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PDTM for the solution of a time-fractional barrier option Black-Scholes model

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Abstract. This paper introduces the Fractional Projected Differential Transform Method (FPDTM) for the solution of the Time-fractional Barrier Option Black-Scholes Pricing Model (TFBOBSPM). The method seeks the solution using sufficient initial (transformed boundary conditions), without any discretization or restrictive assumptions. The efficiency and precision of the proposed methods are tested using illustrative examples. Thus, the FPDTM is suggested for strongly nonlinear differential models with financial applications.

Keywords: Option pricing; Black-Scholes equation; Differential model, Barrier option; fractional calculus

1. Introduction

The interdisciplinary subject of mathematics and finance, also known as mathematical finance, analytical finance, and financial mathematics, emerged at the beginning of the 1980s and the beginning of the 1990s [1]. Financial mathematics primarily uses the modern mathematical methods and techniques of finance such as stocks, bonds, securities, potential investments, options, and other financial instruments to include stochastic analysis, stochastic optimal control, portfolio analytical, non-linear analysis, multivariate statistical analysis, calculus, mathematical system, modern computational methods, and so on [2-7]. Options trading, due to its position in the financial system, is especially relevant [8, 9]. Conventional option contracts are traded, and their prices are typically released on options markets. Nevertheless, more personalized option agreements, including exotic options, are important in order to respond to more advanced risk mitigation approaches. In addition, the Barrier option is referred to as the contingent on a pre-expiration stock price known as the Barrier [8-10]. Barrier options [11-14] are exotics path-dependent and are identical to ordinary options in many respects. You can put or call in Bermuda, American, or European-style exercise. But they only become triggered or extinguished if the underlying level (the barrier) exceeds a predetermined amount. In-options start their lives worthless and only become active in case of breach of a prescribed knock-in barrier price. Out-options start their active lives and become null and void if there is a breach of a certain knock-out barrier limit. Most of these models are differential in nature. Differential equations perform crucial roles in diverse fields like natural sciences, chemistry, physics, economics, and biology in real-life modeling phenomena. As the relevance of ordinary, partial, and integral equations

is growing, seeking approximate solutions has drawn many applied mathematics researchers to establish various methods for solving those equations [15-24]. The majority of the real problems of engineering and technology are influenced by partial differential equations. In certain cases, such equations can be analytically resolved, and in other situations where the equations do not have analytical solutions, numerical techniques could be used. It is a common reality that empirical solutions are very complicated and time-consuming to achieve for any ordinary and partial differential equation. As an option, numerical analysts are searching for numerical methods that can be rendered with a rational error limit as precise as practicable.
A fractional derivative (FD) as a part of fractional calculus (FC), entails a derivative of any random, real, or complex computational mathematics and mathematical analysis form. FD's first mention is in a letter from Gottfried Wilhelm Leibniz to Guillaume de l'Hôpital in 1695 [25]. FC was intitiated and introduced in one of Abel's early papers [26], which shows the concept of integration, the differentiation of the fractional-order, the reciprocal connection of these elements, the notion of differentiation and integration of the fractional-order as the same generalized operation, and also the clear notation of differentiation. Fractional calculus concepts and implementations have grown considerably over the 19th and 20th centuries, and many researchers have described fractional derivatives and integrals [27-33]. The FPDTM is proposed in this study for an effective solution to the TFBOBSM. This approach is defined in terms of Caputo fractional derivatives. This study aims to review, as follows, a time-fractional barrier option pricing model within the Black-Scholes system with a focus on down-and-out call options over a certain barrier term.

2. The Dynamics of the time-fractional Barrier Option

Suppose at time $t<T$, with $S(t)$ as the stock price at time , t such that K and B are the strike price and barrier option respectively, then the corresponding payoff of a down-and-out call option is defined as:

$$f_0^d(S)=(S_T-K)^+, t<T$$

Then, a barrier option is referred to as a traditional option with an additional constraint involving B, such that the following partial Black-Scholes Model (BSM) is satisfied.

$$\begin{cases} P_t + rSP_S + \frac{1}{2}S^2\sigma^2 P_{SS} - rP = 0 \\ P(S,T) = (S-K)^+, S \in (0,\infty), \end{cases} \tag{1}$$

where $P_{(\cdot)}$ denotes partial derivative operator w.r.t. a subscripted variable, $P(S,t)$ is the option value, $\sigma>0$, the volatility parameter, r, the risk-free interest rate, and $P(B,t)=0$ is the extra (additional) condition. For simplicity, we intend to reduce (1) to simplier version based on the following change of variables:

$$\begin{cases} w = \ln(SB^{-1}) \Rightarrow S = Be^w \\ \tau = T - t. \end{cases} \tag{2}$$

Thus,

$$\begin{cases} P_t = \dfrac{\partial P}{\partial t} = -\dfrac{\partial P}{\partial \tau} \\ P_w = \dfrac{\partial P}{\partial w} = \dfrac{\partial P}{\partial S}\dfrac{\partial S}{\partial w} = S\dfrac{\partial P}{\partial S} \\ P_{ww} = \dfrac{\partial^2 P}{\partial w^2} = \dfrac{\partial}{\partial w}\left(\dfrac{\partial P}{\partial w}\right) = S\dfrac{\partial P}{\partial S} + S^2\dfrac{\partial^2 P}{\partial S^2} \end{cases} \tag{3}$$

Putting (2) and (3) in (1), with little algebra, we have:

$$\begin{cases} P_\tau = \left(r - \dfrac{1}{2}\sigma^2\right)P_w + \dfrac{1}{2}\sigma^2 P_{ww} - rP \\ P(w,0) = \left(Be^w - K\right)^+, P(0,\tau) = 0. \end{cases} \tag{4}$$

In fractional term, we proposed the following:

$$\begin{cases} Q_\tau^\alpha = \left(r - \dfrac{1}{2}\sigma^2\right)Q_w + \dfrac{1}{2}\sigma^2 Q_{ww} - rQ, 0 < \alpha \le 1 \\ Q(w,0) = \left(Be^w - K\right)^+, Q(0,\tau) = 0. \end{cases} \tag{5}$$

Note that the classical BSM for European call option is retrievable from (4-6) for $B = 1$ and $\alpha = 0$.

The barrier is noted to be below the initial stock price; if not, the option is worthless. Since this goes beyond the established barrier, the idea of the down-and-out option was cultivated.

Financial models and the likes are, in most cases, in the form of ordinary or partial differential equations [28-32]. Few of these differential models have known exact solutions. However, obtaining the solutions of some of these seems tedious and time-consuming. This is even when the existence of the solutions is guaranteed [33-45]. Thus, a lot of numerical approaches have been proposed and adopted; notwithstanding, better approaches are anticipated. In this regard, the Barrier option model built on the classical BSM is extended to an equivalent time-fractional form, and a fast and efficient semi-analytical method is proposed [28].

2.1 Remarks on the FPDTM

This section introduces the basic principles and methods of the proposed approach (PDTM) with reference to [28, 36].

Let $q(x, t)$ defined on a given domain, G, be an analytic function, at a specified point (x_0, t_0), such that the Taylor series expansion of $q(x,t)$, is ascertained. Then, the projected differential transform of $q(x,t)$ and its inverse projected differential transform are defined and represented respectively as:

$$\left.\begin{aligned} Q(x, l) &= \frac{1}{l!}\left[\frac{\partial^l q(x,t)}{\partial t^l}\right]_{t=t_0} \\ q(x,t) &= \sum_{l=0}^{\infty} Q(x, l)(t - t_0)^l \end{aligned}\right\} \tag{6}$$

The following properties (P1-P5) and theorems associated with the method of the solution are noted as follows in Table 1:

Table 1: Some Basic Properties of the PDTM

Property	Original function form	Projected Transform form
P1	$q(x,\ t)=\alpha q_a(x,\ t)+\beta q_b(x,\ t)$	$Q(x,\ \dot{h})=\alpha Q_a(x,\ l)+\beta Q_b(x,\ l)$
P2	$q(x,\ t)=\alpha\dfrac{\partial^n q_*(x,\ t)}{\partial t^n},$	$l!z(x,\ l)=\alpha(l+n)!Q_*(x,\ l+n)$
P3	$q(x,\ t)=\alpha\dfrac{\partial q_*(x,\ t)}{\partial t}$	$l!Q(x,\ l)=\alpha(l+1)!Q_*(x,\ l+1)$
P4	$q(x,\ t)=f(x)\dfrac{\partial^n q_*(x,\ t)}{\partial x^n}$	$Q(x,\ l)=f(x)\dfrac{\partial^n Q_*(x,\ l)}{\partial x^n}$
P5	$q(x,\ t)=f(x)q_*^2(x,\ t),$	$Q(x,\ l)=f(x)\sum_{i=0}^{l}Q_*(x,\ i)Q_*(x,\ l-i).$

We refer the readers to [28] and the references therein for more details of the PDTM regarding non-integer derivatives.

2. 2 Preliminaries Note

The power of the differential operator is known, in fractional calculus, to be a real or complex integer. Hence the definitions as follows:

Definition 1: [28] Suppose $h(x)$ is defined for $x>0$, and $D(\cdot)$ and $J(\cdot)$ are differential and integration operators respectively, then, in gamma sense, the fractional derivative of $h(x)$ of order $\alpha\in R$ is defined as:

$$D^{\alpha}h(x)=\frac{d^{\alpha}h(x)}{dx^{\alpha}}=\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}x^{k-\alpha} \tag{7}$$

where

$$\left.\begin{array}{l}\Gamma(n)=\int_0^{\infty}e^{-x}t^{n-1}dt,\ \ \mathrm{Re}(n)>0\\ \Gamma(n+1)=n!,\ \Gamma(1/2)=\sqrt{\pi}\end{array}\right\},\tag{8}$$

while the Riemann-Liouville (RL) and Caputo fractional derivatives are respectively defined as:

$$D^{\alpha}h(x)=\frac{d^{\phi}\left(J^{\phi-\alpha}h(x)\right)}{dx^{\phi}} \tag{9}$$

$$D^{\alpha}h(x)=\frac{J^{\phi-\alpha}\left(d^{\phi}h(x)\right)}{dx^{\phi}},\ \ \phi-1<\alpha<\phi,\ \ \phi\in R \tag{10}$$

3. Applications

Here, the considered method is applied to the derived model in (5) for $w>0$; hence, we have:

$$\begin{cases} M\left(w,j\right)=\dfrac{\Gamma\left(1+\alpha\left(j-1\right)\right)}{\Gamma\left(1+\alpha j\right)}\left(\left(r-\dfrac{1}{2}\sigma^2\right)M'\left(x,j-1\right)+\dfrac{1}{2}\sigma^2 M''\left(x,j-1\right)-rM\right), j=1,2,... \\ M\left(w,0\right)=\left(Be^w-K\right)^+, \quad 0<\alpha\le 1. \end{cases} \tag{11}$$

Here $M\left(\cdot\right)$ signifies the projected differential transform of $Q\left(\cdot\right)$. Hence, Using (11), the solution to (5) is easily computable via the following:

$$\begin{aligned} Q\left(w,\tau\right) &= \sum_{i=0}^{\infty} M\left(w,i\right)\tau^{\alpha i} = \left(Be^w-K\right)^+ + \sum_{i=1}^{\infty} M\left(w,i\right)\tau^{\alpha i} \\ &= \left(Be^w-K\right)^+ + \sum_{i=1}^{\infty}\left\{\frac{\Gamma\left(1+\alpha\left(j-1\right)\right)}{\Gamma\left(1+\alpha j\right)}\left(\left(r-\frac{1}{2}\sigma^2\right)M'\left(x,j-1\right)+\frac{1}{2}\sigma^2 M''\left(x,j-1\right)-rM\right)\right\}\tau^{\alpha i}. \end{aligned} \tag{12}$$

4. Conclusions

This research initiated the formulation of the Projected Differential Transformation Method (PDTM) for approximate-analytical approaches to the linear form of the Barrier Option Pricing Model in the framework of the classical Black-Scholes equation. The problem has been solved without a variable-discretizing call. The result obtained suggested that the PDTM was efficient and accurate. The results were presented in a series form with fewer interventions in the computational period. Therefore the method is recommended for application in applied sciences for strongly nonlinear differential and other associated financial models.

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