Remarks on Analytical Solutions of a Non-Homogeneous Gas Dynamic Model via Successive Approximation Method

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Abstract. The Successive Approximation Method (SAM) is introduced in this research to solve the non-homogeneous Gas Dynamic Equation (GDE). This GDE's analytical solution is expressed via the SAM, in series form, with readily computed components. The proposed technique is used directly, without transformation, discretization, linearization, or any restrictive assumptions.

Keywords: Non-linear model**,** Analytical solutions; Transform method, gas dynamics model

1. Introduction

Gas Dynamic Equations (GDEs) are derived from the Boltzmann equation using a method that excludes the a priori assumption of the Enskog theory [1-3]. GDEs are mathematical representations of physical and applied science conservation rules such as momentum conservation, mass conservation, and energy conservation, among others [4, 5]. GDEs are critical in many scientific and engineering fields, including the construction of high-speed aircraft, gas pipelines, hyper loops, jet engines, and rocket motors [10-13]. As a consequence, developing a computer method for calculating the analytic solution to the gas dynamic equation is essential [14-21].
The analytic approaches for these problems are frequently limited and difficult to evaluate. Numerical approaches have been judged more efficient and reliable in solving the gas dynamic models (equations) and other differential models in this regard [22-31].
Here, a one-spatial dimensional version of the classic gas dynamic equation of non-homogeneous type is considered as follows:

$$\begin{cases} m_t - m_{xx} + \dfrac{1}{2} H_1(m) + H_2(m) = f(x,t), \\ m(x,0) = g_1(x), \; m = m(x,t) \end{cases} \tag{1.1}$$

where $H_1(m) = m_x^2(x,t)$, and $H_2(m) = m^2(x,t)$ are non-linear terms of the unknown function, $m = m(x,t)$ (to be determined), while $f(x,t)$ is the source term, respectively.

Different solution experts have recently discussed numerous methods for finding an exact or numerical solution to (1.1). In this work, a novel approach termed Successive Approximation Method (SAM) is applied.

2. Remark on Continuity Condition and the Proposed SIM

The method of solution referred to as Successive Approximation Method (SAM) is introduced here, in line with some basic preliminaries.

2.1 Lipschitzian Continuity Condition

Let $f(t, y)$ be given function, so $f(t, y)$ satisfies a Lipchitz condition with respect to y in a certain region referred to as D in the XY-plane, if there exists a non-negative constant ζ , such that

$$|f(t, y_a) - f(t, y_b)| \le \zeta |y_a - y_b|$$

whenever (t, y_a) and (t, y_b) are in D, and ζ is called the Lipchitz constant.

2.2 Derivation of the Successive Iteration Method

Suppose a first-order non-linear ordinary differential equation (ODE) is given as follows with an initial condition:

$$\begin{cases} \dfrac{dy}{dt} = g(t, y), \\ y(t_0) = y_0. \end{cases} \tag{2.1}$$

Suppose $\int_{t_0}^{t} (\cdot)\, dy = I_{t_0}^{t}(\cdot)$ denotes a one-fold integral operator w.r.t. a concerned variable; thus, by direct integration of both sides (2.1) over $(t_0,\ t)$, we have:

$$\begin{cases} I_{t_0}^{t}(dy) = I_{t_0}^{t}(g(s, y)), \\ y(t_0) = y_0. \end{cases} \tag{2.2}$$

This implies that:

$$\begin{cases} y(t) = y_0 + I_{t_0}^{t}(g(s, y)), \\ y = y(t). \end{cases} \tag{2.3}$$

By iteration, we substitute $y(t) = y_{n+1}(t) = y_{n+1}$ and $y_n(s) = y_n$

Therefore, (2.3) becomes;

$$\begin{cases} y_{n+1} = y_0 + I_{t_0}^{t}(g(s, y_n)), \\ y = y(t). \end{cases} \tag{2.4}$$

3. The Generalized Gas Dynamic Model and the SIM

The SIM is applied as follows to the GDE in (1.1). Let us put (1.1) in an integral form, while the one-fold integral operator, $I_0^t(\cdot)$ is used accordingly. As such:

$$\begin{cases} m = m(x,0) + I_0^t\left(f(x,t) + m_{xx} - \left(\dfrac{1}{2} H_1(m) + H_2(m) \right) \right) \\ \quad = \underbrace{g_1(x) + I_0^t\left(f(x,t) \right)}_{G(x,t)} + I_0^t\left(m_{xx} - \left(\dfrac{1}{2} H_1(m) + H_2(m) \right) \right). \end{cases} \tag{3.1}$$

That is:

$$\left\{ m_{k+1} = G(x) + I_0^t\left((m_k)_{xx} - \left(\frac{1}{2} H_1(m_k) + H_2(m_k) \right) \right) \right. . \tag{3.2}$$

So, by the iterative scheme, we have:

$$\begin{cases} m_0 = G(x) \\ m_1 = I_0^t\left((m_0)_{xx} - \left(\frac{1}{2} H_1(m_0) + H_2(m_0) \right) \right) \\ m_2 = I_0^t\left((m_1)_{xx} - \left(\frac{1}{2} H_1(m_1) + H_2(m_1) \right) \right) \end{cases} \tag{3.3}$$

$$\begin{cases} m_3 = I_0^t\left((m_2)_{xx} - \left(\frac{1}{2} H_1(m_2) + H_2(m_2) \right) \right) \\ m_4 = I_0^t\left((m_3)_{xx} - \left(\frac{1}{2} H_1(m_3) + H_2(m_3) \right) \right) \end{cases}$$

$$\vdots$$

$$\left\{ m_{j+1} = G(x) + I_0^t\left((m_j)_{xx} - \left(\frac{1}{2} H_1(m_j) + H_2(m_j) \right) \right) \right., \; j \geq 0. \tag{3.4}$$

Thus, the desired solution is:

$$m(x,t) = \lim_{N\to\infty}(m_N). \tag{3.5}$$

4. Applications

In this section, some case examples of the GDE are considered via the proposed method in terms of solutions. For the effectiveness and efficiency of the technique, the results are presented graphically compared to their exact solutions.

Case I: Here, we consider a non-homogeneous version of the GDE, with the following data [13]:

$$\left. \begin{aligned} & H_1(m) = (m^2)_x, H_2(m) = -m(1-m), \; m_{xx} = 0, \; f(x,t) = -e^{t-x} \\ & m(x,0) = 1 - e^{-x}, \; m = m(x,t) \end{aligned} \right\}. \tag{4.1}$$

Thus, we have:

$$\begin{cases} m_t + \frac{1}{2}(m^2)_x - m(1-m) = -e^{t-x}, \\ m(x,0) = 1 - e^{-x}, \; m = m(x,t). \end{cases} \tag{4.2}$$

The exact solution of (4.2) is:

$$\Lambda(x,t) = 1 - e^{t-x}. \tag{4.3}$$

Equation (4.3) is rewritten as follows:

$$\begin{cases} m_t = -e^{t-x} - \frac{1}{2}(m^2)_x + m - m^2, \\ m(x,0) = 1 - e^{-x}, \; m = m(x,t). \end{cases} \tag{4.4}$$

Applying the SIM to (4.4), give the following relation:

$$\left\{ m_{j+1} = e^{-x} + I_0^t\left(-e^{t-x} - \left(\frac{1}{2}\left(\left(m_j\right)^2\right)_x + m_j - \left(m_j\right)^2\right)\right),\ j \geq 0. \right. \tag{4.5}$$

Thus, we obtained the following iteratively:

$$\begin{cases} m_0 = 1 - e^{-x} \\ m_1 = 1 - e^{-x} + I_0^t\left(-e^{t-x} - \left(\frac{1}{2}\left(\left(m_0\right)^2\right)_x + m_0 - \left(m_0\right)^2\right)\right) \\ \quad = 1 - \mathrm{e}^{t-x} \end{cases}$$

$$\left.\begin{array}{l} \left\{ m_2 = 1 - e^{-x} + I_0^t\left(-e^{t-x} - \left(\frac{1}{2}\left(\left(m_1\right)^2\right)_x + m_0 - \left(m_1\right)^2\right)\right)\right. \\ \quad = 1 - \mathrm{e}^{t-x} \end{array}\right\}$$

$$\left.\begin{array}{l} m_3 = 1 - e^{-x} + I_0^t\left(-e^{t-x} - \left(\frac{1}{2}\left(\left(m_2\right)^2\right)_x + m_2 - \left(m_2\right)^2\right)\right) \\ \quad = 1 - \mathrm{e}^{t-x} \end{array}\right\}$$

$$\left.\begin{array}{l} m_4 = 1 - e^{-x} + I_0^t\left(-e^{t-x} - \left(\frac{1}{2}\left(\left(m_3\right)^2\right)_x + m_3 - \left(m_3\right)^2\right)\right) \\ \quad = 1 - \mathrm{e}^{t-x} \\ \quad \vdots \end{array}\right\}$$

In fact,

$$m_{j+1} = 1 - e^{t-x},\ j \in [0, \infty) \subset \mathbb{Z}^+.$$

Thus,

$$m(x,t) = 1 - \mathrm{e}^{t-x}. \tag{4.6}$$

The result obtained in (4.6) is consistent with the analytical solution in [13]. However, the technique shown here seems to be simpler and easy. Figures 1-2 illustrate the approximate and exact solutions

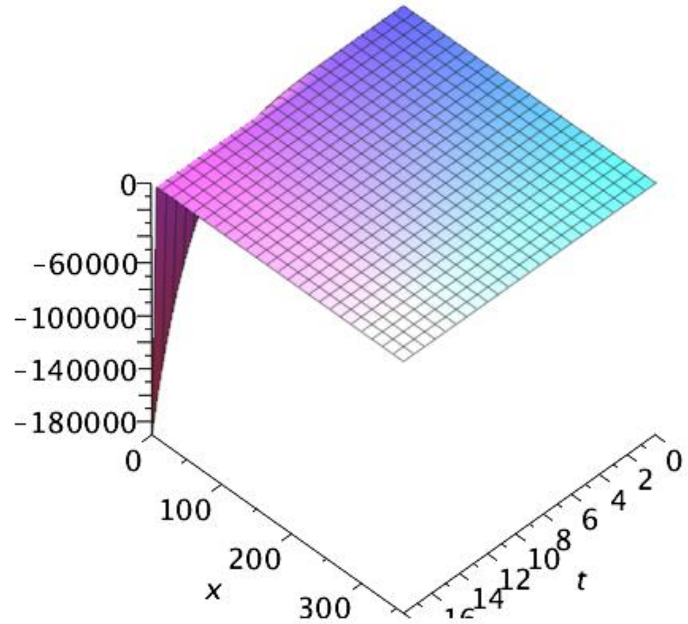

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Figure 1: SIM 7-term Approximate solution (Case I)

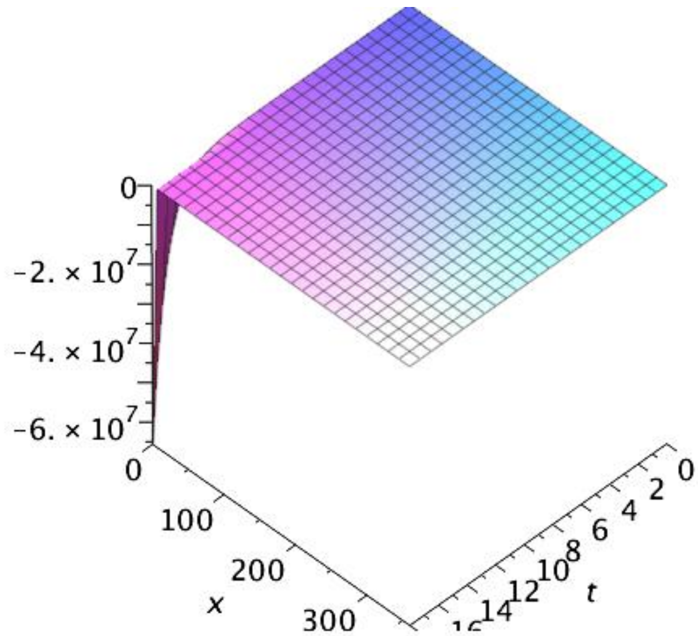

Figure 2: Exact solution (Case I)

5. Conclusions

The Successive Approximation Method (SAM) was used in this work to assess the non-homogeneous non-linear gas dynamic problem. The explicit series solutions to the gas dynamics equation are found to be the same as the findings previously published in the literature. In conclusion, as compared to other techniques, SAM offers more accurate numerical solutions for non-linear problems. It also does not require a huge amount of computer memory, restrictive or limiting assumptions, or discretization procedures. The findings demonstrate that SAM is an effective mathematical technique for solving non-linear partial differential equations. The Maple program was used to do calculations in this study.

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