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A new approach for the solution of the Black-Scholes equation with barrier option constraints

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Abstract. This paper introduces an efficient approach to solve the Black-Scholes Partial Differential Equation (BSPDE) with Barrier Option Constraints (BCOs). The approach of the Laplace –Adomian Decomposition Method (LADM), which is the combination of the Laplace Transform Method (LTM) and the Adomian Decomposition Method (ADM) is employed. The LTM is applied to the BSPDE, and the ADM is used for decomposing the solution of BSPDE into an infinite series. Moreover, the approximate solution obtained via LADM is expressed in the form of a convergent series with computed components. An illustrative example is presented, and the results are compared with the Analytical Value (AV). Hence, LADM is found to be effective and a powerful technique for obtaining an approximate solution of BSPDE with BCOs.

Keywords: Adomian decomposition method, Approximate solution, Laplace transform method, Barrier option constraint, Black-Scholes equation

1. Introduction

In recent years, the valuation of derivative security has become extremely popular; this popularity exceeds that of the stock exchange. This is because diverse derivative security products have been developed as investment instruments. A derivative security is defined as a financial asset whose value is derived in part from the value and characteristics of some other underlying assets. An option is a contract between two parties, the holder and the writer. The holder or buyer of an option purchases the right to buy or sell an underlying asset under some given conditions. On the other hand, the writer or seller of the option, for the price of the option, takes on the obligation to deliver (or buy) the asset under the conditions of the contract. An option comes in two ways, namely, call option that gives the holder right to buy and put option that gives the holder right to sell. According to [1], "a barrier option is a type of derivative where the payoff depends on whether or not the underlying asset has reached or exceeded a predetermined price. A barrier option can be a knock-out, if the underlying exceeds a certain price, limiting profits for the holder and limiting losses for the writer. It can also be a knock-in, if it has no value until the underlying reaches a certain price." Black and Scholes [2] derived the BSPDE with a continuous-time approach. They also developed the Black-Scholes model for the valuation of the European call and put options. There are several methods for the solution of the BSPDE with the European option constraints, such as the LADM [3], Mellin transform method [4], finite difference method [5], projected differential transformation method [6], reduced differential transform [7], ADM [8], homotopy perturbation method [9, 10], just to mention a few. In a similar manner, other possible techniques such as those of [11-14].

In this paper, a new approach for the solution of the BSPDE with up-and-out BCOs is presented. The rest of the paper is structured as follows; Section is the preliminaries, Section 3 presents the analysis of the LADM. In Section 4, the LADM is employed to obtain an approximate solution of BSE with BCOs. In Section 5, LADM is applied for the valuation of a barrier option. Section 6 concludes the paper with the discussion of results.

2. Preliminaries

Consider the well-known BSPDE of the form

$$\frac{\partial V(S,t)}{\partial t}+\frac{\sigma^2 S^2}{2}\frac{\partial^2 V(S,t)}{\partial S^2}+rS\frac{\partial V(S,t)}{\partial S}-rV(S,t)=0 \tag{1}$$

where $V(S,t)$ is the price of the option, S is the asset price, t is the current time, σ is the volatility, r is the risk-free interest rate, $(S,t)\,\epsilon\,\mathbb{R}^{+}\times(0,T)$ and T is the expiry date. Equation (1) is reduced to a heat-like equation with a few terms as follows:

Let

$$S=e^{x},\ t=T-\frac{2\tau}{\sigma^2}\ \text{and}\ V(S,t)=v(x,\tau) \tag{2}$$

Therefore,

$$\frac{\partial V(S,t)}{\partial t}=-\frac{\sigma^2}{2}\frac{\partial v(x,\tau)}{\partial \tau},\ \frac{\partial^2 V(S,t)}{\partial s^2}=-\frac{1}{S}\frac{\partial v(x,\tau)}{\partial x}+\frac{1}{S^2}\frac{\partial^2 v(x,\tau)}{\partial x^2},\ \frac{\partial V(S,t)}{\partial S}=\frac{1}{S}\frac{\partial v(x,\tau)}{\partial x} \tag{3}$$

Substituting (2) and (3) into (1) yields:

$$\frac{\partial v(x,\tau)}{\partial \tau}-\frac{\partial^2 v(x,\tau)}{\partial x^2}-\left(\frac{2r}{\sigma^2}-1\right)\frac{\partial v(x,\tau)}{\partial x}+\frac{2r}{\sigma^2}v(x,\tau)=0 \tag{4}$$

Setting

$$D=\frac{2r}{\sigma^2} \tag{5}$$

Equation (4) becomes

$$\frac{\partial v(x,\tau)}{\partial \tau}-\frac{\partial^2 v(x,\tau)}{\partial x^2}-(D-1)\frac{\partial v(x,\tau)}{\partial x}+Dv(x,\tau)=0 \tag{6}$$

In this paper, a new approach of the LADM for the solution of (6) subject to the up-and-out barrier option constraints [15]

$$v(x,\tau)=0,\ e^{x}\geq e^{B_u},\qquad 0\leq\tau<T \tag{7}$$

$$v(x,0)=(e^{x}-K),\qquad 0<e^{x}<e^{B_u} \tag{8}$$

where B_u is the barrier/ boundary.

3. Analysis of the Laplace-Adomian Decomposition Method

Consider the non-linear PDE of the form

$$G_t\,w(x,t)+N\,w(x,t)+R\,w(x,t)=h(x,t) \tag{9}$$

Subject to

$$w(x,0)=g(x) \tag{10}$$

Where $G_t = \frac{\partial}{\partial t}$, $w(x,t)$ is a function of two variables, N and R are non-linear and linear operators, $h(x,t)$ is a given function and $w(x,0) = g(x)$ is the value of $w(x,t)$ when $t = 0$. By means of the Laplace transform, (9) becomes

$$LT\big(G_t\ w(x,t) + N\,w(x,t) + R\,w(x,t) - h(x,t)\big) = 0 \tag{11}$$

Thus,

$$LT\left(\frac{\partial w(x,t)}{\partial t} + N\,w(x,t) + R\,w(x,t) - h(x,t)\right) = 0 \tag{12}$$

Equation (12) becomes

$$LT\left(\frac{\partial w(x,t)}{\partial t}\right) = LT\big(h(x,t) - N\,w(x,t) - R\,w(x,t)\big) \tag{13}$$

Therefore,

$$LT\left(\frac{\partial w(x,t)}{\partial t}\right) = \big(s\,w(x,s) - w(x,0)\big) \tag{14}$$

Using (14) in (13), one gets

$$sw(x,s) - w(x,0) = LT\big(h(x,t) - N\,w(x,t) - R\,w(x,t)\big) \tag{15}$$

Substituting (10) into (15) yields;

$$sw(x,s) - g(x) = LT\big(h(x,t) - N\,w(x,t) - R\,w(x,t)\big) \tag{16}$$

Simplifying (16) further and by means of the Laplace inverse transform, we obtain

$$LT^{-1}\big(w(x,s)\big) = g(x) + LT^{-1}\big\{s^{-1}\big[LT\big(h(x,t) - N\,w(x,t) - R\,w(x,t)\big)\big]\big\} \tag{17}$$

Therefore,

$$w(x,t) = g(x) + LT^{-1}\big\{s^{-1}\big[LT\big(h(x,t) - N\,w(x,t) - R\,w(x,t)\big)\big]\big\} \tag{18}$$

The function $w(x,t)$ can be decomposed into an infinite series by means of the ADM as follows

$$w(x,t) = \lim_{m\to\infty} \sum_{n=0}^{m} w_n(x,t) \tag{19}$$

Similarly,

$$Nw(x,t) = \lim_{m\to\infty} \sum_{n=0}^{m} B_n \tag{20}$$

Where $B_n = B_n(w_0, w_1, w_2, \ldots, w_n)$ are the Adomian polynomials defined as

$$B_n = \frac{1}{\Gamma(n+1)} \frac{d^n}{d\lambda^n}\left[N\left(\sum_{k=0}^{n} \lambda^k w_k\right)\right]_{\lambda=0},\ \ n \in \mathbb{Z}^+ \tag{21}$$

Substituting (19)and (20) into (18)

$$\lim_{m\to\infty} \sum_{n=0}^{m} w_n(x,t) = g(x) + LT^{-1}\left[\frac{1}{s} LT\big(h(x,t)\big) - \frac{1}{s} LT\left(\lim_{m\to\infty} \sum_{n=0}^{m} B_n\right) - \frac{1}{s} LT\left(R \lim_{m\to\infty} \sum_{n=0}^{m} w_n(x,t)\right)\right] \tag{22}$$

From (22), the recursive relation of the solution is obtained as follows;

$$w_0(x,t) = g(x) + LT^{-1}\left[\frac{1}{s}\, LT\big(h(x,t)\big)\right] \tag{23}$$

$$w_{n+1}(x,t) = -\,LT^{-1}\left[\frac{1}{s}\, LT(B_n) + \frac{1}{s} LT\big(R\, w_n(x,t)\big)\right], n \in \mathbb{Z}^+ \tag{24}$$

Therefore, the solution of (9) via the LADM is obtained as

$$w(x,t) = \sum_{n=0}^{\infty} w_n(x,t) \tag{25}$$

4. Laplace-Adomian Decomposition Method for the Solution of BSPDE with BOCs

By means of LADM, the recursive solution of (6) subject to (8) is obtained as

$$v(x,0) = e^x - K \tag{26}$$

$$v_{n+1} = LT^{-1}\left[\frac{1}{s}LT\left(\frac{\partial^2 v_n}{\partial x^2} + (D-1)\frac{\partial v_n}{\partial x} - Dv_n\right)\right]. \tag{27}$$

Using (26) and (27), the following series is obtained

$$v_1 = LT^{-1}\left[\frac{1}{s}LT\left(\frac{\partial^2 v_0}{\partial x^2} + (D-1)\frac{\partial v_0}{\partial x} - Dv_0\right)\right] = KD\tau \tag{28}$$

$$v_2 = LT^{-1}\left[\frac{1}{s}LT\left(\frac{\partial^2 v_1}{\partial x^2} + (D-1)\frac{\partial v_1}{\partial x} - Dv_1\right)\right] = -\frac{1}{2}KD^2\tau^2 \tag{29}$$

$$v_3 = LT^{-1}\left[\frac{1}{s}LT\left(\frac{\partial^2 v_2}{\partial x^2} + (D-1)\frac{\partial v_2}{\partial x} - Dv_2\right)\right] = \frac{1}{6}KD^3\tau^3 \tag{30}$$

$$v_4 = LT^{-1}\left[\frac{1}{s}LT\left(\frac{\partial^2 v_3}{\partial x^2} + (D-1)\frac{\partial v_3}{\partial x} - Dv_3\right)\right] = -\frac{1}{24}KD^4\tau^4 \tag{31}$$

$$v_5 = LT^{-1}\left[\frac{1}{s}LT\left(\frac{\partial^2 v_4}{\partial x^2} + (D-1)\frac{\partial v_4}{\partial x} - Dv_4\right)\right] = \frac{1}{120}KD^5\tau^5 \tag{32}$$

Continuing this way, we have that

$$v_j = LT^{-1}\left[\frac{1}{s}LT\left(\frac{\partial^2 v_{j-1}}{\partial x^2} + (D-1)\frac{\partial v_{j-1}}{\partial x} - Dv_{j-1}\right)\right] = \frac{(-1)^{j+1}}{j!}KD^j\tau^j \tag{33}$$

Hence, the solution of BSPDE with BCOs is obtained as

$$\begin{aligned} v(x,\tau) &= \lim_{m\to\infty}\sum_{n=0}^{m} v_n(x,t) = v_0(x,t) + v_1(x,t) + v_2(x,t) + \cdots v_j(x,t) + \cdots \\ &= (e^x - K) + KD\tau - \frac{1}{2}KD^2\tau^2 + \frac{1}{6}KD^3\tau^3 - \frac{1}{24}KD^4\tau^4 + \frac{1}{120}KD^5\tau^5 + \cdots + \frac{(-1)^{j+1}}{j!}KD^j\tau^j + \cdots \\ &= e^x - K\exp(-D\tau) \end{aligned} \tag{34}$$

With

$$S = e^x,\ \ \tau = \frac{\sigma^2}{2}(T-t),\ \ \ D = \frac{2r}{\sigma^2}$$

Using (34), the price of the barrier option is obtained as

$$V(s,t) = (S - K\exp(-D\tau)) = S - K\exp\left(-\frac{2r}{\sigma^2}\frac{(T-t)}{2}\sigma^2\right) = S - K\,\mathrm{e}^{-\mathrm{r(T-t)}} \tag{35}$$

With $S < e^{B_u}$

Equation (35) is the LADM formula for the valuation of Barrier option.

5. Application of LADM in the Valuation of Barrier Option

Using (35) and the following parameters:

Table 1: The Parameters

Parameter	Description	Values	Source
S	Underlying asset price	\$80	Assumed
K	Strike Price	\$20, \$30, \$40, \$50	Assumed
T	Expiry date	2, 4, 6, 8 (years)	Assumed
r	Risk-free interest rate	10%, 20%, 30%, 40%, 50%	Assumed
Sigma	Volatility	25%	Assumed
B_u	Barrier/Boundary	\$85	Assumed

The results generated via LADM are displayed in Figures 1-5. The comparative result analyzes of the LADM, and the Analytical Value (AV) [16] is presented in Figures 6-9.

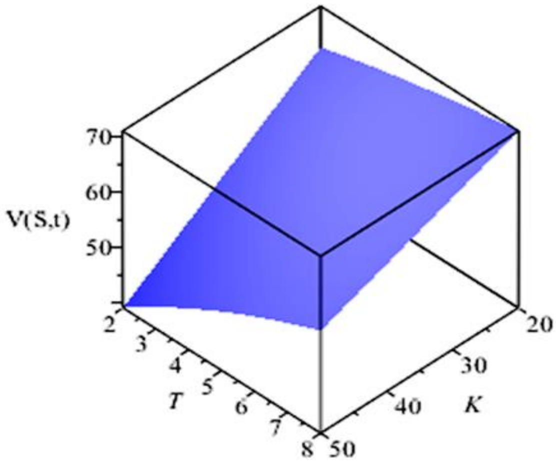

Figure 1: Barrier option prices via LADM with r = 10%

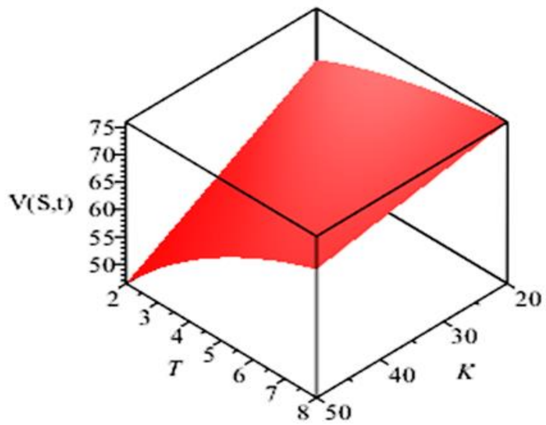

Figure 2: Barrier option prices via LADM with r = 20%

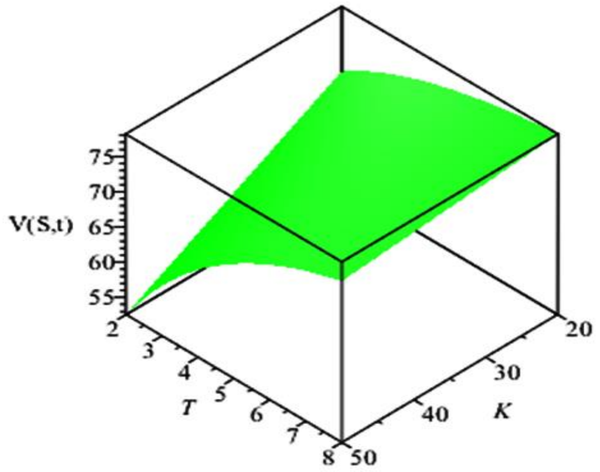

Figure 3: Barrier option prices via LADM with r = 30%

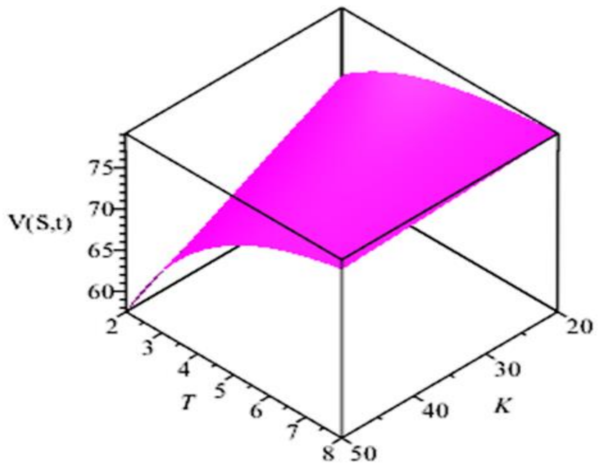

Figure 4: Barrier option prices via LADM with r = 40%

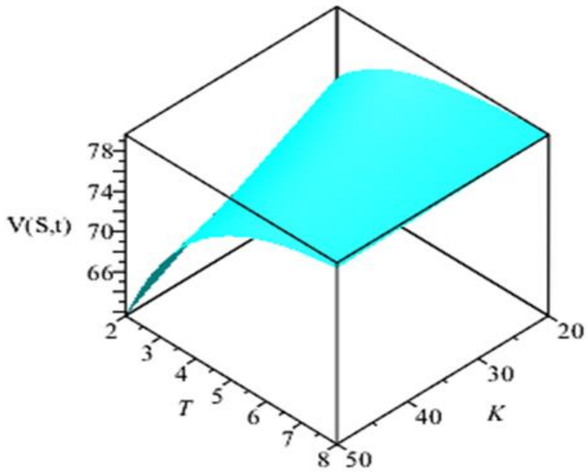

Figure 5: Barrier option prices via LADM with r = 50%

Table 2: Barrier option prices via LADM and AV with $T = 2$

K	LADM	Analytic Value [16]
20	63.6254	63.6232
30	55.4381	53.4336
40	47.2508	45.9519
50	39.0635	38.5482

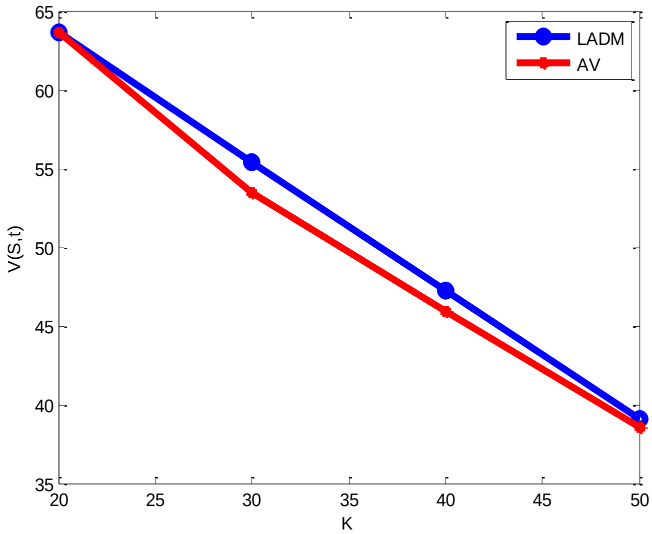

Figure 6: LADM versus AV using Table 2

Table 3: Barrier option prices via LADM and AV with $T = 4$

K	LADM	Analytic Value [16]
20	66.5936	66.5942
30	59.8904	59.9064
40	53.1872	53.3025
50	46.4840	46.9069

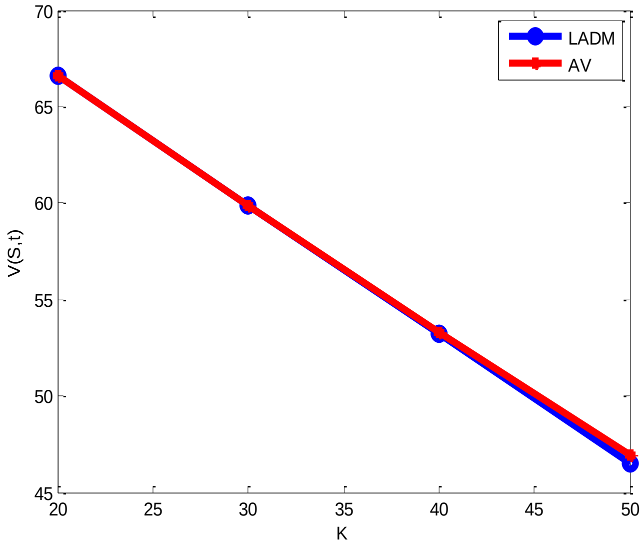

Figure 7: LADM versus AV using Table 3

Table 4: Barrier option prices via LADM and AV with $T = 6$

K	**LADM**	**Analytic Value [16]**
20	69.0238	69.0249
30	63.5357	63.5657
40	58.0475	58.1994
50	52.5594	53.0023

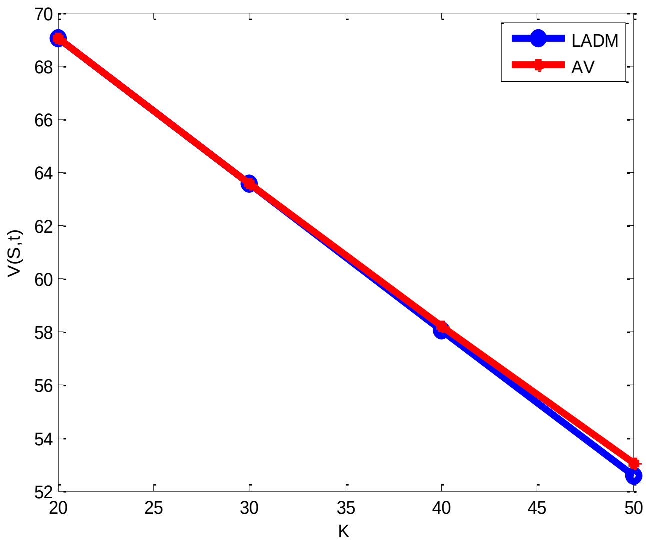

Figure 8: LADM versus AV using Table 4

Table 5: Barrier option prices via LADM and AV with $T = 8$

K	**LADM**	**Analytic Value [16]**
20	71.0134	71.0183
30	66.5201	66.5610
40	62.0268	62.1862
50	57.5336	57.9445

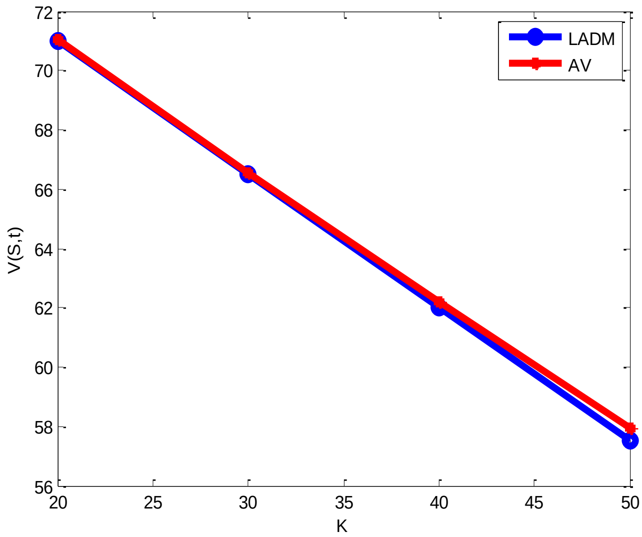

Figure 9: LADM versus AV using Table 5

6. Discussion of Results and Concluding Remarks

In this paper, a new approach, "LADM" is employed for the solution of the BSPDE with BOCs. By varying the risk-free interest rate, it is observed from Figures 1-5 that the price of the barrier call option generated via LADM increases as the risk-free interest rate increases. It is also observed that the plots of LADM follow that of the AV elegantly. The option prices generated via LADM and AV were displayed in Tables 2-5. It is observed from Figures 6-9 that LADM compared favourably and agreed with the AV for different values of T. It is also observed that time to expiry has an influence on the option prices. In other words, an increase in T leads to an increase in the option prices. Moreover, LADM is found to be computationally effective and a good alternative approach for obtaining the approximate solution of the BSPDE with BOCs. Hence, it can be concluded that the LADM is a good tool to be included in the class of methods for the valuation of barrier option.

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