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Fractional coupled decomposition approach for the solution of a linear Klein–Gordon equation

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Abstract. In this article, the solution of the time-fractional linear Klein-Gordon Equation (TFLKGE) is considered. This is in combination with Laplace Decomposition Method (LDM) and Fractional Complex Transform (FCT), herein referred to as FCLDM. In contrast with the standard LDM, this proposed approach demonstrates a feasible improvement and efficiency in use. Considerations are made Illustratively using some examples, and a closed solution of the problem is obtained with ease. The FCLDM is recommended for physical modeling of real-life problems both in pure and applied sciences and for strongly non-linear differential models.

Keywords: Caputo fractional derivative, Fractional Klein-Gordon equation, Fractional order, Laplace decomposition

1 Introduction

One of the crucial tools for seeking the solution of differential equations is the mathematical solution method [1]. The solutions obtained from the differential equation are used as stability analysis, robustness analysis, and so on, even for analytical controllability whose aim is to analyze and control the model involved [2-4]. The Klein-Gordon equation (KGE) proposed by Oscar Klein and Walter Gordon in 1927 is an interesting differential equation [5-8]. KGEs are a simple class of equations of non-linear progression that exist in classical relativistic and quantum mechanics. The

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analysis of solitons and the study of condensed matter has gained a lot of publicity. The KGE is capable of observing quantum motion, actual plasma distribution, and the acceleration of other waves. Many approaches for solving Klein-Gordon equations have been used in recent years, such as the Variational Iteration Method (VIM), Sumudu Decomposition Method (SDM), Laplace Decomposition Method (LDM), Natural Decomposition Method (NDM), Homotopy Perturbation Method (HPM), Sumudu Transformation Method (STM), Adomian Decomposition Method (ADM), and so on [9-17]. The availability of exact or numerical solutions to linear and non-linear differential equations has contributed to direct and semi-analytical methods being developed and implemented [18-26]. In the analysis and simulation of many of the realism problems that arise in applied mathematics and physics, fractional partial differential equations (FPDEs) are frequently used, including fluid dynamics, electrical circuits, induction, damping rules, mathematical biology relaxation processes [18-29]. Fractional derivatives include real-world issues with more detailed representations than integer-order derivatives; they are currently considered an effective method for explaining some physical problems. A critically relevant, useful branch of mathematics, the subject of fractional calculus plays a severe and crucial role in defining a complicated dynamic activity in a wide variety of application fields, helps to understand the essence of matter as well, as simplifying the control design without losing inherited behaviours and demonstrating much more complex structures.

The general time-fractional Klein–Gordon equation to be considered in this work is of the form:

$$\begin{cases} \dfrac{\partial^{2\alpha}\xi}{\partial t^{2\alpha}} = \dfrac{\partial^2 \xi}{\partial x^2} + \sum_{i=1}^{3} \eta_i \xi^i, \ \xi = \xi(x,t), \ 0 < \alpha \le 1 \\ \xi(x,0) = f_1(x), \xi'(x,0) = f_2(x). \end{cases} \tag{1.1}$$

where η_i are constants, α represents the time-fractional order of the derivative, $\xi = \xi(x,t)$ is an unknown analytical function to be determined. The classical Klein–Gordon equation can easily be obtained from (1.1) when $\alpha = 1$.

2 Basic Remarks and Preliminaries

We make some vital remarks as follows.

Remark 1: Jumarie's Fractional Derivative (JFD): Suppose $g(w)$ is a well-defined and continuous real function of w and $D_w^\alpha g = \dfrac{\partial^\alpha g}{\partial w^\alpha}$ denotes Jumarie's Fractional Derivative of $g = g(w)$, of order α w.r.t. w. Then,

$$D_w^{\alpha} g = \begin{cases} \dfrac{1}{\Gamma(-\alpha)} \dfrac{d}{dw} \displaystyle\int_0^w (w-\eta)^{-\alpha-1} \left(g(\eta) - g(0)\right) d\eta, \ \alpha \in (-\infty, 0) \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dw} \displaystyle\int_0^w (w-\eta)^{-\alpha} \left(g(\eta) - g(0)\right) d\eta, \ \alpha \in (0,1) \\ \left(g^{(\alpha-m)}(w)\right)^{(m)}, \ \alpha \in [m, m+1), \ m \geq 1 \end{cases} \tag{2.1}$$

here, $\Gamma(\cdot)$ denotes a gamma function, see details and additional properties in [19], the basic features of JFD are stated as follows via *B1-B5*:

B1: $D_w^{\alpha} c = 0, \ \alpha > 0, \ c$ is a constant,

B2: $D_w^{\alpha}\left(cg(w)\right) = cD_w^{\alpha} g(w), \ \alpha > 0$,

B3: $D_w^{\alpha} w^{\beta} = \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} w^{\beta-\alpha}, \ \beta \geq \alpha > 0,$

B4: $D_w^{\alpha}\left(g_1(w) g_2(w)\right) = D_w^{\alpha} g_1(w)\left(g_2(w)\right) + g_1(w) D_w^{\alpha} g_2(w)$,

B5: $D_w^{\alpha}\left(g\left(w(f)\right)\right) = D_w^1 g \cdot D_f^{\alpha} w$,

Remark 2: FCT and the generalized differential model: Consider a given general time-fractional differential model of the form:

$$h\left(w, D_t^{\alpha} w, D_x^{\beta} w\right) = 0, \ w = w(t,x). \tag{2.2}$$

Then, for some unknown positive constants, a_1 and a_2, the FCT [22] is defined as follows:

$$\begin{cases} T = \dfrac{a_1 t^{\alpha}}{\Gamma(\alpha+1)}, \ \alpha \in (0,1], \\ X = \dfrac{a_2 x^{\beta}}{\Gamma(\beta+1)}, \ \beta \in (0,1], \end{cases} \tag{2.3}$$

such that:

$$\begin{cases} D_t^{\alpha} w(t,x) = D_t^{\alpha} w\left(T(t)\right) = D_T^1 w \cdot D_t^{\alpha} T = a_1 \dfrac{\partial w}{\partial T}, \\ D_x^{\beta} w(t,x) = D_x^{\beta} w\left(X(x)\right) = D_X^1 w \cdot D_x^{\beta} X = a_2 \dfrac{\partial w}{\partial X}, \end{cases}. \tag{2.4}$$

2.1 Laplace Transformation Decomposition

The LADM incorporates the Laplace method and the classical Adomian method of decomposition. With a few iterations, the process gives the solution directly. Consider the general non-linear partial differential equation in general first order: form:

$$\begin{cases} Dp(x,t) + Rp(x,t) + Np(x,t) = g(x,t) \\ g(x,0) = g_1^*, p = p(x,t) \end{cases} \tag{2.5}$$

here D and $\boldsymbol{R}$ are differential operators, while N is the non-linear part of the differential operator and $g = g(x,t)$ is a source term. Equation (2.5) is restructured in the following way:

$$Dp(x,t) = g(x,t) - (Rp(x,t) + Np(x,t)) \tag{2.6}$$

With reference to details in [19], we have:

$$\begin{cases} p = p(x,0) + L^{-1}\left\{ s^{-1}\{L\}g\right\} - L^{-1}\left\{ s^{-1}\{L\ Np\}R\right\} \\ g = g(x,t),\ p = p(x,t). \end{cases} \tag{2.7}$$

The proposed LADM gives the solution to the problem in an infinite series form:

$$p = \sum_{n=0}^{\infty} p_n, \quad p(\cdot) = p(x,t). \tag{2.8}$$

Though, $p_n(x,t)$ is being obtained recursively, $Np(x,t)$ (the non-linear term) is given as:

$$N(p(x,t)) = \sum_{b=0}^{\infty} A_b(p_0, p_1, p_2, \cdots, p_n) \tag{2.9}$$

where A_b denotes the Adomian polynomials given thus:

$$A_c = \frac{1}{c!}\frac{d^c}{d\xi^c}\left(N\left(t, \sum_{k=0}^{c} \xi^k p_k \right)\right)_{\xi=0}, c \geq (\tag{2.10}$$

The recurrence relation form:

$$\begin{cases} p_0 = L^{-1}\left\{ s^{-1}L\{p(x,t)\}\right\} + p(x,0) \\ p_{i+1} = -L^{-1}\left\{ \frac{1}{s} L\{(R(p_i) + NA_i)\}\right\}, i \geq 0. \end{cases} \tag{2.11}$$

3.0 Applications: FCLDM and linear Klein–Gordon equation

In this section, the proposed approach/method is applied to the linear Klein–Gordon equation as follows:

Now, suppose we consider a case where $\eta_1 = 1$, and $\eta_2 = \eta_3 = 0$, the (1.1) becomes:

$$\begin{cases} \frac{\partial^\beta \xi}{\partial t^\beta} = \frac{\partial^2 \xi}{\partial x^2} + \xi, \ \xi = \xi(x,t), \ 0 < \beta \le 2 \\ \xi(x,0) = f_1(x) = 1 + \sin x. \end{cases} \tag{3.1}$$

Solution procedure: By FCT,

$$T = \frac{a_1 t^\beta}{\Gamma(1+\beta)},$$

Hence, we have (according to section 3)

$$\frac{\partial^\beta \xi}{\partial t^\beta} = \frac{\partial \xi}{\partial T} \ \text{ for } \ a_1 = 1. \tag{3.2}$$

Thus, (4.1) becomes:

$$\begin{cases} \frac{\partial \xi}{\partial T} = \frac{\partial^2 \xi}{\partial x^2} + \xi, \ \xi = \xi(x,T), \ 0 < \beta \le 2 \\ \xi(x,0) = 1 + \sin x. \end{cases} \tag{3.3}$$

Thus, by applying the Laplace transform Decomposition Method presented in section 2 to (3.3), the solution of (3.3) is given as:

$$\left. \begin{aligned} \xi(x,T) &= \sin(x) + \left(1 + T + \frac{T^2}{2!} + \frac{T^3}{3!} + \frac{T^4}{4!} + \cdots\right) \\ &= \sin(x) + \exp(T) \end{aligned} \right\} \tag{3.4}$$

Hence, the corresponding exact solution of (3.1) is:

$$\left. \xi(x,t) = \sin(x) + \exp\left(\frac{t^\beta}{\Gamma(1+\beta)}\right) \right\}. \tag{3.5}$$

4 Conclusion

This study has applied the adaptation of the Fractional Coupled Laplace Decomposition Method (FCLDM) to the linear variant of the time-fractional Klein-Gordon equation in terms of approximate -analytical solutions by coupling the Laplace Transform and Adomian Decomposition Method with Fractional Complex Transform (FCT). The problem is resolved without a variable discretization call. The results also show that the proposed FCLDM is effective and precise in terms of application. The solutions were analytically expressed with little or no computational interference. For highly non-linear Klein-Gordon models and other related model variations in implementations, the FCDM is therefore suggested..

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