{cases} \eta_0 = (1 + \sin x) \\ \eta_{n+1} = N^{-1} \left\{ \frac{\omega^2}{s^2} N[(\eta_n)_{xx} + \eta_n] \right\}. \end{cases} \quad (23)$$

As such, for  $n \geq 1$ , we have the following:

$$\begin{aligned} &\left\{ \begin{aligned} \eta_1 &= N^{-1} \left\{ \frac{\omega^2}{s^2} N[(\eta_0)_{xx} + \eta_0] \right\} \\ &= N^{-1} \left\{ \omega^2 s^{-3} \right\} = \frac{t^2}{2!} \end{aligned} \right\}, \left\{ \begin{aligned} \eta_2 &= N^{-1} \left\{ \frac{\omega^2}{s^2} N[(\eta_1)_{xx} + \eta_1] \right\} \\ &= N^{-1} \left\{ \omega^4 s^{-5} \right\} = \frac{t^4}{4!} \end{aligned} \right\}, \\ &\left\{ \begin{aligned} \eta_3 &= N^{-1} \left\{ \frac{\omega^2}{s^2} N[(\eta_2)_{xx} + \eta_2] \right\} \\ &= N^{-1} \left\{ \omega^6 s^{-7} \right\} = \frac{t^6}{6!} \end{aligned} \right\}, \left\{ \begin{aligned} \eta_4 &= N^{-1} \left\{ \frac{\omega^2}{s^2} N[(\eta_3)_{xx} + \eta_3] \right\} \\ &= N^{-1} \left\{ \omega^8 s^{-9} \right\} = \frac{t^8}{8!} \end{aligned} \right\}, \\ &\left\{ \begin{aligned} \eta_5 &= N^{-1} \left\{ \frac{\omega^2}{s^2} N[(\eta_4)_{xx} + \eta_4] \right\} \\ &= N^{-1} \left\{ \omega^{10} s^{-11} \right\} = \frac{t^{10}}{10!} \end{aligned} \right\}, \dots, \left\{ \begin{aligned} \eta_p &= N^{-1} \left\{ \frac{\omega^2}{s^2} N[(\eta_{p-1})_{xx} + \eta_{p-1}] \right\} \\ &= N^{-1} \left\{ \omega^{2p} s^{-2(p+1)} \right\} = \frac{t^{2p}}{(2p)!} \end{aligned} \right\}, p \geq 1. \end{aligned}$$

Therefore,

$$\begin{aligned}
 \eta(x,t) &= \lim_{N \rightarrow \infty} \sum_{j=0}^N \eta_j = 1 + \sin x + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \frac{t^8}{8!} + \frac{t^{10}}{10!} + \dots \\
 &= \sin x + \left( 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \frac{t^8}{8!} + \frac{t^{10}}{10!} + \dots \right) \\
 &= \sin x + \cosh t.
 \end{aligned} \tag{24}$$

Equation (24) corresponds to the exact solution of the classical KGE obtained in [1, 24]. The graphical solutions are presented in Figure 1 and Figure 2 for exact solution and approximate solution respectively.

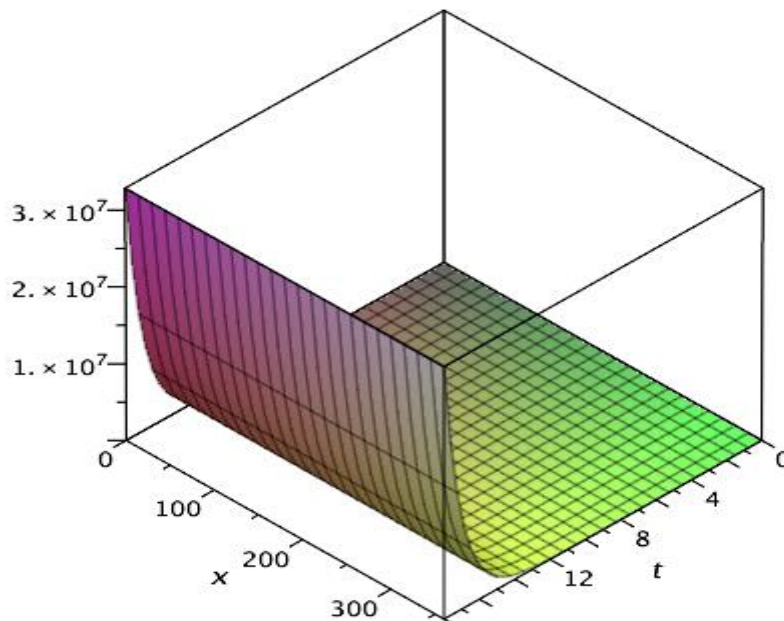


Figure 1: NDM Exact solution of Case 1

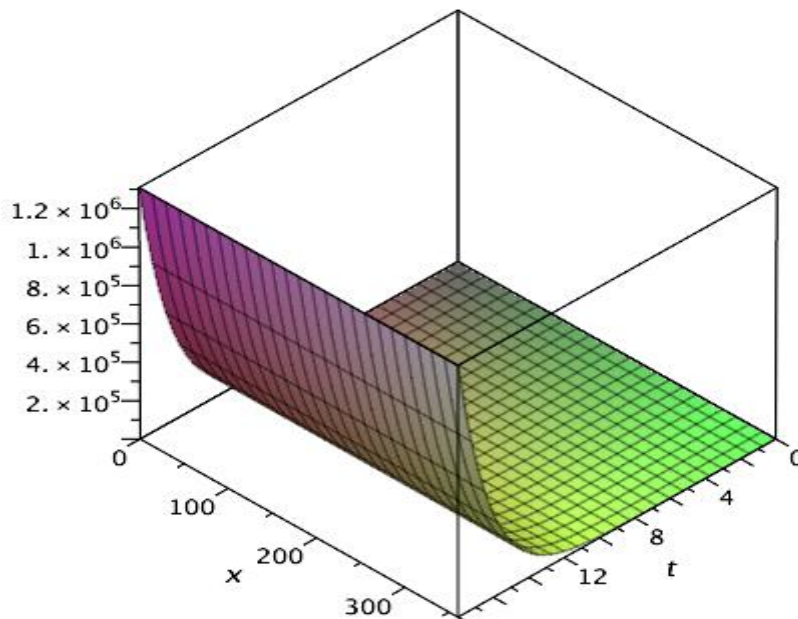


Figure 2: NDM 4-term Approximate solution Case 1

#### 4. Conclusions

This work presented the application of Natural Decomposition Method (NDM) to the linear version of Klein-Gordon equation for approximate-analytical solutions. The problem is solved without a call for variable-discretization. The obtained result showed that the NDM is effective and reliable. The solutions were expressed in closed form with less computational time involvement. Thus, the NDM is recommended for highly nonlinear Klein-Gordon equation and other related differential models in applied sciences.

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