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Implementation of Two-step Hybrid Block Adams Moulton Solution Methods for First Order Delay Differential Equations

C. Chibuisi[1], B. O. Osu[2,4], S. O. Edeki[3*] , G..O. Akinlabi[3], C. Olunkwa[4], and O. P. Ogundile[3]

[1]Department of Insurance, University of Jos, Jos, Nigeria
[2]Department of Mathematics, Michael Okpara University of Agriculture, Nigeria
[3]Department of Mathematics, Covenant University, Ota, Nigeria
[4]Department of Mathematics, Abia State University, Nigeria

Emails:[1]chibuisichygoz@yahoo.com,[2]osu.bright@mouau.edu.ng, [3*]soedeki@yahoo.com

Abstract. In this paper, Hybrid Block Adams Moulton Methods for step number $k=2$ merged with two and three off-grid points were obtained and implemented in solving first order delay differential equations without the use of interpolation condition in evaluating the delay expression. The discrete schemes of these off-grid hybrid block methods were assessed through the continuous development of the linear multistep collocation method using a matrix conversion formula. The results obtained after the implementation of the proposed method in for numerical experiment of some first-order DDEs, the BHAMM2 schemes performed better and faster in satisfying the axioms for convergence and region of absolute stability than the BHAMM3 schemes at fixed step size z when examined with other existing methods.

Keywords: First-order delay differential equations, hybrid block method, off-grid point, Adams Moulton method.

1. Introduction

Scholars [1, 2, 3, 6, 9, 12, 14] use interpolation conditions such as Nordsieck, Hermite, Newton divided difference, and Neville's interpolation in checking the delay expression for numerical solutions of DDEs and experienced some setbacks. One of the setbacks experienced by these scholars was studied by [8] and revealed that the order of the interpolating polynomials has to be the same with the numerical method in carrying out the approximate solutions of delay differential equations (DDEs), which is very hard to arrive at, thereby making the accuracy of the method not to be conserved.

We considered an accurate and efficient formula developed by [13] to overcome this challenge, and this has been carried out appropriately by [4, 5, 11]. Other solution methods can also be adopted [15-20].

In this research, we use the general form of the first-order DDEs formulated by [1] for the proposed method, which is presented as

$$a'(t) = f(t, a(t), a(t-\tau)), \quad \text{for } t > t_0, \tau > 0$$
$$a(t) = \varphi(t), \text{ for } t \le t_0 \tag{1}$$

for $\varphi(t)$ is the elementary function, τ implies the delay, $(t-\tau)$ is the delay expression, and $y(t-\tau)$ is the solution of the delay expression.

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2. Formulation of Multistep Collocation Technique

The *k*-step linear multistep collocation system with *s* collocation points was derived in [1] as

$$y(x)=\sum_{q=0}^{r-1}\delta_q(x)y_{m+q}+z\sum_{q=0}^{s-1}\psi_q(x)f_{m+q}(x,y(x)) \quad (2)$$

with $\delta_q(x)$ and $\psi_q(x)$ as continuous coefficients of the condition defined as

$$\delta_q(x)=\sum_{p=0}^{r+s-1}\delta_{q,p+1}x^p \text{ for } q=\{0,1,\dots,r-1\}$$

(3)

$$z\psi_q(x)=\sum_{p=0}^{r+s-1}z\psi_{q,p+1}x^p \text{ for } q=\{0,1,\dots,s-1\} \quad (4)$$

with $x_0,\dots,x_{s-1}$ as the s collocation points, $x_{n+q}, q=0,1,2,\dots,r-1$ are the r arbitrarily chosen interpolation points and z is the fixed step size.

To get $\delta_q(x)$ and $\psi_q(x)$, [10] arrived at a matrix equation such that:

$$GH=\mathrm{I} \quad (5)$$

where I is the elementary matrix of dimension $(r+s)\times(r+s)$ while G and H are matrices defined as

$$G=\begin{bmatrix} 1 & x_m & x_m^2 & \cdots & x_n^{r+s-1} \\ 1 & x_{m+1} & x_{m+1}^2 & \cdots & x_{m+1}^{r+s-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{m+r-1} & x_{m+r-1}^2 & \cdots & x_{m+r-1}^{r+s-1} \\ 0 & 1 & 2x_0 & \cdots & (r+s-1)x_0^{r+s-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2x_{s-1} & \cdots & (r+s-1)x_{s-1}^{r+s-2} \end{bmatrix} \quad (6)$$

$$H=\begin{bmatrix} \delta_{0,1} & \delta_{1,1} & \cdots & \delta_{r-1,1} & z\psi_{0,1} & \cdots & z\psi_{s-1,1} \\ \delta_{0,2} & \delta_{1,2} & \cdots & \delta_{r-1,2} & z\psi_{0,2} & \cdots & z\psi_{s-1,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \delta_{0,r+s} & \delta_{1,r+s} & \cdots & \delta_{r-1,r+s} & z\psi_{0,r+s} & \cdots & z\psi_{s-1,r+s} \end{bmatrix} \quad (7)$$

From (5), the columns of $H=G^{-1}$ yield the continuous coefficients of the said continuous scheme (2).

2.1 Formulation of two-step BHAMM with two off-grid points (BHAMM2)

Here, the interpolation points, $r=1$ and the collocation points $s=5$ are considered, therefore, (2) becomes

$$y(x)=\delta_1(x)y_{m+1}+z[\psi_0(x)f_m+\psi_{\frac{1}{2}}(x)f_{m+\frac{1}{2}}+\psi_1(x)f_{m+1}+\psi_{\frac{5}{4}}(x)f_{m+\frac{5}{4}}+\psi_2(x)f_{m+2}] \quad (8)$$

The matrix G in (5) becomes

$$G=\begin{pmatrix}1 & x_m+z & (x_m+z)^2 & (x_m+z)^3 & (x_m+z)^4 & (x_m+z)^5\\ 0 & 1 & 2x_m & 3x_m^2 & 4x_m^3 & 5x_m^4\\ 0 & 1 & 2x_m+z & 3(x_m+\frac{1}{2}z)^2 & 4(x_m+\frac{1}{2}z)^3 & 5(x_m+\frac{1}{2}z)^4\\ 0 & 1 & 2x_m+2z & 3(x_m+z)^2 & 4(x_m+z)^3 & 5(x_m+z)^4\\ 0 & 1 & 2x_m+\frac{5}{2}z & 3(x_m+\frac{5}{4}z)^2 & 4(x_m+\frac{5}{4}z)^3 & 5(x_m+\frac{5}{4}z)^4\\ 0 & 1 & 2x_m+4z & 3(x_m+2z)^2 & 4(x_m+2z)^3 & 5(x_m+2z)^4\end{pmatrix} \quad (9)$$

The matrix $H=G^{-1}$ is estimated by Maple 18 where the continuous scheme is gotten using (2) and computing it at $x=x_m, x=x_{m+\frac{1}{2}}, x=x_{m+\frac{5}{4}}$ and $x=x_{m+2}$, we obtained

$$\begin{aligned}
y_m &= y_{m+1}-\frac{4}{25}zf_m-\frac{94}{135}zf_{m+\frac{1}{2}}-\frac{1}{10}zf_{m+1}-\frac{32}{675}zf_{m+\frac{5}{4}}+\frac{1}{270}zf_{m+2}\\
y_{m+\frac{1}{2}} &= y_{m+1}+\frac{17}{1600}zf_m-\frac{241}{1080}zf_{m+\frac{1}{2}}-\frac{63}{160}zf_{m+1}+\frac{74}{675}zf_{m+\frac{5}{4}}-\frac{29}{8640}zf_{m+2}\\
y_{m+\frac{5}{4}} &= y_{m+1}+\frac{29}{25600}zf_m-\frac{157}{17280}zf_{m+\frac{1}{2}}+\frac{369}{2560}zf_{m+1}+\frac{619}{5400}zf_{m+\frac{5}{4}}-\frac{113}{138240}zf_{m+2}\\
y_{m+2} &= y_{m+1}-\frac{1}{25}zf_m+\frac{34}{135}zf_{m+\frac{1}{2}}-\frac{9}{10}zf_{m+1}+\frac{992}{675}zf_{m+\frac{5}{4}}+\frac{59}{270}zf_{m+2}
\end{aligned} \quad (10)$$

2.2 Formulation of two-step BHAMM with three off-grid points (BHAMM3)

In this case, $r=1$ and $s=6$ are considered, therefore, (2) becomes

$$y(x)=\delta_1(x)y_{m+1}+z[\psi_0(x)f_m+\psi_{\frac{1}{2}}(x)f_{m+\frac{1}{2}}+\psi_1(x)f_{m+1}+\psi_{\frac{5}{4}}(x)f_{m+\frac{5}{4}}+\psi_{\frac{3}{2}}(x)f_{m+\frac{3}{2}}+\psi_2(x)f_{m+2} \quad (11)$$

Also the matrix G in (5) becomes

$$G=\begin{pmatrix}1 & x_m+z & (x_m+z)^2 & (x_m+z)^3 & (x_m+z)^4 & (x_m+z)^5 & (x_m+z)^6\\ 0 & 1 & 2x_m & 3x_m^2 & 4x_m^3 & 5x_m^4 & 6x_m^5\\ 0 & 1 & 2x_m+z & 3(x_m+\frac{1}{2}z)^2 & 4(x_m+\frac{1}{2}z)^3 & 5(x_m+\frac{1}{2}z)^4 & 6(x_m+\frac{1}{2}z)^5\\ 0 & 1 & 2x_m+2z & 3(x_m+z)^2 & 4(x_m+z)^3 & 5(x_m+z)^4 & 6(x_m+z)^5\\ 0 & 1 & 2x_m+\frac{5}{2}z & 3(x_m+\frac{5}{4}z)^2 & 4(x_m+\frac{5}{4}z)^3 & 5(x_m+\frac{5}{4}z)^4 & 6(x_m+\frac{5}{4}z)^5\\ 0 & 1 & 2x_m+3z & 3(x_m+\frac{3}{2}z)^2 & 4(x_m+\frac{3}{2}z)^3 & 5(x_m+\frac{3}{2}z)^4 & 6(x_m+\frac{3}{2}z)^5\\ 0 & 1 & 2x_m+4z & 3(x_m+2z)^2 & 4(x_m+2z)^3 & 5(x_m+2z)^4 & 6(x_m+2z)^5\end{pmatrix}. \quad (12)$$

The matrix $H=G^{-1}$ is estimated by Maple 18 where the continuous scheme is gotten using (2) and computing it at $x=x_m, x=x_{m+\frac{1}{2}}, x=x_{m+\frac{5}{4}}, x=x_{m+\frac{3}{2}}$ and $x=x_{m+2}$, we obtained

$$y_m = y_{m+1} - \frac{3}{20} zf_m - \frac{103}{135} zf_{m+\frac{1}{2}} + \frac{1}{5} zf_{m+1} - \frac{64}{135} zf_{m+\frac{5}{4}} + \frac{1}{5} zf_{m+\frac{3}{2}} - \frac{7}{540} zf_{m+2}$$

$$y_{m+\frac{1}{2}} = y_{m+1} + \frac{1}{180} zf_m - \frac{409}{2160} zf_{m+\frac{1}{2}} - \frac{131}{240} zf_{m+1} + \frac{44}{135} zf_{m+\frac{5}{4}} - \frac{73}{720} zf_{m+\frac{3}{2}} + \frac{11}{2160} zf_{m+2}$$

$$y_{m+\frac{5}{4}} = y_{m+1} + \frac{13}{46080} zf_m - \frac{59}{17280} zf_{m+\frac{1}{2}} + \frac{911}{7680} zf_{m+1} + \frac{163}{1080} zf_{m+\frac{5}{4}} - \frac{49}{2880} zf_{m+\frac{3}{2}} + \frac{83}{138240} zf_{m+2}$$

$$y_{m+\frac{3}{2}} = y_{m+1} - \frac{1}{2160} zf_{m+\frac{1}{2}} + \frac{7}{80} zf_{m+1} + \frac{44}{135} zf_{m+\frac{5}{4}} + \frac{7}{80} zf_{m+\frac{3}{2}} - \frac{1}{2160} zf_{m+2}$$

$$y_{m+2} = y_{m+1} + \frac{1}{180} zf_m - \frac{7}{135} zf_{m+\frac{1}{2}} + \frac{7}{15} zf_{m+1} - \frac{64}{135} zf_{m+\frac{5}{4}} + \frac{41}{45} zf_{m+\frac{3}{2}} + \frac{77}{540} zf_{m+2}. \quad (13)$$

3. Convergence analysis

Here, the order and error constant, consistency, zero stability, convergence, and region of absolute stability are discussed.

3.1 Order and Error Constant

As developed by [7], the order and error constant for (10) are obtained as follows

$$b_0 = b_1 = b_2 = b_3 = b_4 = b_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}^T \text{ but } b_6 = \left(-\frac{1}{6400}, \frac{73}{921600}, \frac{49}{3686400}, -\frac{41}{57600} \right)^T.$$

Therefore, (10) has an order, $\Omega = 5$ and error constants

$$b_6 = -\frac{1}{6400}, \frac{73}{921600}, \frac{49}{3686400}, -\frac{41}{57600}$$

By using the same approach for (13) and can be presented as follows

$$b_0 = b_1 = b_2 = b_3 = b_4 = b_5 = b_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$$

but $b_7 = \left(\frac{13}{322560}, -\frac{157}{15482880}, -\frac{407}{495452160}, \frac{1}{5160960}, -\frac{5}{193536} \right)^T$

Therefore, (13) has order $\Omega = 6$ and error constants

$$b_7 = \frac{13}{322560}, -\frac{157}{15482880}, -\frac{407}{495452160}, \frac{1}{5160960}, -\frac{5}{193536}$$

3.2 Consistency

As stated by [7], (10) and (13) are said to be consistent since the orders $\Omega \geq 1$.

3.3 Zero Stability

For (10), the zero stability can be investigated as follows

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{m+\frac{1}{2}} \\ y_{m+1} \\ y_{m+\frac{5}{4}} \\ y_{m+2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{m-\frac{5}{4}} \\ y_{m-1} \\ y_{m-\frac{1}{2}} \\ y_m \end{pmatrix}$$

$$+h\begin{pmatrix} -\frac{94}{135} & -\frac{1}{10} & -\frac{32}{675} & \frac{1}{270} \\ -\frac{241}{1080} & -\frac{63}{160} & \frac{74}{675} & -\frac{29}{8640} \\ -\frac{157}{17280} & \frac{369}{2560} & \frac{619}{5400} & -\frac{113}{138240} \\ \frac{34}{135} & -\frac{9}{10} & \frac{992}{675} & \frac{59}{270} \end{pmatrix}\begin{pmatrix} f_{m+\frac{1}{2}} \\ f_{m+1} \\ f_{m+\frac{5}{4}} \\ f_{m+2} \end{pmatrix}+h\begin{pmatrix} 0 & 0 & 0 & -\frac{4}{25} \\ 0 & 0 & 0 & \frac{17}{1600} \\ 0 & 0 & 0 & \frac{29}{25600} \\ 0 & 0 & 0 & -\frac{1}{25} \end{pmatrix}\begin{pmatrix} f_{m-\frac{5}{4}} \\ f_{m-1} \\ f_{m-\frac{1}{2}} \\ f_m \end{pmatrix}$$

where

$$U_2^{(1)}=\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix},\quad U_1^{(1)}=\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $V_2^{(1)}=\begin{pmatrix} -\frac{94}{135} & -\frac{1}{10} & -\frac{32}{675} & \frac{1}{270} \\ -\frac{241}{1080} & -\frac{63}{160} & \frac{74}{675} & -\frac{29}{8640} \\ -\frac{157}{17280} & \frac{369}{2560} & \frac{619}{5400} & -\frac{113}{138240} \\ \frac{34}{135} & -\frac{9}{10} & \frac{992}{675} & \frac{59}{270} \end{pmatrix}$

$$\begin{aligned}\rho(\xi) &= \det(\xi U_2^{(1)} - U_1^{(1)}) \\ &= \left|\xi U_2^{(1)} - U_1^{(1)}\right| = 0\end{aligned} \tag{14}$$

We have,

$$\rho(\xi)=\left|\xi\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}-\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right|=\left|\begin{pmatrix} 0 & -\xi & 0 & 0 \\ \xi & -\xi & 0 & 0 \\ 0 & -\xi & \xi & 0 \\ 0 & -\xi & 0 & \xi \end{pmatrix}-\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right|$$

$$\Rightarrow \rho(\xi)=\begin{vmatrix} 0 & -\xi & 0 & 1 \\ \xi & -\xi & 0 & 0 \\ 0 & -\xi & \xi & 0 \\ 0 & -\xi & 0 & \xi \end{vmatrix}.$$

Using Maple (18) software,

$$\rho(\xi)=\xi^3(\xi-1)$$

$$\Rightarrow \xi^3(\xi-1)=0$$

$\Rightarrow \xi_1 = 1, \xi_2 = 0, \xi_3 = 0, \xi_4 = 0$. Considering that $\left|\xi_j\right| \le 1,\ \mathrm{j} = 1, 2, 3, 4$ the discrete schemes in (10) is zero stable.

For (13), the zero stability can be investigated as follows

$$\begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{m+\frac{1}{2}} \\ y_{m+1} \\ y_{m+\frac{5}{4}} \\ y_{m+\frac{3}{2}} \\ y_{m+2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{m-\frac{3}{2}} \\ y_{m-\frac{5}{4}} \\ y_{m-1} \\ y_{m-\frac{1}{2}} \\ y_m \end{pmatrix}$$

$$+h \begin{pmatrix} -\frac{103}{135} & \frac{1}{5} & -\frac{64}{135} & \frac{1}{5} & -\frac{7}{540} \\ -\frac{409}{2160} & \frac{131}{240} & \frac{44}{135} & -\frac{73}{720} & \frac{11}{2160} \\ -\frac{59}{17280} & \frac{911}{7680} & \frac{163}{1080} & -\frac{49}{2880} & \frac{83}{138240} \\ -\frac{1}{2160} & \frac{7}{80} & \frac{44}{135} & \frac{7}{80} & -\frac{1}{2160} \\ -\frac{7}{135} & \frac{7}{15} & -\frac{64}{135} & \frac{41}{45} & \frac{77}{540} \end{pmatrix} \begin{pmatrix} f_{m+\frac{1}{2}} \\ f_{m+1} \\ f_{m+\frac{5}{4}} \\ f_{m+\frac{3}{2}} \\ f_{m+2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{3}{20} \\ 0 & 0 & 0 & 0 & \frac{1}{180} \\ 0 & 0 & 0 & 0 & \frac{13}{46080} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{180} \end{pmatrix} \begin{pmatrix} f_{m-\frac{3}{2}} \\ f_{m-\frac{5}{4}} \\ f_{m-1} \\ f_{m-\frac{1}{2}} \\ f_m \end{pmatrix}$$

where $U_2^{(2)} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix}, U_1^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

and $V_2^{(2)} = \begin{pmatrix} -\frac{103}{135} & \frac{1}{5} & -\frac{64}{135} & \frac{1}{5} & -\frac{7}{540} \\ -\frac{409}{2160} & \frac{131}{240} & \frac{44}{135} & -\frac{73}{720} & \frac{11}{2160} \\ -\frac{59}{17280} & \frac{911}{7680} & \frac{163}{1080} & -\frac{49}{2880} & \frac{83}{138240} \\ -\frac{1}{2160} & \frac{7}{80} & \frac{44}{135} & \frac{7}{80} & -\frac{1}{2160} \\ -\frac{7}{135} & \frac{7}{15} & -\frac{64}{135} & \frac{41}{45} & \frac{77}{540} \end{pmatrix}$

$$\left.\begin{aligned}\rho(\xi) &= \det \xi\left(U_2^{(2)} - U_1^{(2)}\right) \\ &= \left|\xi U_2^{(2)} - U_1^{(2)}\right| = 0.\end{aligned}\right\} \tag{15}$$

We have,

$$\rho(\xi) = \left|\xi\begin{pmatrix}0 & -1 & 0 & 0 & 0\\ 1 & -1 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0\\ 0 & -1 & 0 & 1 & 0\\ 0 & -1 & 0 & 0 & 1\end{pmatrix} - \begin{pmatrix}0 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}\right| = \left|\begin{pmatrix}0 & -\xi & 0 & 0 & 0\\ \xi & -\xi & 0 & 0 & 0\\ 0 & -\xi & \xi & 0 & 0\\ 0 & -\xi & 0 & \xi & 0\\ 0 & -\xi & 0 & 0 & \xi\end{pmatrix} - \begin{pmatrix}0 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}\right|$$

$$\Rightarrow \rho(\xi) = \begin{pmatrix}0 & -\xi & 0 & 0 & 1\\ \xi & -\xi & 0 & 0 & 0\\ 0 & -\xi & \xi & 0 & 0\\ 0 & -\xi & 0 & \xi & 0\\ 0 & -\xi & 0 & 0 & \xi\end{pmatrix}.$$

Using Maple (18) software,

$$\rho(\xi) = \xi^4(\xi - 1)$$

$$\Rightarrow \xi^4(\xi - 1) = 0$$

$\Rightarrow \xi_1 = 1, \xi_2 = 0, \xi_3 = 0, \xi_4 = 0, \xi_5 = 0$. Considering that $\left|\xi_j\right| \le 1,\ j = 1,2,3,4,5$, (13) is zero stable.

3.4 Convergence

Both (10) and (13) are convergent by satisfying the necessary and sufficient condition regarding the convergence of a numerical method of being consistent and zero stable.

3.5 Region of Absolute Stability

The V-stability and R-stability of the proposed method shall be obtained from the DDEs of the form

$$\begin{aligned} a'(t) &= \gamma a(\mathrm{t}) + \omega a(t - \tau), & t \ge t_0 \\ a(t) &= g(t), & t \le t_0 \end{aligned} \tag{16}$$

where $g(t)$ is the initial function γ, ω are the coefficients, $\tau = mh, m \in \mathbb{Z}^+$ and z is the step size or length. Let $E_1 = z\gamma$ and $E_2 = z\omega$, then the V -and R -stability of (10) and (13) are worked-out and are represented in figure 1 to 4 below using Maple 18 and MATLAB.

Let $Y_{M+4} = \begin{pmatrix} y_{m+\frac{1}{2}} \\ y_{m+1} \\ y_{m+\frac{5}{4}} \\ y_{m+2} \end{pmatrix}, Y_M = \begin{pmatrix} y_{m-\frac{5}{4}} \\ y_{m-1} \\ y_{m-\frac{1}{2}} \\ y_m \end{pmatrix}, F_{M+4} = \begin{pmatrix} f_{m+\frac{1}{2}} \\ f_{m+1} \\ f_{m+\frac{5}{4}} \\ f_{m+2} \end{pmatrix}$ and $F_M = \begin{pmatrix} f_{m-\frac{5}{4}} \\ f_{m-1} \\ f_{m-\frac{1}{2}} \\ f_m \end{pmatrix}$

Since $U_2^{(1)} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, U_1^{(1)} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and

$$V_2^{(1)} = \begin{pmatrix} -\frac{94}{135} & -\frac{1}{10} & -\frac{32}{675} & \frac{1}{270} \\ -\frac{241}{1080} & -\frac{63}{160} & \frac{74}{675} & -\frac{29}{8640} \\ -\frac{157}{17280} & \frac{369}{2560} & \frac{619}{5400} & -\frac{113}{138240} \\ \frac{34}{135} & -\frac{9}{10} & \frac{992}{675} & \frac{59}{270} \end{pmatrix}$$

we have,

$$U_2^{(1)} Y_{M+2} = U_1^{(1)} Y_{M+1} - z \sum_{i=1}^{2} V_i^{(1)} F_{M+i} \quad (17)$$

For (13), we have

$$Y_{M+5} = \begin{pmatrix} y_{m+\frac{1}{2}} \\ y_{m+1} \\ y_{m+\frac{5}{4}} \\ y_{m+\frac{3}{2}} \\ y_{m+2} \end{pmatrix}, Y_M = \begin{pmatrix} y_{m-\frac{3}{2}} \\ y_{m-\frac{5}{4}} \\ y_{m-1} \\ y_{m-\frac{1}{2}} \\ y_m \end{pmatrix}, F_{M+5} = \begin{pmatrix} f_{m+\frac{1}{2}} \\ f_{m+1} \\ f_{m+\frac{5}{4}} \\ f_{m+\frac{3}{2}} \\ f_{m+2} \end{pmatrix} \text{ and } F_M = \begin{pmatrix} f_{m-\frac{3}{2}} \\ f_{m-\frac{5}{4}} \\ f_{m-1} \\ f_{m-\frac{1}{2}} \\ f_m \end{pmatrix}$$

Since $U_2^{(2)} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix}, U_1^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

and $V_2^{(2)} = \begin{pmatrix} -\frac{103}{135} & \frac{1}{5} & -\frac{64}{135} & \frac{1}{5} & -\frac{7}{540} \\ -\frac{409}{2160} & \frac{131}{240} & \frac{44}{135} & -\frac{73}{720} & \frac{11}{2160} \\ -\frac{59}{17280} & \frac{911}{7680} & \frac{163}{1080} & -\frac{49}{2880} & \frac{83}{138240} \\ -\frac{1}{2160} & \frac{7}{80} & \frac{44}{135} & \frac{7}{80} & -\frac{1}{2160} \\ -\frac{7}{135} & \frac{7}{15} & -\frac{64}{135} & \frac{41}{45} & \frac{77}{540} \end{pmatrix}$

we have,

$$U_2^{(2)}Y_{M+2} = U_1^{(2)}Y_{M+1} - z\sum_{i=1}^{2} V_i^{(2)} F_{M+i} \tag{18}$$

The determinants of V- and R-stability are formulated by merging (17) and (18) to (16) and (10) and (13) to (16) as stated below

$$\Psi^{(1)}(\xi) = \det\left[\left(U_2^{(1)} - E_1V_2^{(1)}\right)\xi^{2+r} - \left(U_1^{(1)} - E_1V_1^{(1)}\right)\xi^{1+r} - E_2\sum_{i=1}^{2} V_i^{(1)}\xi^{i}\right] \tag{19}$$

$$\Psi^{(2)}(\xi) = \det\left[\left(U_2^{(2)} - E_1V_2^{(2)}\right)\xi^{2+r} - \left(U_1^{(2)} - E_1V_1^{(2)}\right)\xi^{1+r} - E_2\sum_{i=1}^{2} V_i^{(2)}\xi^{i}\right] \tag{20}$$

and

$$\pi^{(1)}(\xi) = \det\left[U_2^{(1)}\xi^{2+r} - U_1^{(1)}\xi^{1+r} - E_2\sum_{i=1}^{2} V_i^{(1)}\xi^{i}\right] \tag{21}$$

$$\pi^{(2)}(\xi) = \det\left[U_2^{(2)}\xi^{2+r} - U_1^{(2)}\xi^{1+r} - E_2\sum_{i=1}^{2} V_i^{(2)}\xi^{i}\right], \tag{22}$$

Using MATLAB and Maple 18, the region of V- and R-stability for (10) and (13) are represented in Figures.1 to 4.

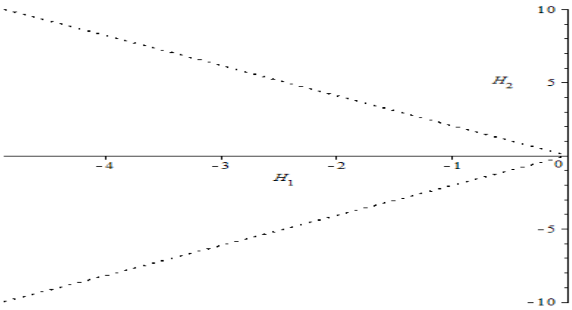

Figure.1.V-stability BHAMM2 in (10)

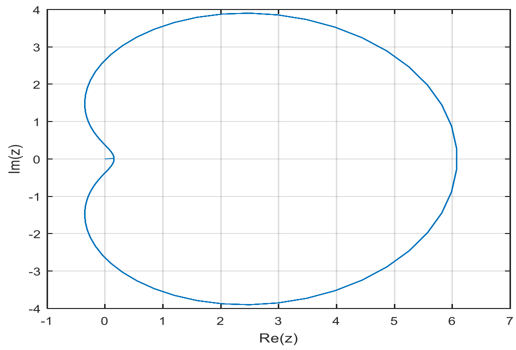

Figure.3. R-stability BHAMM2 in (10)

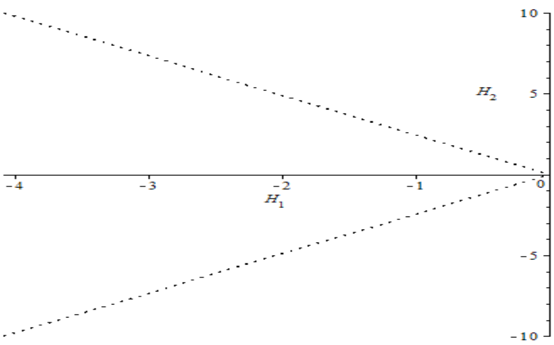

Figure.2. V-stability BHAMM3 in (13)

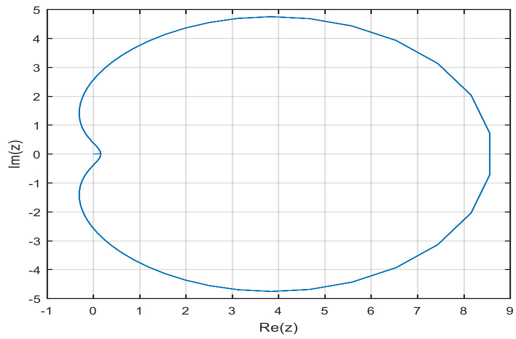

Figure.4.R-stability BHAMM3 in (13)

4. Numerical Computations

Some first-order DDEs shall be evaluated by applying the discrete schemes of the BHAMM2 and BHAMM3.

4.1 Numerical and Illustrative Cases

Example 1

$$a'(t) = -a(t-1+e^{-t}) + \sin(t-1+e^{-t}) + \cos(t),\ 0 \le t \le \frac{300}{10}$$

$$a(t) = \sin(t),\ \ t \le 0,$$

with an exact Solution $a(t) = \sin(t)$.

Example 2

$$a^{'}(t) = -1000a(t) + a(t-(\ln(1000-1))),\ 0 \le t \le \frac{300}{10}$$

$$a(t) = e^{-t}, t \le 0,$$

with an exact solution $a(t) = e^{-t}$.

5. Analysis and Comparison of Results

Here, (10) and (13), will be applied in executing the two examples above by analyzing their absolute errors. Tables 1 to 4 show the summarized results,

Table 1: Absolute Errors of BHAMM2 and BHAMM3 using Example 1.

T	BHAMM2 Error	BHAMM3 Error
0.01	1.51328E-06	1.53328E-04
0.02	2.42656E-06	2.35545E-04
0.03	6.29916E-06	6.28804E-05
0.04	8.37204E-06	8.08763E-05
0.05	1.46024E-06	1.4418E-04
0.06	1.7834E-06	1.71942E-04
0.07	2.64197E-06	2.59198E-05
0.08	3.08088E-07	2.96716E-05
0.09	4.17463E-07	4.07889E-06
0.10	4.72911E-07	4.55148E-06
0.11	6.05761E-07	5.90193E-06
0.12	6.72745E-07	6.47175E-07
0.13	8.29016E-08	8.06039E-07
0.14	9.07507E-08	8.72719E-07
0.15	0.10871E-08	1.05534E-07
0.16	1.17711E-08	1.13169E-07
0.17	1.38002E-08	1.33799E-07
0.18	1.48143E-09	1.42399E-07
0.19	1.70755E-09	1.65389E-07
0.20	1.82036E-09	1.74949E-07
0.21	2.06960E-09	2.00290E-08
0.22	2.19377E-11	2.10807E-08
0.23	2.46602E-11	2.38488E-08
0.24	2.60149E-11	2.49958E-08
0.25	2.89665E-12	2.79969E-09
0.26	3.04337E-12	2.92387E-09
0.27	3.36133E-12	3.24716E-09
0.28	3.51924E-13	3.38077E-09
0.29	3.85985E-13	3.72710E-09
0.30	4.02890E-13	3.87009E-09

Table 2: Absolute Errors of BHAMM2 and BHAMM3 using Example 2.

T	BHAMM2 Error	BHAMM3 Error
0.01	8.50832E-10	4.50832E-08
0.02	1.70676E-11	1.09324E-09
0.03	8.51492E-11	7.51492E-10
0.04	5.23231E-12	2.04768E-11
0.05	7.13984E-12	1.00714E-11
0.06	1.84249E-12	1.88425E-10
0.07	8.94052E-13	2.05948E-10
0.08	4.86636E-13	1.08664E-10
0.09	7.28772E-13	2.71228E-10
0.10	2.43596E-13	6.6404E-09
0.11	1.03472E-13	9.65282E-10
0.12	3.71716E-14	1.38284E-09
0.13	7.94387E-14	1.20561E-11
0.14	6.01194E-14	1.98806E-10
0.15	4.74942E-14	4.74942E-10
0.16	2.03379E-14	5.33789E-09
0.17	4.03616E-14	1.96384E-10
0.18	3.88728E-15	2.61127E-10
0.19	8.56638E-15	5.66377E-10
0.2	7.79818E-15	2.77982E-11
0.21	4.29813E-15	1.29813E-10
0.22	1.36248E-15	1.03752E-09
0.23	3.96666E-15	4.96666E-10
0.24	2.06655E-15	2.63345E-09
0.25	4.28595E-16	2.85951E-10
0.26	1.96434E-16	1.99643E-10
0.27	5.63147E-16	6.31468E-10
0.28	2.55573E-16	7.44275E-09
0.29	2.14347E-16	1.21435E-11
0.3	1.28172E-16	1.61828E-09

The notations used in the comparison of BHAMM with other existing methods are listed below:

BHAMM2 (resp. **BHAMM3)** imply Block Hybrid Adams Moulton Methods for step number $k = 2$ with two off-grid points (resp. Block Hybrid Adams Moulton Methods for step number $k = 2$ with three off-grid points) while **RBBDFM (**resp. **CBBDFM)** imply Reformulated Block Backward Differentiation Formulae for step numbers $k = 3$ and 4 in [13] (resp. Conventional BBDF for step numbers $k = 2$ and 3) in [1]**. ME** implies Maximum Error.

Table 3. ME of BHAMM2 and BHAMM3 $k = 2$ [13, 1] for fixed step size z = 0.01 (Example 1)

Numerical Method	Compared MEs with [13, 1]

BHAMM2 ME for k = 2	4.02890E-13
BHAMM3 ME for k = 2	3.87009E-09
RBBDFM ME for k = 3	1.61E-07
RBBDFM ME for k = 4	1.54E-08
CBBDFM ME for k = 2	1.66E-05
CBBDFM ME for k = 3	2.22E-07

CPU time of BHAMM2 for k = 2 is 0.210s, BHAMM3 k = 2 is 0.340s

Table 4. ME of BHAMM2 and BHAMM3 $k=2$ [13, 1] for fixed step size z = 0.01 (Example 2)

Numerical Method	Compared MEs with [13, 1]
BHAMM2 ME for k = 2	5.63147E-16
BHAMM3 ME for k = 2	2.77982E-11
RBBDFM ME for k = 3	1.61E-07
RBBDFM ME for k = 4	1.54E-08
CBBDFM ME for k = 2	1.66E-05
CBBDFM ME for k = 3	2.22E-07

CPU time of BHAMM2 for k = 2 is 0.215s, BHAMM3 k = 2 is 0.39s

6. Conclusions

The discrete schemes of the BHAMM2 and BHAMM3 were obtained through their individual continuous formulations and were analyzed to be convergent, V-stable, and R-stable. Also, it was revealed in tables 1 to 2 that the BHAMM2 scheme performed better than the BHAMM3 scheme for step number $k=2$ respectively and even better when compared with other existing methods, as shown in tables 3 to 4. Hence, it is suggested that two-step BHAMM schemes for two and three off-grid collocation points are appropriate for computing DDEs numerically. It is also suggested that the BHAMM schemes of lesser off-grid collocation points perform better than the BHAMM schemes of higher off-grid collocation points. Further research needs to be executed for step number $k=3,4,5...$ on the derivation of discrete schemes of BHAMM for computing DDEs without the use of interpolation conditions in obtaining the delay expression.

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