Time-fractional classical Black-Scholes option pricing model via He-separation of variable transformation method for exact solutions

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Time-fractional classical Black-Scholes option pricing model via He-separation of variable transformation method for exact solutions

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Abstract. This paper considers the exact solutions of the time-fractional classical Black-Scholes option pricing model through the application of a method known as He-Separation of Variable Transformation Method (HSVTM). The HSVTM combines the basic properties of the He's polynomials, the Homo-separation variable, the modified DTM, which increases the efficiency and effectiveness of the proposed method. The proposed method is direct and straight forward. Hence, it is recommended for obtaining solutions to financial models resulting from either Ito or Stratonovich Stochastic Differential Equations (SDEs).

1. Introduction

The Black-Scholes Pricing Model (BSPM) for European option pricing and valuation plays a notable role in risk and portfolio management [1-11]. Though, some of the BSM underlying assumptions when relaxed leads to more complex and nonlinear versions. Hence, the need for effective and efficient numerical, semi-approximate methods of solution. The solution of the Black-Scholes model is used for describing the value of option mainly of European type. The solution solves the model of the form:

$$\frac{\partial f}{\partial \tau}+\frac{1}{2}S^2\sigma^2\frac{\partial^2 f}{\partial S^2}+rS\frac{\partial f}{\partial S}-rf=0 \tag{1}$$

with the following as defined: $f=f\left(S,\tau\right)$ represents the value of the underlying S, at a particular time, τ such that $\tau\in\left[0,t\right]$, $f\in C^{2,1}\left[R\times[0,T]\right],\left(S,\tau\right)\in R^{+}\times\left(0,T\right)$, for the underlying asset $S=S(t)$, the volatility is σ, r is taken as the risk-free interest rate, meanwhile, the maturity time is T.

Other solution methods suffice; for instance, the Picard iteration method (PIM) was initiated by a group of people Augustin–Louis Cauchy, Emile Picard, Rudolf Lipschitz, Ernst Lindelöf. The theorem is imperative in the existence and uniqueness of first-order equations solutions when given initial

conditions. This method has been used for many years and has been proven to be a promising method in modeling. The Picard iteration method (PIM) was used by several researchers [5, 6].
In this work, we will look at a generalization of (1) regarding fractional order in terms of real and complex order of the derivatives. This will be regarded as a non-integer (time-fractional) Black-Scholes model (TFBSM) following the form:

$$\frac{\partial^\alpha \Psi}{\partial \tau^\alpha}+m_1\left(S,\delta\right)\frac{\partial^2\Psi}{\partial S^2}+m_2\left(S,r\right)\frac{\partial \Psi}{\partial S}=r\Psi \text{ , } \alpha>0 \tag{2}$$

subject to an attributed initial or boundary conditions, $m_i\left(\cdot,\cdot\right)$, $i=0,1,2,3,\cdots$, are non-zero functions.
Numerical methods and a lot of solution methods have been considered for solving related problems [12-18]. He's polynomials method was initiated in [19, 20] by Ghorbani *et al.* Recently, Ghandehari and Ranjbar [21] presented the exact solution of option pricing model built on Fractional Black-Scholes (FBS) equation by means of a modified Homotopy Perturbation Method (HPM). In their method [21], they obtained the exact solutions basically with the aid of green function by combining the separation of variables method with HPM. Ouafoudi and Gao [22] introduced two solution methods viz: modified HPM and Homotopy Perturbation combined with Sumudu transform for handling the same option pricing model as considered in [21]. Both views of [21] and [22] required the application of the green function. The new approach in this present work aims at providing exact solutions to the time-fractional classical Black-Scholes option pricing model through the He-Separation of Variable Transformation Method (HSVTM). The HSVTM combines the basic features of the He's polynomials, the Homo-separation variable, and the modified Differential Transform Method without the concept and application of the green function. Here, the fractional derivative is defined in the sense of Caputo.

2. Remarks on the He's Polynomial Solution Method

Suppose a general form is considered as follows:

$$\Lambda\left(\psi\right)=0 \tag{3}$$

for a differential or an integral operator, Λ and $H\left(\psi, p\right)$ denotes a convex homotopy given as:

$$H\left(\psi,p\right)=p\Lambda\left(\psi\right)+\left(1-p\right)\Omega\left(\psi\right) \tag{4}$$

where $\Omega\left(\psi\right)$ is a known operator (functional) with ψ_0 as a solution. Therefore, we get:

$$\left.\begin{aligned} H\left(\psi,0\right)&=\Omega\left(\psi\right)\\ H\left(\psi,1\right)&=\Lambda\left(\psi\right)\end{aligned}\right\} \tag{5}$$

where the parameter $p\in\left(0,1\right]$ is embedded. According to HPM in [19, 20], the parameter, p is used in the expansion of:

$$\psi=\sum_{j=0}^{\infty}p^j\psi_j=\psi_0+p\psi_1+p^2\psi_2+\cdots. \tag{6}$$

From (6) we have the solution as $p\to1$. Though, the convergence of (2.4) as $p\to1$ has already been considered in [23]. Related convergence theorems are referred [24, 25].
The method considers $N\left(\psi\right)$ as the nonlinear term given as:

$$N(\psi)=\sum_{j=0}^{\infty} p^j H_j = H_o + p^1 H_1 + p^2 H_2 + p^3 H_3 + \cdots \tag{7}$$

where the He's polynomials, H_k's can be obtained using:

$$H_k(\psi_0,\psi_1,\psi_2,\psi_3,\cdots,\psi_k)=\frac{1}{k!}\frac{\partial^k}{\partial p^k}\left(N\left(\sum_{j=0}^{k} p^j\psi_j\right)\right)_{p=0},\ n\ge 0. \tag{8}$$

3. Applications

In this section, the following time-fractional Black-Scholes equation is considered.
Problem 1: A linear Black-Scholes equation of the following form is considered:

$$\frac{\partial^\alpha w}{\partial t^\alpha}+x^2\frac{\partial^2 w}{\partial x^2}+\frac{1}{2}x\frac{\partial w}{\partial x}-w=0 \tag{9}$$

subject to:

$$w(x,0)=\max\left(x^3,0\right)=\begin{cases} x^3, \text{ for } x>0 \\ 0, \text{ for } x\le 0\end{cases}. \tag{10}$$

Procedure w.r.t Problem 1:
Choose $w_*(x,t)$ as an initial approximation to (9) such that:

$$w_*(x,t)=\max\left(x^3,0\right)\lambda_1(t)+3\max\left(x^2,0\right)\lambda_2(t). \tag{11}$$

Hence, (3.1) becomes:

$$\begin{aligned}
0 &= \frac{\partial^\alpha}{\partial t^\alpha}\left(\max\left(x^3,0\right)\lambda_1(t)+3\max\left(x^2,0\right)\lambda_2(t)\right) \\
&\quad + x^2\frac{\partial^2}{\partial x^2}\left(\max\left(x^3,0\right)\lambda_1(t)+3\max\left(x^2,0\right)\lambda_2(t)\right) \\
&\quad + \frac{1}{2}x\frac{\partial}{\partial x}\left(\max\left(x^3,0\right)\lambda_1(t)+3\max\left(x^2,0\right)\lambda_2(t)\right) \\
&\quad - \left(\max\left(x^3,0\right)\lambda_1(t)+3\max\left(x^2,0\right)\lambda_2(t)\right)
\end{aligned} \tag{12}$$

$$\begin{aligned}
&= \frac{\partial^\alpha}{\partial t^\alpha}\left(\max\left(x^3,0\right)\lambda_1(t)+3\max\left(x^2,0\right)\lambda_2(t)\right)+x^2\frac{\partial^2}{\partial x^2}\left(\max\left(x^3,0\right)\lambda_1(t)+3\max\left(x^2,0\right)\lambda_2(t)\right) \\
&\quad +\frac{1}{2}x\frac{\partial}{\partial x}\left(\max\left(x^3,0\right)\lambda_1(t)+3\max\left(x^2,0\right)\lambda_2(t)\right)-\left(\max\left(x^3,0\right)\lambda_1(t)+3\max\left(x^2,0\right)\lambda_2(t)\right).
\end{aligned}$$

$$\Rightarrow \left\{\begin{aligned}
0 &= \left(x^3\frac{d^\alpha}{dt^\alpha}\lambda_1(t)+3x^2\frac{d^\alpha}{dt^\alpha}\lambda_2(t)\right)+\left(6x^3\lambda_1(t)+6x^2\lambda_2(t)\right) \\
&\quad +\frac{3x^3}{2}\lambda_1(t)+3x^2\lambda_2(t)-x^3\lambda_1(t)-3x^2\lambda_2(t)
\end{aligned}\right.$$

$$=\left(x^3\frac{d^\alpha}{dt^\alpha}\lambda_1(t)+3x^2\frac{d^\alpha}{dt^\alpha}\lambda_2(t)\right)+6x^2\lambda_2(t)+\frac{13x^3}{2}\lambda_1(t).$$

Thus,

$$0\equiv x^3\left(\frac{d^\alpha}{dt^\alpha}\lambda_1(t)+\frac{13}{2}\lambda_1(t)\right)+3x^2\left(\frac{d^\alpha}{dt^\alpha}\lambda_2(t)+2\lambda_2(t)\right). \tag{13}$$

We therefore obtain the FODE system:

$$\begin{cases} \frac{d^{\alpha}}{dt^{\alpha}}\lambda_1(t)+\frac{13}{2}\lambda_1(t)=0 \\ \lambda_1(0)=1 \end{cases}, \tag{14}$$

$$\begin{cases} \left(\frac{d^{\alpha}}{dt^{\alpha}}\lambda_2(t)+2\lambda_2(t)\right)=0 \\ \lambda_2(0)=0 \end{cases}. \tag{15}$$

From (15), it is obvious that $\lambda_2(t)=0$. But solving (14) using the transformation properties [14] with $\Xi_1(h)$ as the differential transform of $\lambda_1(t)$ gives:

$$\Xi_1(1+h)=\frac{\Gamma(1+\alpha h)}{\Gamma(1+\alpha(1+h))}\left(-6.5\Xi_1(h)\right). \tag{16}$$

Thus,

$$\Xi_1(p)=\frac{(-6.5)^p}{\Gamma(1+p\alpha)},\ p\geq 1. \tag{17}$$

$$\Rightarrow \begin{aligned} \lambda_1(t)&=\sum_{p=0}^{\infty}\Xi_1(p)t^{\alpha p} \\ &=\sum_{p=0}^{\infty}\frac{\left(-6.5t^{\alpha}\right)^p}{\Gamma(1+p\alpha)}=E_{\alpha}\left(-6.5t^{\alpha}\right). \end{aligned} \tag{18}$$

So, using (15) and (16) in (17) gives:

$$\begin{aligned} w_*(x,t)&=\max\left(x^3,0\right)\sum_{p=0}^{\infty}\frac{\left(-6.5t^{\alpha}\right)^p}{\Gamma(1+p\alpha)} \\ &=\max\left(x^3,0\right)E_{\alpha}\left(-6.5t^{\alpha}\right)=x^3E_{\alpha}\left(-6.5t^{\alpha}\right). \end{aligned} \tag{19}$$

4. Conclusions

This paper presented the exact solutions of the time-fractional classical Black-Scholes option pricing model through the He-Separation of Variable Transformation Method (HSVTM) by combing the basic properties of the He's polynomials, the Homo-separation variable, and the modified DTM. The engendered fractional derivative is defined in the sense of Caputo. The HSVTM is direct and simple in application, and no knowledge of green function, linearization, or Lagrange multiplier is required. Hence, it is recommended for obtaining solutions to financial models resulting from either Ito or Stratonovich Stochastic Differential Equations (SDEs).

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