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Laplace decomposition method for series solutions of systems of linear partial differential models

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Abstract Most of the real-life problems experienced in the industry these days cannot be expressed as a single differential equation but as a system of differential equations. Such structures of linear partial differential models are considered in this work with the aid of the Laplace Adomian decomposition method (LADM) in terms of solutions. Various examples are taken into consideration. The results are easily obtained, are in good agreement when compared to their exact forms. In addition, the proposed method seems effective and efficient; the solutions are graphically presented.

Keywords: Differential models, Laplace Transform, Adomian decomposition, Approximate solution

1. Introduction

A differential equation is usual within the mathematics field; this study deals with linear partial differential equations systems. Most physical problems cannot be described with a single differential equation that has one unknown [1-4]. In order to obtain approximate or theoretical solutions to various forms of differential equations or systems, in the case of the solution exiting, reliable solution approaches are needed. Several researchers have developed iterative methods or modified existing methods for efficiency and reliability. These include the New Iterative Method (NIM), Picard Iterative Method (PIM), Variational Iterative Method (VIM), ADM, Homotopy Perturbation Method (HPM), Boundary Value Methods (BVMs), and so on [5-10]. The majority of natural occurrences are linear and non-linear. A good number of researchers have used the Laplace decomposition algorithm and other solution methods [11-19]. This work aims to apply the precise and powerful method for solving a system of linear partial differential equations. Accordingly, the objectives are to: solve systems of linear partial differential equations via the Laplace Adomian decomposition method; and compare the result with already existing exact solutions.

2. The Adomian Decomposition Method

The Adomian decomposition method is a step-by-step numerical method that can be used to resolve differential equations. This method is iterative in nature with an algorithm.
We would examine a differential equation of the form.

$$L\xi(x,y)+R\xi(x,y)+N\xi(x,y)=q(x,y) \tag{2.1}$$

where L is a linear differential operator, R is taken to be the differential operator less than D,

More often than not, $L=\frac{d^n}{dx^n}(.)$ is the nth order differential operator, which means that its inverse L^{-1} is the nth order integration operator.

Therefore, we would have:

$$L^{-1}\left[L\xi(x,y)+R\xi(x,y)+N\xi(x,y)\right]=L^{-1}q(x,y) \tag{2.2}$$

$$L^{-1}L\xi(x,y)=t-\phi \tag{2.3}$$

for ϕ signifies the initial values.

Substituting (3.3) in (3.2) we have,

$$y=\beta(y)-L^{-1}\left[N\xi(x,y)+R\xi(x,y)\right] \tag{2.4}$$

where $\beta(y)=L^{-1}g+\phi$ which signifies a function obtained by integrating the source term with respect to the initial condition(s).

The ADM simplifies the solution $y(t)$ in series form

$$y=\sum_{n=0}^{\infty}y_n \tag{2.5}$$

The non-linear term is expressed as:

$$Nh(x,t)-\sum_{m=0}^{\infty}A_m=0 \tag{2.6}$$

$$A_m=\frac{1}{m!}\frac{d^m}{dx^m}\left(f\left(t.\sum_{k=0}^{\infty}x^k y_k\right)\right)_{k=0},\ m\geq 0 \tag{2.7}$$

$$\sum_{n=0}^{\infty}y_n=\alpha(y)-L^{-1}\left(R\left(\sum_{n=0}^{\infty}y_n\right)+\sum_{n=0}^{\infty}A_n\right) \tag{2.8}$$

By recursion relation, we have

$$y_0(x)=a(x)$$

$$y_{n+1}(x)=-L^{-1}\left[Ry_n+A_n\right],n\geq 0$$

Hence, the solution is

$$\left.\begin{aligned} y(x)&=\sum_{k=0}^{M}y_k(x)\\ &=\lim_{n\to\infty}\left(\sum_{n=0}^{\infty}y_n\right)\end{aligned}\right\}$$

3. The Solution method Laplace ADM (LADM)

The LADM is a semi-analytical method that combines the ADM and the Laplace transform method. With little iteration, LADM solves linear and nonlinear differential equations. Let us examine the general first-order non-linear PDE of the form:

$$\begin{cases} Lu+Ru+Nu=q(x,y)\\ u=u(x,y)\end{cases} \tag{3.1}$$

Rearranging (3.1) we have,

$$Lu = q(x,y) - (Ru + Nu) \tag{3.2}$$

Taking the inverse Laplace transform of (3.2)

$$u = u(x,0) + L^{-1}\left\{s^{-1}L\{q\}\right\} - L^{-1}\left\{s^{-1}L\{Nu+Ru\}\right\}, q = q(x,y), \tag{3.3}$$

Taking the inverse Laplace transform of (3.3)

$$u = u(x,0) - L^{-1}\left\{s^{-1}L\{Nu+Ru\}\right\} \tag{3.4}$$

LADM puts forward its solution as an infinite series as shown in (3.4)

$$u = \sum_{n=0}^{\infty} u_n, u(\cdot) = u(x,y) \tag{3.5}$$

$$Nu(x,y) = \sum_{b=0}^{\infty} A_b(u_0, u_1, u_2, \ldots, u_n) \tag{3.6}$$

where A_b represents the Adomian polynomials of the form

$$A_b = \frac{1}{b!}\frac{d^b}{d\lambda^b}\left(N\left(t, \sum_{k=0}^{\infty}\lambda^k h_k\right)\right)_{\lambda=0}, b \geq 0 \tag{3.7}$$

Putting (3.5) and (3.6) in (3.7) gives

$$\sum_{b=0}^{\infty} ub(x,y) = \left(u(x,0) + L^{-1}\left\{s^{-1}L\{q(x,y)\}\right\} - Q\right) \tag{3.8}$$

where

$$Q = L^{-1}\left\{\frac{1}{s}L\left\{\left(\sum_{b=0}^{\infty} A_b + R\sum_{b=0}^{\infty} u_b(x,y)\right)\right\}\right\} \tag{3.9}$$

From (3.9) the solution can be obtained through the following recurrence relations

$$u_0 = L^{-1}\left\{s^{-1}L\{u(x,y)\}\right\} + u(x,0) \tag{3.10}$$

$$u_{m+1} = -L^{-1}\left\{\frac{1}{s}L\{(Ru_m + NA_m)\}\right\}, m \geq 0 \tag{3.11}$$

While $u(x,y)$ is given as:

$$u(x,y) = \lim_{j\to\infty}\sum_{n=0}^{j} u_n(x,y) \tag{3.12}$$

4. LADM and Systems of Two Linear PDEs

In this section, the linear partial differential systems would be solved using the proposed LADM.

4.1 Case I Consider the linear system

$$\begin{cases} q_x + p_t = 0 \\ p_x + q_t = 0 \end{cases} \tag{4.1}$$

such that

$$\begin{cases} p(x,0) = \exp(x) \\ q(x,0) = \exp(-x) \end{cases} \tag{4.2}$$

and the the exact solution are:

$$\begin{cases} p(x,t) = \exp(x)\cosh(t) + \exp(-x)\sinh(t) \\ q(x,t) = \exp(-x)c\operatorname{osh}(t) - \exp(x)\sinh(t) \end{cases} \tag{4.3}$$

Solution to Case I
The system in (4.1) can be re-expressed as:

$$\begin{cases} p_t = -q_x \\ q_t = -p_x \\ q(x,0) = e^{-x}, p(x,0) = e^x,\ p = p(x,t),\ q = q(x,t),\ h_z = \dfrac{\partial h}{\partial z} \end{cases} \tag{4.4}$$

The Laplace transform of (4.4), with $L = H$ and $L^{-1} = H^{-1}$ yields:

$$\begin{cases} H\{p_t\} = H\{-q_x\} \\ H\{q_t\} = H\{-p_x\} \end{cases} \tag{4.5}$$

$$\Rightarrow \begin{cases} p(x,s) = \dfrac{1}{s}\left(u(x,0) + H\{-q_x\}\right) \\ q(x,s) = \dfrac{1}{s}\left(q(x,0) + H\{-p_x\}\right) \end{cases} \tag{4.6}$$

Taking the inverse LT of both sides of (4.6) gives:

$$\begin{cases} p(x,t) = p(x,0) + H^{-1}\left[\dfrac{1}{s}H\{-q_x\}\right] \\ q(x,t) = q(x,0) + H^{-1}\left[\dfrac{1}{s}H\{-p_x\}\right] \end{cases} \tag{4.7}$$

By ADM, the solution is given as:

$$\begin{cases} p = \sum_{n=0}^{\infty} p_n, \\ q = \sum_{n=0}^{\infty} q_n. \end{cases} \tag{4.8}$$

Thus, (4.7) becomes

$$\begin{cases} \sum_{n=0}^{\infty} p_n = p(x,0) + H^{-1}\left[\dfrac{1}{s}H\left\{-\left(\sum_{n=0}^{\infty} q_n\right)_x\right\}\right] \\ \sum_{n=0}^{\infty} q_n = q(x,0) + H^{-1}\left[\dfrac{1}{s}H\left\{-\left(\sum_{n=0}^{\infty} p_n\right)_x\right\}\right] \end{cases} \tag{4.9}$$

Comparing the terms in (4.9) gives the following recurrence relations

$$\begin{cases} p_0 = p(x,0) = e^x \\ p_{k+1} = H^{-1}\left[\dfrac{1}{s}H\{-(q_k)_x\}\right] \end{cases}$$

$$\begin{cases} q_0 = q(x,0) = e^{-x} \\ q_{k+1} = H^{-1}\left[\frac{1}{s} H\left\{-\left(p_k\right)_x\right\}\right] \end{cases}$$

When $k = 0$, we have

$$\begin{cases} p_1 = H^{-1}\left[\frac{1}{s} H\left\{-\left(q_0\right)_x\right\}\right] = t\exp(-x) \\ q_1 = H^{-1}\left[\frac{1}{s} H\left\{-\left(p_0\right)_x\right\}\right] = -t\exp(x) \end{cases}$$

When $k = 1$, we have

$$\begin{cases} p_2 = H^{-1}\left[\frac{1}{s} H\left\{-\left(q_1\right)_x\right\}\right] = \frac{t^2}{2!}\exp(x) \\ q_2 = H^{-1}\left[\frac{1}{s} H\left\{-\left(p_1\right)_x\right\}\right] = \frac{t^2}{2!}\exp(-x) \end{cases}$$

When $k = 2$, we have

$$\begin{cases} p_3 = H^{-1}\left[\frac{1}{s} H\left\{-\left(q_2\right)_x\right\}\right] = \frac{t^3}{3!}\exp(-x) \\ q_3 = H^{-1}\left[\frac{1}{s} H\left\{-\left(p_2\right)_x\right\}\right] = \frac{-t^3}{3!}\exp(x) \end{cases}$$

$$\left.\begin{aligned} \Rightarrow p(x,t) &= p_0 + p_1 + p_2 + \cdots + p_n + \cdots \\ &= \exp(x)\left(1 + \frac{t^2}{2!} + \cdots\right) + \exp(-x)\left(t + \frac{t^3}{3!} + \cdots\right) \\ &= \mathrm{e}^{(x)}\cosh t + \mathrm{e}^{(-x)}\sinh t \end{aligned}\right\} \tag{4.10}$$

and

$$\left.\begin{aligned} q(x,t) &= q_0 + q_1 + q_2 + \cdots + q_n + \cdots \\ &= \exp(-x)\left(1 + \frac{t^2}{2!} + \cdots\right) - \exp(x)\left(t + \frac{t^3}{3!} + \cdots\right) \\ &= \mathrm{e}^{(-x)}\cosh t - \mathrm{e}^{(x)}\sinh t \end{aligned}\right\} \tag{4.11}$$

$$\therefore \quad (p,q) = \left(\exp(x)\cosh t + \exp(-x)\sinh t,\ \exp(-x)\cosh t - \exp(x)\sinh t\right) \tag{4.12}$$

Equation (4.11) denotes the LADM solution of case 1. These are plotted in Figures 1 to 4 in comparison with the exact solutions in (4.2).

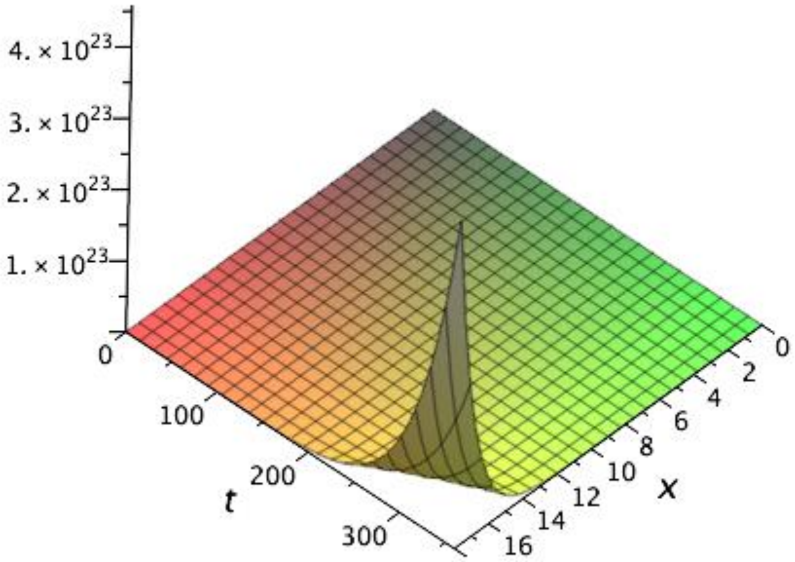

Figure 1: Case 1 Approximate solution for $p(x,t)$

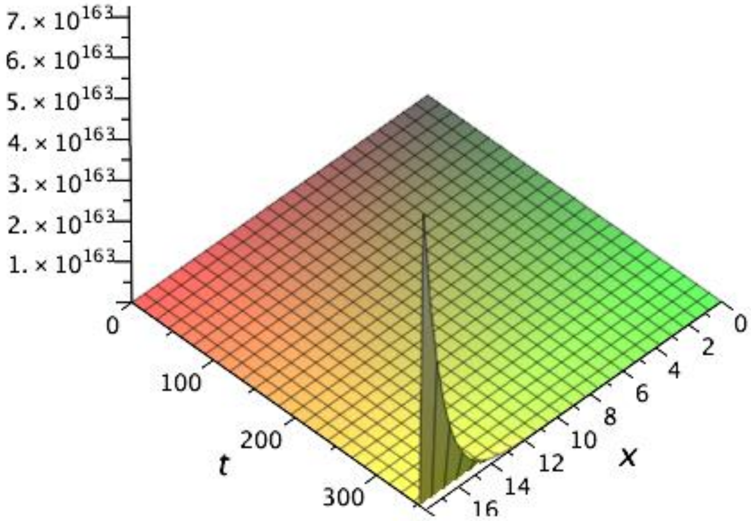

Figure 2: Case 1 Exact solution for $p(x,t)$

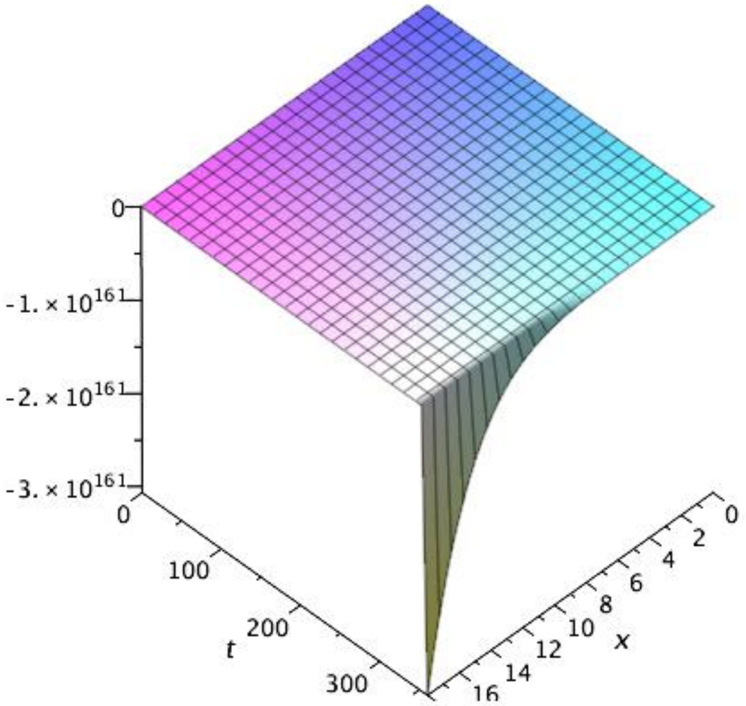

Figure 3: Case 1 Approximate solution for $q(x,t)$

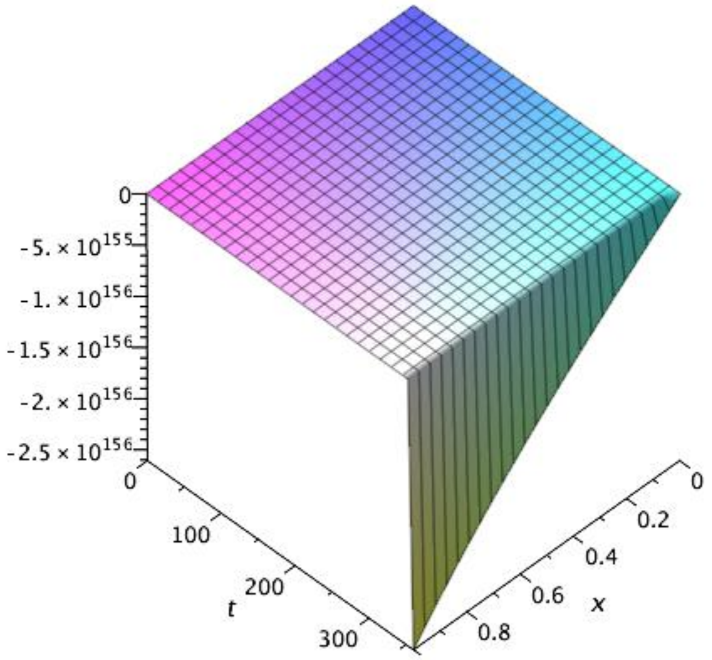

Figure 4: Case 1 Exact solution for $q(x,t)$

4.2 Case II

The following linear system of PDE is considered

$$\begin{cases} p_t + p_x + 2q = 0 \\ q_t + q_x - 2p = 0 \end{cases} \tag{4.13}$$

such that:

$$\begin{cases} q(x,0) - \sin x = 0 \\ p(x,0) - \cos x = 0 \end{cases} \tag{4.14}$$

and

$$\begin{cases} q = \sin(x+t) \\ p = \cos(x+t),\ q = q(x,t),\ p = p(x,t) \end{cases} \tag{4.15}$$

represents the exact solution.

Solution to Case II:
Following the same approach as in case 1, we obtained the following:

$$\left.\begin{aligned} p(x,t) &= \sum_{n=0}^{\infty} p_n \\ &= p_0 + p_1 + p_2 + \cdots + p_n + \cdots \\ &= \cos x - t\sin x - \frac{t^2}{2!}\cos x + \frac{t^3}{3!}\sin x + \cdots \\ &= \cos x\left(1 - \frac{t^2}{2!} + \cdots\right) - \sin x\left(t - \frac{t^3}{3!} + \cdots\right) \\ &= \cos x(\cos t) - \sin x(\sin t) \end{aligned}\right\} \tag{4.16}$$

and

$$\left.\begin{aligned} q(x,t) &= \sum_{n=0}^{\infty} q_n \\ &= q_0 + q_1 + q_2 + \cdots + q_n + \cdots \\ &= \sin x\left(1 - \frac{t^2}{2!} + \cdots\right) + \cos x\left(t - \frac{t^3}{3!} + \cdots\right) \\ &= \sin x(\cos t) + \cos x(\sin t) \end{aligned}\right\} \tag{4.17}$$

$$\therefore \quad (p,q) = (\cos(x+t), \sin(x+t)) \tag{4.18}$$

Equation (4.18) denotes the LADM solution of case II. These are plotted in Figures 5 to 8 in comparison with the exact solutions in (4.15).

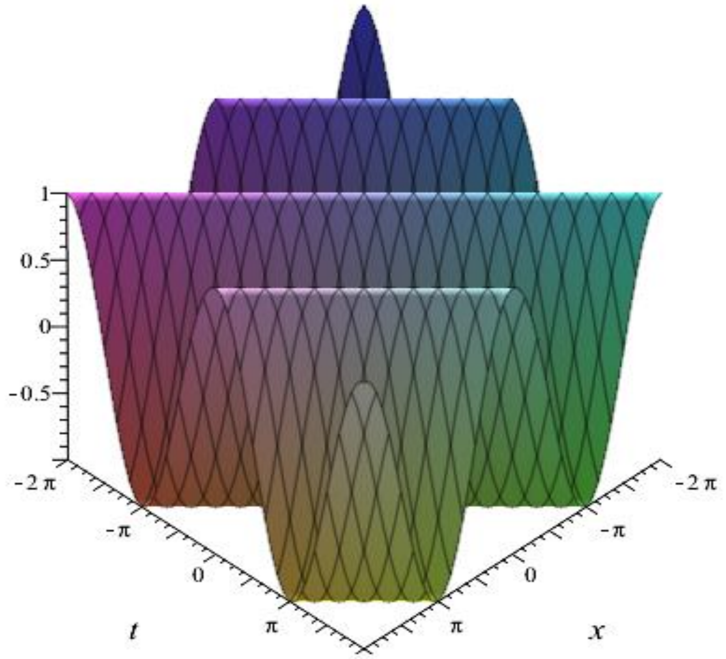

Figure 5: Case 2 Exact solution for $p(x,t)$

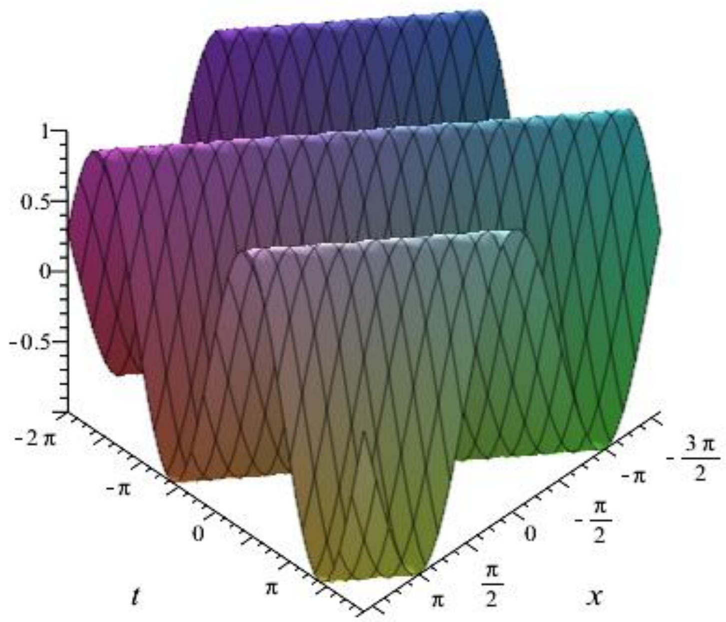

Figure 6: Case 2 Approximate solution for $p(x,t)$

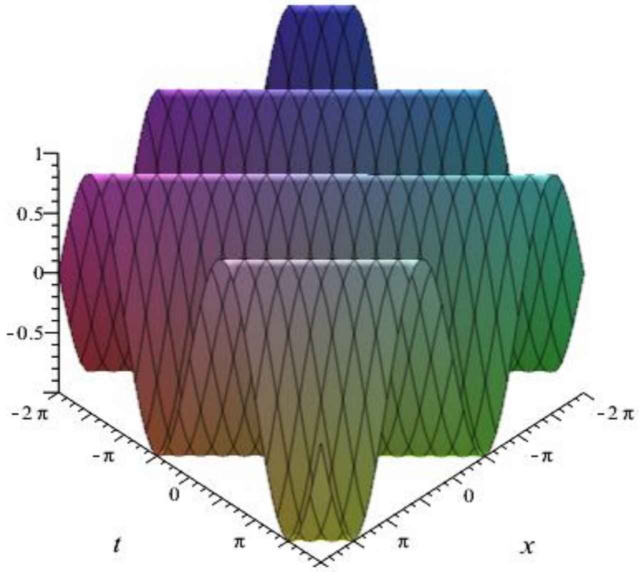

Figure 7: Case 2 Exact solution for $q(x,t)$

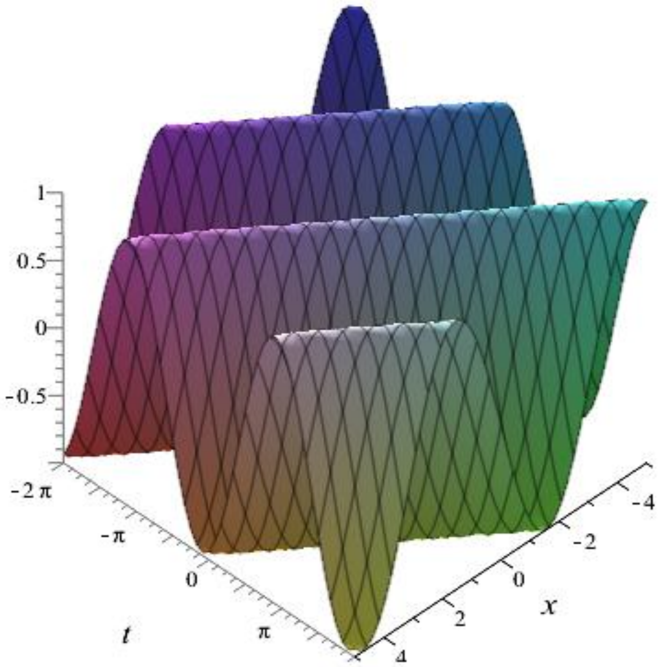

Figure 8: Case 2 Approximate solution for $q(x,t)$

5. Concluding Remark

This research has successfully considered anproximate-analytical solutions of some certain partial differential models using a transform method termed Laplace Adomian decomposition technique (LADM). The approximate solutions obtained using the technique are compared to the exact solution graphically, and the method yields the same result as the exact solution. Thus, the LADM is highly recommended for higher-order differential models in pure and applied.

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