On the Solvability of a Resonant p-Laplacian Third-order Integral m-Point Boundary Value Problem

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Abstract— **In this work, we establish conditions for the existence of at least one solution for a p-Laplacian third order integral and m-point boundary value problem at resonance. The Ge and Ren extension of Mawhin's coincidence theory will be used to obtain existence results for the p-Laplacian problem at resonance.**

Index Terms— **Coincidence degree, resonance, m-point, integral boundary value problem, p-Laplacian.**

1 Introduction

This work deals with the following p-Laplacian third order integral and m-point boundary value problem at resonance

$$(\phi_p(u''(t)))' = w(t, u(t), u'(t), u''(t)), \quad t \in (0,1), \quad (1)$$

subject to the boundary conditions

$$\begin{aligned} \phi_p(u''(0)) &= \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \phi_p(u''(t))dt, \\ u''(1) &= 0,\ u'(1) = \beta u'(\eta), \end{aligned} \quad (2)$$

where the function $w : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous, $\phi_p(s) = |s|^{p-2}s,\ p > 1$, the inverse of ϕ_p^{-1} is ϕ_q, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$, $\beta > 0$, $\alpha_i (1 \le i \le m) \in \mathbb{R}$ and $\eta \in (0,1)$. Since we require a nontrivial kernel for our quasi-linear operator, the condition $\sum_{i=1}^{m} \alpha_i \xi_i = 1$ is critical. The integral in (2) is the Riemann-Stieltjes integral.

A boundary value problem $Lu = u'''(t) = 0$ is said to be at resonance if L is non-vertible else it is a non-resonance problem where L is a linear operator. Since the establishment of the coincidence degree theory by Mawhin, for boundary value problems at ressonance [13], many authors have studied resonant problems when the differential operator is linear (see [1, 3, 5, 6, 8, 9, 12]). When the differential operator is nonlinear, like in p-Laplace boundary value problems the Mawhin coincidence degree theory fails while the extension of the theorem by Ge and Ren [4] is used (see [7, 2, 15, 10]).

Inspired by the above works, this paper uses the Ge and Ren extension of the coincidence degree theory [4] to establish the existence of solutions for the problems (1)-(2) at resonance.

The rest of the paper is organized as follows. Section 2 gives necessary definitions, lemmas and theorem that are needed tor the work. In section 3, we obtain existence results for (1)-(2) while an example will be given in section 4 to corroborate our result.

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2 Preliminaries

In this section, we will give necessary lemmas, definitions and theorems.

Definition 1. Given two Banach spaces, U and Z with norms $\|\cdot\|_U$ and $\|\cdot\|_Z$ respectively, a continuous operator

$$M : \text{dom } M \subset U \to Z$$

is said to be quasi-linear if

(i) $\text{Im } M$ is a closed subset of Z;

(ii) $\ker M$ is linearly homeomorphic to $\mathbb{R}^n,\ n < \infty$.

Definition 2. ([10]) Let $\Omega \subset U$ be a bounded open set with the origin $\sigma \in \Omega$. The nonlinear operator $N_\lambda : \overline{\Omega} \to Z,\ \lambda \in [0,1]$ is said to be M-compact in $\overline{\Omega}$ if there exist $Z_1 \subset Z$ with $\dim Z_1 = \dim \ker M$ and a continuous, compact operator $T : \overline{\Omega} \times [0,1] \to U_2$ such that for $\lambda \in [0,1]$,

(i) $(I - Q)N_\lambda \subset \text{Im } M \subset (I - Q)Z$;

(ii) $QN_\lambda u = 0,\ \lambda \in (0,1) \Leftrightarrow QNu = 0,\ \forall u \in \Omega$;

(iii) $T(\cdot, 0) \equiv 0$ and $T(\cdot, \lambda)|_{\sum_\lambda} = (I - P)_{\sum_\lambda}$;

(iv) $M[P+T(\cdot,\lambda)] = (I-Q)N_\lambda$, $\lambda \in [0,1]$;

where $U_2 \in U$ is a complement space of $\ker M$, i.e. $U = \ker M \oplus U_2$; P, Q are projectors such that $\ker M = \text{Im } P$, $\text{Im } Q = Z_1$, $N = N_1$, and $\sum_\lambda = \{u \in \overline{\Omega} : Mu = N_\lambda u\}$.

Lemma 3. [16] The following are true for ϕ_p:

1. ((i)) ϕ_p is continuous, invertible and monotonically increasing. In addition, $\phi_p^{-1} = \phi_q$ and for $q > 1$ then $\frac{1}{p} + \frac{1}{q} = 1$;

(ii) For all $y, z, \geq 0$,

$$\begin{aligned} \phi_p(y+z) &\leq \phi_p(y) + \phi_p(z), && \text{if } 1 < p < 2, \\ \phi_p(y+z) &\leq 2^{p-2}(\phi_p(y) + \phi_p(z)), && \text{if } p \geq 2. \end{aligned}$$

Theorem 1. ([4]) Let U and Z be Banach spaces, and $\Omega \subset U$ a bounded open nonempty set. Also $M : \text{dom } M \subset U \to Z$ is quasi-linear and $N_\lambda : \overline{\Omega} \to Z$, $\lambda \in [0,1]$ is M-compact in $\overline{\Omega}$. Assume the following conditions are satisfied

(i) $Mu \neq N_\lambda u$ for every $(u,\lambda) \in [(\text{dom } M \backslash \ker M) \cap \partial\Omega] \times (0,1)$;

(ii) $QNu \neq 0$ for every $u \in \ker M \cap \partial\Omega$;

(iii) $\deg(JQN, \Omega \cap \ker M, 0) \neq 0$, where $J : \text{Im } Q \to \ker M$ is a homeomorphism.

Then, the abstract equation $Mu = Nu$ has at least one solution in $\overline{\Omega}$.

Let

$U = \{u \in C^2[0,1] : \phi_p(u''(t)) \in C^1[0,1],\ u(t) \text{ satisfies (2)}\}$

where the norms $\|z\|_\infty = \max_{t\in[0,1]} |x(t)|$ and $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty\}$ are defined on U.
Let $Z = L^1[0,1]$ with the norm on Z denoted by $\|\cdot\|_1$.
The quasi-linear operator $M : \text{dom } M \subset U \to Z$ will be defined by

$$M : u \mapsto Mu = (\phi_t(u''(t))',\ t \in [0,1],$$

where $\text{dom } M = \Big\{u \in U \cap C^2[0,+\infty) : \phi_p(u''(0)) = \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \phi_p(u''(t))dt,\ u''(1) = 0,\ u'(1) = \beta u'(\eta)\Big\}$. Also, the nonlinear operator $N_\lambda : U \to Z$, $\lambda \in [0,1]$ will be defined by

$$(N_\lambda u)t = \lambda q(t, u(t), u'(t), u''(t)),\ t \in [0,1],$$

thus problem (1)-(2) may be written in the form

$$Mu = N_\lambda u.$$

Lemma 2. If $\sum_{i=1}^m \alpha_i\xi_i = 1$ then there exists $r \in \{1,2,\ldots,m-1\}$, such that

$$\sum_{i=1}^m \alpha_i \xi_i^{r+2} \neq 0.$$

Proof. Since $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$, and $\sum_{i=1}^m \alpha_i\xi_i = 1$ then there exists $i \in [1,m]$ such that $\alpha_i \neq 0$, hence $\sum_{i=1}^m \alpha_i \neq 0$. Assuming

$$\sum_{i=1}^m \alpha_i \xi_i^{r+2} = 0,\ r = 0,1,\ldots,m-2,$$

we have

$$\begin{pmatrix} \xi_1^2 & \xi_2^2 & \cdots & \xi_m^2 \\ \xi_1^3 & \xi_2^3 & \cdots & \xi_m^3 \\ \vdots & \vdots & \ddots & \vdots \\ \xi_1^m & \xi_2^m & \cdots & \xi_m^m \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since

$$\begin{aligned} &\det \begin{pmatrix} \xi_1^2 & \xi_2^2 & \cdots & \xi_m^2 \\ \xi_1^3 & \xi_2^3 & \cdots & \xi_m^3 \\ \vdots & \vdots & \ddots & \vdots \\ \xi_1^m & \xi_2^m & \cdots & \xi_m^m \end{pmatrix} \\ &= \xi_1^2\xi_2^2\cdots\xi_m^2 \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \xi_1 & \xi_2 & \cdots & \xi_m \\ \vdots & \vdots & \ddots & \vdots \\ \xi_1^{m-2} & \xi_2^{m-2} & \cdots & \xi_m^{m-2} \end{pmatrix} \\ &= \left(\prod_{i=1}^m \xi^2\right) \prod_{1\leq i<j\leq m} (\xi_j - \xi_i) \neq 0, \end{aligned}$$

then, $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$, which contradicts $\sum_{i=1}^m \alpha_i \neq 0$. Hence, Lemma 2 holds.

Lemma 3. If $\sum_{i=1}^m \alpha_i\xi_i = 1$, then, the operator $M : \text{dom } M \subset U \to Z$ is quasi-linear.

Proof. By simple calculation, we see that

$$\ker M = \{u \in \text{dom } M : u = d, d \in \mathbb{R}\}.$$

We will now show that

$$\text{Im } M = \left\{y \in Z : \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \int_0^x y(v)dvdx = 0\right\}. \quad (3)$$

The p-Laplacian problem

$$\phi_p(u''(t)))' = y(v) \quad (4)$$

has a solution $u(t)$ that satisfies (2) when

$$\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \int_0^x y(v)dvdx = 0. \quad (5)$$

The solution of (4), $u(t)$ that satisfies (2) can be written as

$$u(t) = u(1) + u'(1)(t-1) - \int_t^1 \int_s^1 \phi_q\left(\int_x^1 y(v)dv\right) dxds. \quad (6)$$

Applying the boundary condition (2) and $\sum_{i=1}^m \alpha_i \xi_i = 1$ to (6) we obtain

$$\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \int_0^x y(v)dvdx = 0,$$

which satisfies (3) and

$$u(t) = d + \frac{\beta(t-1)}{1-\beta} \int_\eta^1 \phi_q \int_x^1 y(v)dvdxs - \int_t^1 \int_s^1 \phi_q\left(\int_x^1 y(v)dv\right) dxds,$$

where d is an arbitrary constant and $u(t)$ is the solution to (4) satisfying (2). Since $\ker M = 1 < \infty$ and $M(U \cap \text{dom } M) \subset Z$ is closed, the operator M is quasi-linear.

Lemma 4. The nonlinear operator N_λ is M-compact, if $w \in C([0,1] \times \mathbb{R}^3, \mathbb{R})$.

Proof. We define projectors $P : U \to U_1$ as $Pu = u(1)$ for all $u \in U$ and $Q : Z \to Z_1$ as

$$Qy = \frac{(r+1)(r+2)}{\sum_{i=1}^m \alpha_i \xi^{r+2}} \left(\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \int_0^x y(v)dvdx\right) t^r,$$

$t \in [0,1]$, $\forall\, y \in Z$, where Z_1 is the complement space of Im M in Z. Let $\overline{\Omega} \subset U$ be bounded, then we will define $T : \overline{\Omega} \times [0,1] \to \ker P$ as

$$T(u,\lambda)(t) = \frac{\beta(t-1)}{1-\beta} \int_\eta^1 \left(\phi_q \int_x^1 [(I-Q)N_\lambda u](v)dv\right) dx - \int_t^1 \int_s^1 \phi_q\left(\int_x^1 [(I-Q)N_\lambda u](v)dv\right) dxds,\ t \in [0,1]. \quad (7)$$

$T(\cdot,\lambda)$ is continuous and relatively compact since $w \in C([0,1] \times \mathbb{R}^3, \mathbb{R})$, and $\lambda \in [0,1]$. We will now show in the following four steps that N_λ is M-compact.

Step 1: Let $y \in Z$, then

$$\begin{aligned} Q^2 y &= Q(Qy) = Qy(Q) \\ &= Qy\left[\frac{(r+1)(r+2)}{\sum_{i=1}^m \alpha_i \xi^{r+2}} \left(\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \int_0^x v^r dvdx\right)\right] \\ &= Qy, \quad t \in [0,1], \end{aligned}$$

hence $Q^2 = Q$. Therefore $Q(I-Q)N_\lambda(\overline{\Omega}) = (Q-Q)N_\lambda(\overline{\Omega}) = 0$. This implies that $Q(I-Q)N_\lambda(\overline{\Omega}) \subset \ker Q = \text{Im } M$. Now, if $g \in \text{Im } M$, then $Qg = 0$. We can write g as $g = g - Qg = (I-Q)g$, thus $g \in (I-Q)Z$. Therefore (i) of definition 2.2 is satisfied.

Step 2: If $QNu = 0$, then $Nu = Nu - QNu = (I-Q)Nu = 0$. Since $Nu \neq 0$, $(I-Q)$ is a zero operator. Hence $(I-Q)N_\lambda u = 0$ and $QN_\lambda u = 0$. Using same logic it can also be shown that when $QN_\lambda u = 0$, $QNu = 0$. Hence (ii) of definition 2.1 is satisfied.

Step 3: Here we show that (iii) of definition 2 holds. From (7), we have

$$T(u,\lambda)(t) = \lambda \frac{\beta(t-1)}{1-\beta} \int_\eta^1 \left(\phi_q \int_x^1 [(I-Q)Nu](v)dv\right) dx - \lambda \int_t^1 \int_s^1 \phi_q\left(\int_x^1 [(I-Q)Nu](v)dv\right) dxds,$$

hence $T(\cdot,0) = 0$.

Also for $u \in \sum_\lambda = \{u \in \overline{\Omega} : Mu = N_\lambda u\}$ or $\{u \in \overline{\Omega} : (\phi_p(u''))' = \lambda w(t,u(t),u'(t),u''(t))\}$, we have

$$\begin{aligned} T(u,\lambda)(t) &= \frac{\beta(t-1)}{1-\beta} \int_\eta^1 \left(\phi_q \int_x^1 (\phi_p(u''(v)))'(v)dv\right) dx \\ &- \int_t^1 \int_s^1 \phi_q\left(\int_x^1 (\phi_p(u''(v)))'(v)dv\right) dxds \\ &= -\frac{\beta(t-1)}{1-\beta} \int_\eta^1 u''(x)dxds + \int_t^1 \int_s^1 u''(x)dxds \\ &= \frac{\beta(t-1)}{1-\beta}[u'(\eta) - u'(1)] + u'(1)(1-t) - u(1) + u(t) \\ &= u'(1)(t-1) + u'(1)(1-t) - u(1) + u(t) \\ &= [(I-P)u](t). \end{aligned}$$

Step 4: Now for all $u \in U \cap \text{dom } M$, we have

$$\begin{aligned} M[P + T(\cdot,\lambda)]u &= u(1) \\ &+ \frac{\beta(t-1)}{1-\beta} \int_\eta^1 \left(\phi_q \int_x^1 [(I-Q)N_\lambda u](v)dv\right) dx \\ &- \int_t^1 \int_s^1 \phi_q\left(\int_x^1 [(I-Q)N_\lambda u](v)dv\right) dxds \\ &= (I-Q)N_\lambda u(t). \end{aligned}$$

Since conditions (i) - (iv) of Definition 2 are satisfied in $\overline{\Omega}$, then N_λ is M-compact .

3 Existence Results

Theorem 2 Let $w : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ be continuous function. The p-Laplacian boundary value problem (1)-(2) with $\sum_{i=1}^m \alpha_i \xi_1 = 1$,

$$\phi_q(2) 2^{2q-4}(\|x\|_\infty^{q-1} + \|y\|_\infty^{q-1} + \|z\|_\infty^{q-1}) < 1 \quad \text{for } p < 2 \quad (8)$$

and

$$\phi_q(2)(\|x\|_\infty^{q-1} + \|y\|_\infty^{q-1} + \|z\|_\infty^{q-1}) < 1 \quad \text{for } p \geq 2 \quad (9)$$

has at least one solution in $C^2[0,1]$, if the following conditions hold

(C_1) There exist function $x, y, z, h \in C([0,1],[0,\infty))$ such that for all $(a,b,c) \in \mathbb{R}^3$, $t \in [0,1]$

$$\begin{aligned}|w(t,a,b,c)| &\le x(t)\phi_p(|a|) + y(t)\phi_p(|b|)\\ &\quad + z(t)\phi_p(|c|) + h(t).\end{aligned} \tag{10}$$

(C_2) There exists a constant $D > 0$, such that for any $u \in$ dom M, if $|u(t)| > D$, or $|u'(t)| > D$, or $|u''(t)| > D$, for every $t \in [0,1]$ then

$$QNu(t) \neq 0,\ t \in [0,1]. \tag{11}$$

(C_3) There exists a constant $F > 0$ such that for $d \in \mathbb{R}$, if $|d| > F$, then either

$$d \cdot \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \int_0^t w(v,d,0,0)dvdt < 0, \tag{12}$$

or

$$d \cdot \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \int_0^t w(v,d,0,0)dvdt > 0. \tag{13}$$

Proof. We set

$$\Omega_1 = \{u \in \operatorname{dom} M\ \ \ker M : Mu = N_\lambda u,\ \lambda \in [0,1]\}.$$

If $u \in \Omega_1$, then $Mu = N_\lambda u$ and $\lambda \neq 0$, then $Nu \in \operatorname{Im} M = \ker Q$ and$QNu(t) = 0$. From (C_2), it follows that there exits $t_0, t_1, t_2 \in [0,1]$ such that $|u(t_0)| \le D$, $|u'(t_1)| \le D$ and $|u''(t_2)| \le D$. By the absolute continuity of u, u', we have $u(t) = u(t_0) + \int_{t_0}^t u'(v)dv$ i.e,

$$|u(t)| = \left|u(t_0) + \int_{t_0}^t u'(v)dv\right| \le D + \int_{t_0}^t |u'(v)|dv.$$

Hence, $\|u\|_\infty \le D + \|u'\|_\infty$. Also, since $u'(t) = u(t_1) + \int_{t_1}^t u''(v)dv$, then

$$|u'(t)| = \left|u(t_1) + \int_{t_1}^t u''(v)dv\right| \le D + \int_{t_1}^t |u''(v)|dv$$

Hence, $\|u'\|_\infty \le D + \|u''\|_\infty$. Thus,

$$\|u\|_\infty \le 2D + \|u''\|$$

.

Therefore,

$$\begin{aligned}\|u\| &= \max_{t\in[0,1]}\{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty\}\\ &\le 2D + \|u''\|_\infty.\end{aligned} \tag{14}$$

Now,

$$\begin{aligned}|u''(t)| &= \phi_q\left|\phi_p(|u''(t_2)|) + \int_{t_2}^t u'''(v)dv\right|\\ &\le \phi_q\left[\phi_p(|u''(t_2)|) + \int_{t_2}^t |N_\lambda u(v)|dv\right]\\ &\le \phi_q[\phi_p(D) + \|Nu\|_1].\end{aligned}$$

Suppose $\|Nu\|_1 \le \phi_q(D)$, then

$$\|u''\|_\infty \le \phi_q(2\|Nu\|\|_1).$$

For $1 < p < 2$, considering (10) and lemma 3, we have

$$\begin{aligned}\|u''\|_\infty &\le \phi_q(2\|Nu\|\|_1)\\ &\le \phi_q(2)[2^{q-2}(\phi_q(\|x\|_\infty\|u\|_\infty^{q-1} + \|y\|_\infty\|u'\|_\infty^{q-1})\\ &+ \phi_q(\|z\|_\infty\|u''\|_\infty^{q-1} + \|h\|_\infty))]\\ &\le \phi_q(2)2^{2q-4}[\|x\|_\infty^{q-1}\|u\|_\infty\\ &+ \|y\|_\infty^{q-1}\|u'\|_\infty\\ &+ \|z\|_\infty^{q-1}\|u''\|_\infty + \|h\|_\infty^{q-1}]\\ &\le \phi_q(2)2^{2q-4}[\|u\|(\|x\|_\infty^{q-1} + \|y\|_\infty^{q-1}\\ &+ \|z\|_\infty^{q-1} + \|h\|_\infty^{q-1}).\end{aligned}$$

From (14), we have

$$\begin{aligned}\|u\| &\le 2D + \|u''\|_\infty\\ &= 2D + \phi_q(2)2^{2q-4}[\|u\|(\|x\|_\infty^{q-1} + \|y\|_\infty^{q-1}\\ &+ \|z\|_\infty^{q-1} + \|h\|_\infty^{q-1})\end{aligned}$$

or

$$\|u\| \le \frac{2D + \phi_q(2)2^{2q-4}\|h\|_\infty^{q-1}}{1 - \phi_q(2)2^{2q-4}[\|x\|_\infty^{q-1} + \|y\|_\infty^{q-1} + \|z\|_\infty^{q-1}]} \tag{15}$$

Let $D_1 = \frac{2D+\phi_q(2)2^{2q-4}\|h\|_\infty^{q-1}}{1-\phi_q(2)2^{2q-4}[\|x\|_\infty^{q-1}+\|y\|_\infty^{q-1}+\|z\|_\infty^{q-1}]}$, in view of (8), we see that $D_1 > 0$ and $\|u\| \le D_1$. Hence, Ω_1 is bounded.
For $p \ge 2$,

$$\begin{aligned}\|u''\|_\infty &\le \phi_q(2\|Nu\|\|_1)\\ &\le \phi_q(2)[\|x\|_\infty^{q-1}\|u\|_\infty\\ &+ \|y\|_\infty^{q-1}\|u'\|_\infty + \|z\|_\infty^{q-1}\|u''\|_\infty + \|h\|_\infty^{q-1}]\\ &\le \phi_q(2)[\|u\|(\|x\|_\infty^{q-1} + \|y\|_\infty^{q-1} + \|z\|_\infty^{q-1} + \|h\|_\infty^{q-1}).\end{aligned}$$

From (14), we have

$$\begin{aligned}\|u\| &\le 2D + \|u''\|_\infty\\ &= 2D + \phi_q(2)[\|u\|(\|x\|_\infty^{q-1} + \|y\|_\infty^{q-1}\\ &+ \|z\|_\infty^{q-1} + \|h\|_\infty^{q-1})\end{aligned}$$

or

$$\|u\| \le \frac{2D + \phi_q(2)\|h\|_\infty^{q-1}}{1 - \phi_q(2)[\|x\|_\infty^{q-1} + \|y\|_\infty^{q-1} + \|z\|_\infty^{q-1}]} \tag{16}$$

Let $D_1 = \frac{2D+\phi_q(2)\|h\|_\infty^{q-1}}{1-\phi_q(2)[\|x\|_\infty^{q-1}+\|y\|_\infty^{q-1}+\|z\|_\infty^{q-1}]}$, in view of (9), we see that $D_1 > 0$ and $\|u\| \le D_1$. Hence, Ω_1 is bounded.
We next let

$$\Omega_2 = \{u \in \ker M : Nu \in \operatorname{Im} M\}.$$

If $u \in \Omega_2$, then $u \in \ker M$ and u can be defined as $u(t) = \omega,\ t \in [0,1]$, ω is an arbitrary constant.
Since $QNu = 0$, then

$$\sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \int_0^x w(v,d,0,0)dvdt = 0.$$

From (C_3), it follows that $\|u\| = \omega \leq F$. Hence, Ω_2 is bounded.
Let the isomorphism $J : \operatorname{Im} Q \to \ker L$ be defined as

$$J(dt^r) = d, \ d \in \mathbb{R}.$$

If $d \cdot \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \int_0^x w(v, d, 0, 0) dv dt < 0$, we define

$$\Omega_3 = \{u \in \ker M : \lambda J^{-1} u = (1-\lambda) QNu, \ \lambda \in [0,1]\}.$$

For $u \in \Omega_3$, we have

$$\lambda dt^r = t^r(1-\lambda)\frac{(r+1)(r+2)}{\sum_{i=1}^m \alpha_i \xi^{r+2}} \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \int_0^x w(v,d,0,0) dv dt.$$

When $\lambda = 1$, $d = 0$. However, when $|d| > F$, in view of (11), we obtain

$$\lambda d^2 t^r = t^r d(1-\lambda)\frac{(r+1)(r+2)}{\sum_{i=1}^m \alpha_i \xi^{r+2}} \sum_{i=1}^m \alpha_i \int_0^{\xi_i} \int_0^x w(v,d,0,0) dv dt < 0,$$

which contradicts $\lambda d^2 t^r > 0$. Therefore $|u| = |d| \leq F$, implying that $\|u\| \leq F$. Hence Ω_3 is bounded.
If $d \cdot \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \int_0^x w(v, d, 0, 0) dv dt > 0$, we define

$$\Omega_3 = \{u \in \ker M : \lambda J^{-1} u = -(1-\lambda) QNu, \ \lambda \in [0,1]\}.$$

Similar arguments can be used to show that Ω_3 is bounded. This concludes the proof of Theorem 2.

Finally, we will show that all the conditions of Theorem 1 are satisfied. Take an open bounded set $\Omega \subset U$ such that $U_{i=1}^3 \overline{\Omega}_i \subset \Omega$. Lemma 3 shows that M is a quasi-linear operator while Lemma 4 shows that N_λ is M-compact on $\overline{\Omega}$. Thus conditions (i) and (ii) of Theorem 1 are satisfied. Finally, we show that (iii) also holds. Set $E(u, \lambda) = \pm \lambda u + (1-\lambda) JQNu$, $J(dt^r) = d$. When $\lambda = 0$, $JQNu \neq 0$, for $\lambda = 1$, $E(u, 1) = \pm Idu \neq 0$. For $\lambda \in (0,1)$, from (C_3), we see that $E(u, 0) \neq 0$. Then based on the above argument, for every $u \in \ker M \cap \partial\Omega$, $E(u, \lambda) \neq 0$. Therefore, the homotopy property of degree gives

$$\begin{aligned}\deg(JQN|_{\ker M}, \Omega \cap \ker M, 0) &= \deg(E(\cdot, 0), \Omega \cap \ker M, 0)\\ &= \deg(E(\cdot, 1), \Omega \cap \ker M, 0)\\ &= \deg(\pm Id, \Omega \cap \ker M, 0) = \pm 1\\ &\neq 0.\end{aligned}$$

Therefore condition (iii) of Theorem 1 holds and problem (1)-(2) has at least one solution in $\overline{\Omega}$.

4 Example

We will consider the following p-Laplacian boundary value problem

$$(\phi_3(u''(t)))' = t + 5u(t)^2 + 12\cos(u'(t)^2) + 12u''(t)^2, \ t \in (0,1), \tag{17}$$

$$\begin{aligned}&\phi_3(u''(0)) = 6\int_0^{\frac{1}{3}} \phi_3 u''(t) dt - 2\int_0^{\frac{1}{2}} \phi_3 u''(t) dt,\\ &u''(1) = 0, \ u'(1) = 3u'\left(\frac{1}{2}\right),\end{aligned} \tag{18}$$

where $p = 3 > 2$, $q = \frac{2}{3}$, $\alpha_1 = 6$, $\alpha_2 = -2$, $\xi_1 = \frac{1}{3}$, $\xi_2 = \frac{1}{2}$, $\eta = \frac{1}{2}$, and $\beta = 3$. Also,

$$w(t, a, b, c) = t + 5a^2 + 12(\cos b^2) + 12c^2.$$

The resonance condition is fulfilled since, $\alpha_1 + \alpha_2 = 4 - 2 = 2 \neq 0$ and
$\alpha_1\xi_1 + \alpha_2\xi_2 = (4)\left(\frac{1}{2}\right) + (-2)\left(\frac{1}{2}\right) = 1$. Now

$$\begin{aligned}|w(t,a,b,c)| &\leq |t| + 5|a|^2 + 12|\cos b^2| + 12|c|^2\\ &= 1 + 5|a|^2 + 12 + 12|c|^2\\ &= 13 + 5|a|^2 + 12|c|^2.\end{aligned}$$

Since $x(t) = 5$, $y(t) = 0$, $z(t) = 12$, $t \in (0,1)$, then

$$\begin{aligned}\phi_q(2)[\|x\|_\infty^{q-1} + \|y\|_\infty^{q-1} + \|z\|_\infty^{q-1}] &= 2^{-\frac{1}{3}}[5^{-\frac{1}{3}} + 12^{-\frac{1}{3}}]\\ &= 0.6934(0.5848 + 0.4368) = 0.7083 < 1.\end{aligned}$$

Therefore, condition (E_1) is satisfied.
Next we show that condition (E_2) holds. Let $D = 3$. and $u \in \operatorname{dom} M$. if $|u(t)| > D$, $t \in (0,1)$, then either $u(t) > D$ or $u(t) < -D$.
For $u(t) > D$, we have

$$\begin{aligned}&\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \int_0^t w(v, u, u', u'') dv dt\\ &= 4\int_0^{\frac{1}{2}} \int_0^t \left(v + 5u^2 + 12(\cos(u')^2 + 12(u'')\right) dv dt\\ &- 2\int_0^{\frac{1}{2}} \int_0^t \left(v + 5u^2 + 12\cos(u')^2 + 12(u'')^2\right) dv dt\\ &> 4\int_0^{\frac{1}{2}} \int_0^t \left(v + 5D^2 - 12 + 12D^2\right) dv dt\\ &- 2\int_0^{\frac{1}{2}} \int_0^t \left(v + 5D^2 - 12 + 12D^2\right) dv dt\\ &> \frac{17}{4}D^2 - \frac{47}{24} > 0.\end{aligned}$$

Similarly, if $u(t) < -D$, then

$$\begin{aligned}
&\sum_{i=1}^{m}\alpha_i\int_0^{\xi_i}\int_0^t w(v,u,u',u'')dvdt\\
&= 4\int_0^{\frac{1}{2}}\int_0^t\Big(v+5u^2+12\cos(u')^2+12(u'')^2\Big)dvdt\\
&- 2\int_0^{\frac{1}{2}}\int_0^t\Big(v+5u^2+12\cos(u')^2+12(u'')^2\Big)dvdt\\
&< 4\int_0^{\frac{1}{2}}\int_0^t\Big(v-5D^2+12-12D^2\Big)dvdt\\
&- 2\int_0^{\frac{1}{2}}\int_0^t\Big(v-5D^2+12-12D^2\Big)dvdt\\
&< \frac{73}{24}-\frac{17}{4}D^2 < 0
\end{aligned}$$

Therefore, condition (E_2) holds.
Finally, we will show that condition (E_3) holds. Here,

$$\begin{aligned}
&d\cdot\sum_{i=1}^{m}\alpha_i\int_0^{\xi_i}\int_0^t w(v,d,0,0)dvdt\\
&= d\Big[4\int_0^{\frac{1}{2}}\int_0^t\Big(v+\frac{1}{5}d\Big)dvdt - 2\int_0^{\frac{1}{2}}\int_0^t\Big(v+\frac{1}{5}d\Big)dvdt\Big]\\
&= d\Big[\frac{1}{20}d+\frac{1}{24}\Big]
\end{aligned}$$

Let $F=\frac{1}{6}>0$, then for $c\in\mathbb{R}$, such that $|d|>F$, then either $d>F$ or $d<-F$. For $d>F$, we have

$$d\cdot\sum_{i=1}^{m}\alpha_i\int_0^{\xi_i}\int_0^t w(v,d,0,0)dvdt > 0,$$

while for $d<F$,

$$d\cdot\sum_{i=1}^{m}\alpha_i\int_0^{\xi_i}\int_0^t w(v,d,0,0)dvdt < 0.$$

Thus, Condition (E_3) is holds. The p-Laplacian problem (13) - (14) has at least one solution in $C^2[0,1]$ since it satisfies Theorem 2.

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