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AN EFFICIENT BLOCK SOLVER OF TRIGONOMETRICALLY FITTED METHOD FOR STIFF ODEs

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Abstract

An efficient block solver of trigonometrically fitted method for stiff ODEs has been developed. This block solver utilizes a special trigonometrically fitted method as the basis function approximation with the introduction of varying step, varying order and suitably varying step size. The idea of interpolation and collocation is utilized

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out via trigonometrically fitted method. Some theoretical properties of block solver are also investigated. To demonstrate the efficiency and accuracy of the method, we solve some examples of stiff ODEs.

1. Introduction

A large number of real life applications of differential equations come with different functions to be tracked unitedly as functions of time. Some systems of ordinary differential equations might be employed to stimulate the physical process. Assume that a situation occurs that allows diverse solution functions which possess quite different behaviour that makes the choice of the step size in the computational solution tough. For instance, a single element of a function might need a lesser step size in the computational solution since it is changing quickly, whereas some other elements might change tardily and do not need lesser step sizes for their calculations. Such a system is called *stiff*. Stiff differential equations are qualified as the analytical solutions which has a terminal figure of the kind e^{-ct}, where c is a large prescribed constant quantity. This is ordinarily a component of the solution named the *transient solution*. The most essential component of the solution is named the *steady-state solution*. A transient component of a stiff equation quickly decays to zero as t heightens [3, 5, 6].

We consider the stiff general one-dimensional systems with changeless constants:

$$y' = Ay + B(x), \quad y(a) = y_0, \tag{1}$$

where A is an $m \times m$ matrix with actual entries and $B(x)$, y, y' are m-vectors.

The exact solution to (1) is

$$y(x) = \sum_{i=1}^{m} \alpha_i e^{\lambda_i x} c_i + y_p(x), \tag{2}$$

where λ_i, $i = 1(1)m$ are the eigenvalues of A, with c_i, $i = 1(1)m$ the matching eigenvectors. $y_p(x)$ is a special solution to (1), and α_i, $i = 1(1)m$

are the actual constants that are unambiguously determined by the related initial conditions $y(a) = y_0$ [10, 13, 14].

The block solver of (1) is formed as the computation of block predictor-block corrector mode of

$$A^{(0)} Y_m = \sum_{i=1}^{k} A^{(i)} Y_{m-i} + h \sum_{i=0}^{k} B^{(i)} F_{m-i}, \tag{3}$$

$$A^{*(0)} Y_m = \sum_{i=2}^{k} A^{(i)} Y_{m-i} + h \sum_{i=1}^{k} B^{(i)} F_{m+i}, \tag{4}$$

where $Y_n = \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ \vdots \\ y_{n+r} \end{bmatrix}$, $F_n = \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ \vdots \\ f_{n+r} \end{bmatrix}$ (for $n = mr$, $m = 0, 1, \ldots$), $A^{(0)}$ and $B^{(0)}$ are $r \times r$ matrices.

It is presumed that (3) and (4) is annealed so that $A^{(0)}$ is an identity element matrix. A block solver is a *block predictor mode* if and only if the constant matrix $B^{(0)}$ is a zero matrix. Otherwise, it is referred to as a *block corrector mode* [24].

Definition 1.1. Stiffness takes place whenever the linear constant quantity system of all its eigenvalues possesses a negatively charged real component and the stiffness proportion is large [13, 14].

Definition 1.2. Stiffness takes place whenever more or less elements of the analytical solution radioactively decay majorly more quickly than others [13, 14].

Definition 1.3. Stiffness takes place whenever stability requirements, rather than those of precision, restrain the step size [13, 14].

Definition 1.4. Whenever a computational method with a bounded region of absolute constancy is employed to a system with any initial precondition which is pushed to employ in a certain interval of integration. The step size which is too small in relation to the smoothness of the analytical solution in that interval of the system is said to be *stiff* in that interval [13, 14].

Definition 1.5. The initial value problem

$$y' = f(x, y), \quad y(a) = y_0, \quad y = (y_1, y_2, \ldots, y_m)^T, \quad y_0 = (\eta_1, \eta_2, \ldots, \eta_m)^T$$

is stated to be *stiff oscillatory* whenever the eigenvalues $\lambda_j = u_j + iv_j$, $j = 1(1)m$ of the Jacobian $J = \left(\frac{\partial f}{\partial dy}\right)$ have the succeeding attributes:

$$u_j < 0, \quad j = 1(1)m,$$

$$\max_{1 \le j \le m} |u_j| > \min_{1 \le j \le m} |u_j|,$$

or whenever the stiffness ratio satisfies

$$S = \max_{i, j} \left|\frac{u_i}{u_j}\right| > 1. \text{ See [7].} \tag{5}$$

Theorem 1.1 (Weierstrass approximation theorem). *Let* $f : R \to R$ *be continuous and* 2π*-periodic. Then for each* $\varepsilon > 0$, *there exists a trigonometric polynomial* $P(x) = \sum_{j=-n}^{k} c_j e^{ijx}$ *such that for all* x, $|f(x) - P(x)| < \varepsilon$. *Tantamountly, for any such f, there is a sequence of polynomials* P_n *converging uniformly to f* [2].

Theorem 1.2. *An A-stable linear multi-step scheme*:

(i) *must be implicit*, *and*

(ii) *the most precise A-stable linear multi-step scheme* $y_{n+1} = y_n + \frac{h}{2}(f_n + f_{n+1})$ *of order* $p = 2$ *has an error constant* $c_3 = -\frac{1}{12}$, *see* [7, 13, 14].

The Dahlquist barrier of Theorem 1.2 can be outwitted by accepting unconventional numeric integrators, some of which are

- nonlinear multistep schemes,
- multiderivative multistep schemes,
- exponentially fitting, and
- extrapolation process [7].

Successful contributions to stiff problems involve the diagonally implicit block backward differentiation formula with optimal stability properties for stiff ODEs. D12BBDF applies diverse step size to get results [10]. The block method for generalized multistep Adams and backward differentiation formulae in solving first order ODEs is developed in [11]. BGMBDF implemented the step reduction with varying step size to achieve the tolerance level [11]. The adaptive order of block backward differentiation formulae for stiff ODEs is developed utilizing uniform step size. The ABBDF is done without the use of tolerance level [12]. An accurate block solver for stiff initial value problems has been implemented in a varying step size approach for optimal execution. This idea demands dividing and amending by a product of 1.7 or preserving the current step size. The VSSBBDF explores the combination of varying the step size and utilizing the tolerance level [15]. These methods used by [10-12, 15] were carried out using fixed step size, variable step size and tolerance level. The nature of the behaviour of the type of problem is not considered. On the other hand, the EBSTFM introduces the trigonometrically fitted method as the basis function approximation to fit the nature of the behaviour of the type of problem. Again, the variable step, variable order and suitable variable step size were suggested by [13, 14, 16-20] to enhance the efficiency and accuracy of the method. This idea distinguishes EBSTFM from other implemented methods. References [16-18] suggested a BHMTB for solving oscillatory IVPs. The BHMTB satisfies *A*-stability and possesses the potential for handling stiff IVPs. Also, BHMTB comes with the ability of being self-initiating with good accuracy of order 4. Thus, needs solely two functions valuation for each integration step. The EBSTFM suggested the idea of variable step, variable order and suitable variable step size along with tolerance to enhance better efficiency and accuracy compared to [16-18] which were implemented with uniform step size.

Solving stiff ODEs using block implicit method, block predictor-block corrector method and block hybrid predictor-corrector method is very cumbersome due to the stability requirements posed by Dahlquist theorem while others pose the challenges of stiffness ratio, solution decay and finite region of absolute stability. The Dahlquist theorem can be bypassed with the introduction of trigonometrically fitted method to approximate in accordance with the exact solution of the stiff problem. References [13, 14] proposed the inclusion of varying step, varying order and varying step size to improve the accuracy, enhance efficiency and maximize error [7, 13, 14].

The motivation of this work originates from the suggestion to yield high efficiency and accuracy via the introduction of varying step, varying order with varying step size. References [13, 14] proposed the exponentially method to bypass the Dahlquist barriers. Thus, this study implements trigonometrically fitted method in accordance with the exact solution to outwit the Dahlquist barriers [7, 13, 14].

2. Methods

The formulation of the methods depends on [7, 13, 14]. References [13, 14] suggested the idea of varying order, varying step with varying step size as the central device to greater efficiency and accuracy resulting to lesser maximum errors. The paper [7] proposed the exponentially fitted method to overcome the Dahlquist barriers. This study suggests and initiates a remodification of the block solver in form of block predictor-block corrector mode. The block predictor mode has point of interpolation as y_{n-1} and points of collocation as f_n, f_{n-1}, f_{n-2}, f_{n-3}, while the block corrector mode has point of interpolation as y_{n-2} and points of collocation as f_{n+1}, f_{n+2}, f_{n+3}. The points of interpolation and collocation of block predictor-block corrector mode will be implemented for varying step. The structure for deriving the block predictor mode is defined as:

Point of interpolation - y_{n-1}.

Points of collocation - f_n, f_{n-1}, f_{n-2}, f_{n-3}.

Block solver points of evaluation - y_{n+1}, y_{n+2}, y_{n+3}.

Basis function approximation of trigonometrically fitted method is as follows:

$$y(x) = a_0 + a_1\left(\frac{x - x_n}{h}\right) + a_2\left(\frac{x - x_n}{h}\right)^2 + a_3\left(w\left(\frac{x - x_n}{h}\right) - w^3\left(\frac{x - x_n}{h}\right)^3\right)$$

$$+ a_4\left(1 - w^2\left(\frac{x - x_n}{h}\right)^2 + w^3\left(\frac{x - x_n}{h}\right)^4\right).$$

Utilizing the combination of the point of interpolation, points of collocation and block solver points of evaluation will yield the expression written in Mathematica matrix format as

$$\text{matrixa}\left\{\left\{1, -1, 1, -w + \frac{w^3}{6}, 1 - \frac{w^2}{2} + \frac{w^4}{24}\right\}, \{0, 1, 0, w, 0\},\right.$$

$$\left\{0, 1, -2, w - \frac{w^3}{2}, \frac{w^2}{2} - \frac{w^4}{6}\right\}, \left\{0, 1, -4, w - 2w^3, 2w^2 - \frac{4w^4}{3}\right\},$$

$$\left.\left\{0, 1, -6, w - \frac{9w^3}{2}, 3w^2 - \frac{9w^4}{2}\right\}\right\},$$

$b = \{y[n-1],\ f[n],\ f[n-1],\ f[n-2],\ f[n-3]\};$

$\{a[0],\ a[1],\ a[2],\ a[3],\ a[4]\} = \text{Inverse [matrixa]}.b.$

The Mathematica matrix is solved using Mathematica Kernel 9. The unknowns of a_i, $i = 0, 1, \ldots, 4$ are substituted into the trigonometrically fitted method to yield the continuous scheme. This continuous scheme is evaluated using the block solver points of evaluation to generate the block predictor mode as

$$y_{n+1} = y_{n-1} + h\left[\left(\frac{1}{w^4} - \frac{2}{w^3} - \frac{-4w^5 - 2w^7}{2w^7} - \frac{-w^5 - \frac{11w^7}{6}}{2w^7} - \frac{2w^3 - \frac{3w^7}{4}}{2w^7}\right)f_n\right.$$

$$+\left(\frac{-3}{w^4} + \frac{5}{w^3} - \frac{5}{w^2} - \frac{-6w^3 - \frac{19w^7}{12}}{2w^7} - \frac{3w^5 + 3w^7}{2w^7}\right)f_{n-1}$$

$$+\left(\frac{3}{w^4} - \frac{4}{w^3} + \frac{4}{w^2} - \frac{-3w^5 - \frac{3w^7}{2}}{2w^7} - \frac{6w^3 + \frac{5w^7}{12}}{2w^7}\right)f_{n-2}$$

$$\left.+\left(\frac{-1}{w^4} + \frac{1}{w^3} - \frac{1}{w^2} - \frac{-2w^3 - \frac{w^7}{12}}{2w^7} - \frac{w^5 + \frac{w^7}{3}}{2w^7}\right)f_{n-3}\right],$$

$$y_{n+2} = y_{n-1} + h\left[\left(\frac{16}{w^4} - \frac{16}{w^3} - \frac{-2w^5 - \frac{11w^7}{3}}{w^7} - \frac{-4w^5 - 2w^7}{w^7} - \frac{2w^3 - \frac{3w^7}{4}}{2w^7}\right)f_n\right.$$

$$+\left(\frac{-48}{w^4} + \frac{40}{w^3} - \frac{10}{w^2} - \frac{-6w^3 - \frac{19w^7}{12}}{2w^7} - \frac{6w^5 + 6w^7}{w^7}\right)f_{n-1}$$

$$+\left(\frac{48}{w^4}-\frac{32}{w^3}+\frac{8}{w^2}-\frac{-6w^5-3w^7}{w^7}-\frac{6w^3+\frac{5w^7}{12}}{2w^7}\right)f_{n-2}$$

$$\left.+\left(\frac{-16}{w^4}+\frac{8}{w^3}-\frac{2}{w^2}-\frac{-2w^3-\frac{w^7}{12}}{2w^7}-\frac{2w^5+\frac{2w^7}{3}}{w^7}\right)f_{n-3}\right],$$

$$y_{n+3}=y_{n-2}+h\left[\left(\frac{81}{w^4}-\frac{54}{w^3}-\frac{-9w^5-\frac{33w^7}{2}}{w^7}-\frac{-12w^5-6w^7}{2w^7}-\frac{2w^3-\frac{3w^7}{4}}{2w^7}\right)f_n\right.$$

$$+\left(\frac{-243}{w^4}+\frac{135}{w^3}-\frac{15}{w^2}-\frac{-6w^3-\frac{19w^7}{12}}{2w^7}-\frac{27w^5+27w^7}{2w^7}\right)f_{n-1}$$

$$+\left(\frac{243}{w^4}-\frac{108}{w^3}+\frac{12}{w^2}-\frac{-27w^5-\frac{27w^7}{2}}{2w^7}-\frac{6w^3+\frac{5w^7}{12}}{2w^7}\right)f_{n-2}$$

$$\left.+\left(\frac{-81}{w^4}+\frac{27}{w^3}-\frac{3}{w^2}-\frac{-2w^3-\frac{w^7}{12}}{2w^7}-\frac{9w^5+3w^7}{2w^7}\right)f_{n-3}\right]. \quad (6)$$

Similarly, the derivation of the block corrector mode is done as follows:

Point of interpolation - y_{n-2}.

Points of collocation - $f_{n+1},\ f_{n+2},\ f_{n+3}$.

Block solver points of evaluation - $y_{n+1},\ y_{n+2},\ y_{n+3}$.

Basis function approximation of trigonometrically fitted method is given as follows:

$$y(x) = a_0 + a_1\left(\frac{x - x_n}{h}\right) + a_2\left(w\left(\frac{x - x_n}{h}\right) - w^3\left(\frac{x - x_n}{h}\right)^3\right)$$

$$+ a_3\left(1 - w^2\left(\frac{x - x_n}{h}\right)^2\right).$$

Utilizing the combination of the point of interpolation, points of collocation and block solver points of evaluation will yield the expression written in Mathematica matrix format as

$$\text{matrixa } \left\{\left\{1, -2, -2w + \frac{4w^3}{3}, 1 - 2w^2\right\}, \left\{0, 1, w - \frac{w^3}{2}\right\},\right.$$

$$\left.\left\{0, 1, w - \frac{w^3}{2}, -w^2\right\}, \left\{0, 1, w - \frac{9w^3}{2}, -3w^2\right\}\right\},$$

$$b = \{y[n-2], f[n+1], f[n+2], f[n+3]\};$$

$$\{a[0], a[1], a[2], a[3]\} = \text{Inverse [matrixa]}.b.$$

The Mathematica matrix is solved using Mathematica Kernel 9. The unknowns of a_i, $i = 0, 1, \ldots, 3$ are substituted into the trigonometrically fitted method to yield the continuous scheme. This continuous scheme is evaluated using the block solver points of evaluation to generate the block corrector mode as

$$y_{n+1} = y_{n-2} + h\left[\left(\frac{-1}{w^3} + \frac{5}{2w^2} - \frac{-\frac{5w^3}{2} - \frac{37w^5}{3}}{w^5} - \frac{w^3 - 3w^5}{w^5}\right)f_{n+1}\right.$$

$$+ \left(\frac{2}{w^3} - \frac{4}{w^2} - \frac{2w^3 + 3w^5}{w^5} - \frac{-4w^3 + \frac{50w^5}{3}}{w^5}\right)f_{n+2}$$

$$+\left(\frac{-3}{w^3}+\frac{3}{2w^2}-\frac{\frac{3w^2}{2}-\frac{19w^5}{3}}{w^5}-\frac{-w^3-w^5}{w^5}\right)f_{n+3}\Bigg],$$

$$y_{n+2} = y_{n-2} + h\Bigg[\left(\frac{-4}{w^3}+\frac{20}{w^2}-\frac{-\frac{5w^3}{2}-\frac{37w^5}{3}}{w^5}-\frac{-2w^3-6w^5}{w^5}\right)f_{n+1}$$

$$+\left(\frac{8}{w^3}-\frac{32}{w^2}-\frac{4w^3+6w^5}{w^5}-\frac{-4w^3+\frac{50w^5}{3}}{w^5}\right)f_{n+2}$$

$$+\left(\frac{-4}{w^3}+\frac{12}{w^2}-\frac{\frac{3w^2}{2}-\frac{19w^5}{3}}{w^5}-\frac{-2w^3-2w^5}{w^5}\right)f_{n+3}\Bigg],$$

$$y_{n+3} = y_{n-2} + h\Bigg[\left(\frac{-9}{w^3}+\frac{135}{2w^2}-\frac{-\frac{5w^3}{2}-\frac{37w^5}{3}}{w^5}-\frac{-3w^3-3w^5}{w^5}\right)f_{n+1}$$

$$+\left(\frac{18}{w^3}-\frac{108}{w^2}-\frac{6w^3+9w^5}{w^5}-\frac{-4w^3+\frac{50w^5}{3}}{w^5}\right)f_{n+2}$$

$$+\left(\frac{-9}{w^3}+\frac{81}{2w^2}-\frac{\frac{3w^3}{2}-\frac{19w^5}{3}}{w^5}-\frac{-3w^3-3w^5}{w^5}\right)f_{n+3}\Bigg]. \quad (7)$$

Equations (6) and (7) are called the *block predictor-block corrector mode* otherwise referred to as block solver which is formulated via the implementation of varying step and trigonometrically fitted method. The point of interpolation of the block predictor mode and block corrector mode differs. The block predictor mode is of order four (4) which requires the values of y_i, $i = 0, 1, 2$ to initiate the process while the block corrector is of order three (3) which requires the values of y_i, $i = 0, 1$ to start the process. This defines the variable step implementation. The number of unknowns of the block predictor mode is five (5) while the number of unknowns of the block corrector solver is four (4). This justifies the variable order. The symbol w represents the frequency [16-23].

2.1. Theoretical investigation of the method

Definition 2.1. The *order* of the operator L_h is the highest r such that whenever $y(x)$ possesses a continuous $(r+1)$th derivative,

$$L_h(y(x)) = 0(h^{r+1}). \tag{8}$$

Whenever we presume a continuous $(r+2)$th derivative for y, we can replace the Taylor's series for y and y' with remainder $0(h^{r+1})$.. Whenever the terms $h^0, h^2, \ldots, h^{r+1}$ are assembled unitedly, we arrive at

$$L_h(y(x)) = \sum_{q=0}^{r+1} C_q h^q y^{(q)}(x) + 0(h^{r+1}),$$

where

$$C_q = \begin{cases} \sum_{i=0}^{k} \alpha_i, & q = 0, \\ \sum_{i=0}^{k} \left[\frac{(-i)^q}{q!} \alpha_i - \frac{(-i)^{q-1}}{(q-1)!} \beta_i \right], & q > 0. \end{cases} \tag{9}$$

The linear equations $C_q = 0$, $q \leq r$, are the equations which decide an rth order method.

Given the number of truncation errors, put in for each step, is given by

$$\frac{C_{r+1}}{\sum_{i=0}^{k} \beta_i} h^{r+1} y^{(r+1)} + 0(h^{r+2}).$$

Therefore, the natural standardization becomes

$$\sum_{i=0}^{k} \beta_i = 1 \ [8]. \tag{10}$$

Theorem 2.1. *The multistep method* (3) *is of order* p, *if and only if one of the following equivalent conditions is satisfied*:

(i) $\sum_{i=0}^{k} \alpha_i = 0$ *and* $\sum_{i=0}^{k} \alpha_i i^q = q \sum_{i=0}^{k} \beta_i i^{q-1}$ *for* $q = 1, \ldots, p$;

(ii) $\rho(e^h) - h\sigma(e^h) = O(h^{p+1})$ *as* $h \to 0$;

(iii) $\dfrac{\rho(\varsigma)}{\log \varsigma} - \sigma(\varsigma) = O((\varsigma - 1)^p)$ *as* $\varsigma \to 1$,

where the linear difference operator L is specified by

$$L(y, x, h) = \sum_{i=0}^{k} (\alpha_i y(x + ih) - h\beta_i y'(x + ih)) \ [9]. \tag{11}$$

Proof. Expanding $y(x + ih)$ and $y'(x + ih)$ into Taylor series and introducing these series (truncated whenever essential into (11)), we have

$$L(y, x, h) = \sum_{i=0}^{k} \left(\sum_{q \geq 0} \frac{i^q}{q!} h^q y^{(q)}(x) - h\beta_i \sum_{r \geq 0} \frac{i^r}{r!} h^r y^{(r+1)}(x) \right)$$

$$= y(x) \sum_{i=0}^{k} \alpha_i + \sum_{q \geq 1} \frac{h^q}{q!} y^{(q)}(x) \left(\sum_{i=0}^{k} \alpha_i i^q - q \sum_{i=0}^{k} \beta_i i^{q-1} \right). \tag{12}$$

This agrees with the condition (i) as $L(y, x, h) = O(h^{p+1})$ for all sufficiently regular functions $y(x)$.

It continues to show that the three preconditions of Theorem 2.1 are tantamount. The identity operator

$$L(\exp, 0, h) = \rho(e^h) - h\sigma(e^h),$$

where exp represents the exponential function, unitedly with

$$L(\exp, 0, h) = \sum_{i=0}^{k} \alpha_i + \sum_{q \geq 1} \frac{h^q}{q!} \left(\sum_{i=0}^{k} \alpha_i i^q - q \sum_{i=0}^{k} \beta_i i^{q-1} \right),$$

which succeeds from (12) and proves the equivalence of the conditions (i) and (ii).

By using the translation $\varsigma = e^h$ (or $h = \log \varsigma$), condition (ii) will be spelt out in the form

$$\rho(\varsigma) - \log \varsigma * \sigma(\varsigma) = O((\log \varsigma)^{p+1}) \text{ as } \varsigma \to 1.$$

But this condition above is equivalent to (iii), because

$\log \varsigma = (\varsigma - 1) + O((\varsigma - 1)^2)$ as $\varsigma \to 1$, see [9] for more information.

2.2. Convergence

Convergence for variable step size Adams method was first considered in [26]. In order to show the convergence for the general case, we present the vector $Y_n = (y_{n+k-1}, ..., y_{n+1}, y_n)^T$. The method

$$y_{n+k} + \sum_{i=0}^{k-1} \alpha_i y_{n+i} = h_{n+k-1} \sum_{i=0}^{k} \beta_{in} f_{n+i} \tag{13}$$

then turns tantamount to

$$Y_{n+1} = (A_n \otimes I) Y_n + h_{n+k-1} \Phi_n (x_n, Y_n, h_n), \tag{14}$$

where A_n is established by the comrade matrix:

$$A_n = \begin{pmatrix} -\alpha_{k-1,n} & \cdots & \cdots & -\alpha_{1,n} & -\alpha_{0,n} \\ 1 & 0 & \cdots & 0 & 0 \\ & \ddots & \ddots & \vdots & \vdots \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \end{pmatrix}$$

and

$$\Phi_n(x_n, Y_n, h_n) = (e_1 \otimes I)\Psi_n(x_n, Y_n, h_n).$$

The value $\Psi = \Psi_n(x_n, Y_n, h_n)$ is specified by

$$\Psi = y_{n+k} + \sum_{i=0}^{k-1} \beta_{in} f(x_{n+i}, y_{n+i}) + \beta_{kn} f(x_{n+k}, h)\Psi - \sum_{i=0}^{k} \alpha_{in} y_{n+i}.$$

We proceed by writing

$$Y(x_n) = (y(x_{n+k-1}), \ldots, y(x_{n+1}), y(x_n))^T,$$

the precise values to be estimated by Y_n. The convergence theorem can immediately be developed as succeeds, see [9] for more information.

Theorem 2.2. *Assume that*

- *the method* (3) *is stable of order* p, *and has bounded coefficients* α_{in} *and* β_{in};
- *the starting values satisfy* $\| Y(x_n) - Y_0 \| = O(h_0^p)$;
- *the step size ratios are bounded* $\left(\frac{h_n}{h_{n-1}} \leq \Omega\right)$.

Then the method is convergent of order p, *i.e., for each differential equation*

$$y' = f(x, y), \quad y(x_0) = y_0$$

with f *sufficiently differentiable, the global error satisfies*

$$\| Y(x_n) - y_n \| \leq Ch^p \text{ for } x_n \leq \hat{x},$$

where $h = \max h_i$, *see* [9] *for more information.*

Proof. Since the method is of order p and the coefficients and step size ratios are bounded, the formula

$$y(x_{n+k}) + \sum_{i=0}^{k-1} \alpha_{in} y(x_{n+i}) - h_{n+k-1} \sum_{i=0}^{k} \beta_{in} y'(x_{n+i}) = O(h_0^{p+1})$$

proves that the local error

$$\delta_{n+1} = Y(x_{n+1}) - (A_n \otimes I)Y(x_n) - h_{n+k-1}\Phi_n(x_n, Y(x_n), h_n) \tag{15}$$

satisfies

$$\delta_{n+1} = O(h_n^{p+1}). \tag{16}$$

Deducting (13) from (15), we get

$$Y(x_{n+1}) - Y_{n+1} = (A_n \otimes I)(Y(x_n) - Y_n)$$
$$+ h_{n+k-1}(\Phi_n(x_n, Y(x_n), h_n) - \Phi_n(x_n, Y_n, h_n)) + \delta_{n+1}$$

and by induction,

$$Y(x_{n+1}) - Y_{n+1}$$
$$= ((A_n \dots A_0) \otimes I)(Y(x_0) - Y_0)$$
$$+ \sum_{i=0}^{n} h_{i+k-1}((A_n \dots A_{i+1}) \otimes I)(\Phi_i(x_i, Y(x_i), h_i) - \Phi_i(x_i, Y_i, h_i))$$
$$+ \sum_{i=0}^{n} ((A_n \dots A_{i+1}) \otimes I)\delta_{i+1}.$$

As in the proof of Theorem 2.1, we derive that the Φ_n satisfies a uniform Lipschitz precondition with respect to Y_n. Unitedly with stability and (16), we have

$$\| Y(x_{n+1}) - Y_{n+1} \| \leq \sum_{i=0}^{n} h_{i+k-1} L \| Y(x_i) - Y_i \| + C_1 h^p.$$

In order to figure out this inequality, we bring in the sequence $\{\varepsilon_n\}$ specified by

$$\varepsilon_0 = \| Y(x_0) - Y_0 \|,$$

$$\varepsilon_{n+1} = \sum_{i=0}^{n} h_{i+k-1} L \varepsilon_i + C_1 h^p. \tag{17}$$

A simple induction statement proves that

$$\| Y(x_n) - Y_n \| \leq \varepsilon_n. \tag{18}$$

From (17), we get for $n \geq 1$,

$$\varepsilon_{n+1} = \varepsilon_n + h_{n+k-1} L \varepsilon_n \leq \exp(L h_{n+k-1}) \varepsilon_n$$

so that, in addition,

$$\varepsilon_n \leq \exp((\hat{x} - x_0) L) \varepsilon_1 = \exp((\hat{x} - x_0) L) \cdot (h_{k-1} L \| Y(x_0) - Y_0 \| + C_1 h^p).$$

The inequality unitedly with (18) completes the proof of Theorem 2.2, see [9] for more information.

2.3. Variable step size implementation for block solver

This aspect utilizes the global errors of the block predictor mode of order four (4) and block corrector mode of order three (3) to initiate the procedure. The block predictor mode of order four (4) and the block corrector mode of order three (3) will be used to implement the variable step size of the block solver. The global error of the block predictor-block corrector mode is estimated as:

$$|\, y(x_{n+1}) - \bar{y}_{n+1}^{[\tilde{l}_1]} \,| \approx \frac{|\, z(x_{n+1}) - \bar{y}_{n+1}^{[\tilde{q}_1]} \,|}{h} \approx \frac{7170}{7247h} |\, y_{n+1}^{[\bar{l}_1]} - \bar{y}_{n+1}^{[\bar{q}_1]} \,|,$$

$$|\, y(x_{n+2}) - \bar{y}_{n+2}^{[\tilde{l}_2]} \,| \approx \frac{|\, z(x_{n+2}) - \bar{y}_{n+2}^{[\bar{q}_2]} \,|}{h} \approx \frac{7440}{7973h} |\, y_{n+2}^{[\bar{l}_2]} - \bar{y}_{n+2}^{[\bar{q}_2]} \,|,$$

$$|\, y(x_{n+3}) - \bar{y}_{n+3}^{[\tilde{l}_3]} \,| \approx \frac{|\, z(x_{n+3}) - \bar{y}_{n+3}^{[\bar{q}_3]} \,|}{h} \approx \frac{5490}{7739h} |\, y_{n+3}^{[\bar{l}_3]} - \bar{y}_{n+3}^{[\bar{q}_3]} \,|, \tag{19}$$

where z establishes the exact solution to the first derivative equation conforming to the initial condition $z(x_n) = y(x_n)$.

Imagine if we rebuild the process with a new step size qh to produce new estimates of $y_{n+1}^{[\bar{l}_1]}$, $\bar{y}_{n+1}^{[\bar{q}_1]}$, $y_{n+2}^{[\bar{l}_2]}$, $\bar{y}_{n+2}^{[\bar{q}_2]}$, $y_{n+3}^{[\bar{l}_3]}$ and $\bar{y}_{n+3}^{[\bar{q}_3]}$. To check and control the global error in ε, we select q such that

$$\frac{|z(x_n + qh) - \bar{y}_{n+1}^{[\bar{q}_1]} \text{ (employing the step size } qh)|}{qh} < \varepsilon,$$

$$\frac{|z(x_n + qh) - \bar{y}_{n+2}^{[\bar{q}_2]} \text{ (employing the step size } qh)|}{qh} < \varepsilon,$$

$$\frac{|z(x_n + qh) - \bar{y}_{n+3}^{[\bar{q}_3]} \text{ (employing the step size } qh)|}{qh} < \varepsilon. \tag{20}$$

Utilizing the principal local truncation errors of the block predictor-block corrector mode together with (20) results

$$\frac{|z(x_n + qh) - \bar{y}_{n+1}^{[\bar{q}_1]} \text{ (utilizing } qh)|}{qh} = \frac{239}{24} = |z^{(4)}(\bar{x}_n)| q^4 h^4$$

$$\approx \frac{239}{24}\left[\frac{720}{7247}|y_{n+1}^{[\bar{l}_1]} - \bar{y}_{n+1}^{[\bar{q}_1]}|\right] q^4 h^4,$$

$$\frac{|z(x_n + qh) - \bar{y}_{n+2}^{[\bar{q}_2]} \text{ (utilizing } qh)|}{qh} = \frac{31}{4} = |z^{(4)}(\bar{x}_n)| q^4 h^4$$

$$\approx \frac{31}{4}\left[\frac{720}{7973}|y_{n+2}^{[\bar{l}_2]} - \bar{y}_{n+2}^{[\bar{q}_2]}|\right] q^4 h^4,$$

$$\frac{|z(x_n + qh) - \bar{y}_{n+3}^{[\bar{q}_3]} \text{ (utilizing } qh)|}{qh} = \frac{61}{8} = |z^{(4)}(\bar{x}_n)| q^4 h^4$$

$$\approx \frac{61}{8}\left[\frac{720}{7739}|y_{n+3}^{[\bar{l}_3]} - \bar{y}_{n+3}^{[\bar{q}_3]}|\right] q^4 h^4. \tag{21}$$

So, we select q with

$$\frac{239}{24}\left[\frac{720}{7247}|y_{n+1}^{[\bar{l}_1]} - \bar{y}_{n+1}^{[\bar{q}_1]}|\right] q^4 h^4 = \frac{7170}{7247} \frac{|y_{n+1}^{[\bar{l}_1]} - \bar{y}_{n+1}^{[\bar{q}_1]}|}{h} < \varepsilon,$$

$$\frac{31}{3}\left[\frac{720}{7973}\left| y_{n+2}^{[\bar{l}_2]} - \bar{y}_{n+2}^{[\bar{q}_2]}\right|\right] q^4 h^4 = \frac{7440}{7973}\frac{\left| y_{n+2}^{[\bar{l}_2]} - \bar{y}_{n+2}^{[\bar{q}_2]}\right|}{h} < \varepsilon,$$

$$\frac{61}{8}\left[\frac{720}{7739}\left| y_{n+3}^{[\bar{l}_3]} - \bar{y}_{n+3}^{[\bar{q}_3]}\right|\right] q^4 h^4 = \frac{5490}{7739}\frac{\left| y_{n+3}^{[\bar{l}_3]} - \bar{y}_{n+3}^{[\bar{q}_3]}\right|}{h} < \varepsilon. \tag{22}$$

Accordingly, change the variation in step size from h to qh, where

$$q < \left(\left(\frac{7170}{7247}\right)\frac{h\varepsilon}{\left| y_{n+1}^{[\bar{l}_1]} - \bar{y}_{n+1}^{[\bar{q}_1]}\right|}\right)^{\frac{1}{4}} \approx 0.989375\left(\frac{h\varepsilon}{\left| y_{n+1}^{[\bar{l}_1]} - \bar{y}_{n+1}^{[\bar{q}_1]}\right|}\right),$$

$$q < \left(\left(\frac{7440}{7973}\right)\frac{h\varepsilon}{\left| y_{n+1}^{[\bar{l}_1]} - \bar{y}_{n+1}^{[\bar{q}_1]}\right|}\right)^{\frac{1}{4}} \approx 0.933149\left(\frac{h\varepsilon}{\left| y_{n+2}^{[\bar{l}_2]} - \bar{y}_{n+2}^{[\bar{q}_2]}\right|}\right),$$

$$q < \left(\left(\frac{5490}{7739}\right)\frac{h\varepsilon}{\left| y_{n+1}^{[\bar{l}_1]} - \bar{y}_{n+1}^{[\bar{q}_1]}\right|}\right)^{\frac{1}{4}} \approx 0.709394\left(\frac{h\varepsilon}{\left| y_{n+3}^{[\bar{l}_3]} - \bar{y}_{n+3}^{[\bar{q}_3]}\right|}\right). \tag{23}$$

Nevertheless, the process of implementing the block predictor-block corrector mode of the block solver involves the use of (22) or (23). Again, the block predictor-block corrector mode together with the newly suited step size must be resolved iteratively until the tolerance level is ascertained. Again, this process is repeated if it fails until a newly chosen step size conforms to the tolerance level. If the newly chosen step size succeeds, then it becomes the suitable variable step size to yield the desired results with better accuracy and efficiency. Variable step size procedures involve changing the step repeatedly during the process of the loop until the tolerance level is achieved.

A step size variation for block solver is more expensive in terms of functional measures in comparison to a multistep scheme, see [1, 4-6, 19-23] for details.

3. Stiff Problems

This aspect will be carried out employing the suited variable step size and block predictor-block corrector mode as proposed earlier. Three stiff ODEs were considered and solved at some selected tolerance level of 10^{-2}, 10^{-4} and 10^{-6}. The efficient block solver is compared with some stiff BBDF adopting the approach of using the tolerance level. The following stiff ODEs will be considered and solved as the numerical examples [11, 13-15]. The execution of this method is carried out under the Mathematica platform of Mathematica Kernel 9.

Stiff Problem 3.1

$$y_1' = -21y_1 + 19y_2 - 20y_3,$$

$$y_2' = 19y_1 - 21y_2 + 20y_3,$$

$$y_3' = 40y_1 - 40y_2 - 40y_3,$$

$$y_1(0) = 1,$$

$$y_2(0) = 0,$$

$$y_3(0) = -1, \quad 0 \le x \le 10.$$

Exact solution

$$y_1(x) = \frac{1}{2}[e^{-2x} + e^{-40x}(\cos 40x + \sin 40x)],$$

$$y_2(x) = \frac{1}{2}[e^{-2x} - e^{-40x}(\cos 40x + \sin 40x)],$$

$$y_3(x) = 2e^{-40x}\left[-\frac{1}{2}(\cos 40x - \sin 40x)\right].$$

Author: [15].

Stiff Problem 3.2

$$y_1' = y_2,$$

$$y_2' = -y_1.$$

Exact solution

$$y_1(x) = \sin x,$$

$$y_2(x) = \cos x, \quad 0 \leq x \leq 16\pi.$$

Author: [11].

Stiff Problem 3.3

$$y(x) = -2\pi \sin(2\pi x) - \frac{1}{\varepsilon}(y - \cos(2\pi x)).$$

Exact solution

$$y(x) = \cos(2\pi x), \quad 0 \leq x \leq 10.$$

Author: [15].

4. Results

The numerical results are shown in Tables 1-3. The *EBSTFM* considers three stiff problems whose exact solution is trigonometrical in nature. The results of *EBSTFM* were compared with *BGMBDF* and *VSSBBDF* to demonstrate the efficiency and accuracy of the method. The results in Tables 1-3 were computed under the platform of Mathematica Kernel 9. The *EBSTFM* yields better results in Tables 1 and 2 compared to *VSSBBDF* and *BGMBDF* while *VSSBBDF* yields better results in Table 3 compared to *EBSTFM*. The betterment of the *EBSTFM* of Tables 1 and 2 occurs as a result of the ability to implement the trigonometrically fitted method and finding a suited variable step size as suggested by [13, 14]. *VSSBBDF* yields better results compared to *EBSTFM* due to its ability to find a suited varying step size to satisfy the tolerance level [11, 15].

Table 1. Stiff Problem 1

$Mthd_{used}$	Max_{error}	TOL
VSSBBDF	$5.27731e-006$	10^{-2}
EBSTFM	$5.14453e-005$	
EBSTFM	$1.29594e-005$	
EBSTFM	$2.99998e-006$	
VSSBBDF	$7.42231e-008$	10^{-4}
EBSTFM	$5.30633e-009$	
EBSTFM	$1.54404e-009$	
EBSTFM	$3.e-010$	
VSSBBDF	$7.93169e-010$	10^{-6}
EBSTFM	$5.30909e-013$	
EBSTFM	$1.54644e-013$	
EBSTFM	$2.9976e-014$	

Table 2. Stiff Problem 2

$Mthd_{used}$	Max_{error}	TOL
BGMBDF	$1.50633e-001$	10^{-2}
EBSTFM	$7.6004e-008$	
EBSTFM	$2.34258e-004$	
BGMBDF	$7.87780e-004$	10^{-4}
EBSTFM	$7.59999e-014$	
EBSTFM	$2.34374e-006$	

Table 3. Stiff Problem 3

$Mthd_{used}$	Max_{error}	TOL
VSSBBDF *EBSTFM*	$1.23612e-005$ $1.96072e-004$	10^{-2}
VSSBBDF *EBSTFM*	$6.74134e-007$ $1.94861e-006$	10^{-4}
VSSBBDF *EBSTFM*	$3.19856e-009$ $1.94861e-008$	10^{-6}

The following terminologies are used in showing the results in Tables 1-3:

EBSTFM : Efficient block solver of trigonometrically fitted method

TOL : The tolerance level used

Max_{error} : Maximum error

$Mthd_{used}$: Method used

BGMBDF : Block method for generalized multistep Adams and backward differentiation formulae [11]

VSSBBDF : Variable step size algorithm based on block backward differentiation formula [15].

5. Conclusion

An efficient block solver of trigonometrically fitted method has been suggested for solving stiff problems appearing in the areas of applied and natural sciences. The variable step size, variable order and suitable variable step size suggested for this study are considered to be very proficient, efficient and accurate. The *EBSTFM* is designed to provide solutions to oscillating and vibrating problems in applied and natural sciences. The results of *EBSTFM* are provided in Tables 1-3 which show the efficiency and accuracy. The *EBSTFM* has three results in Table 1 since it has three exact solutions while it has two results in Table 2 since it has two exact results and one result in Table 3 since it has one exact solution. The *EBSTFM* shows better performance than *VSSBBDF* and *BGMBDF* as shown in Tables 1 and 2 due to its ability to find a suitable variable step size to satisfy the tolerance level. On the other hand, the results of *VSSBBDF* perform better than *EBSTFM* as shown in Table 3 as a result of its inability to determine a suitable variable step size to satisfy the tolerance level. The performance demonstrated by *EBSTFM* utilizes the trigonometrically fitted method in accordance with the behaviour of the problem compared with other methods using linear difference operator and divided differences. Both the linear difference operator and divided differences are at variance with the behaviour of the problem. The *EBSTFM* possesses the advantage to determine the suited step size to enhance convergence at all tolerance levels. Thus, *EBSTFM* is seen to compete favourably with *VSSBBDF* and *BGMBDF* by implementing the variable step, variable order and suited variable step size. Further study will be carried out by reducing the step with varying order and suited varying step size.

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References

[1] U. M. Ascher and R. P. Linda, Computer Methods for Ordinary Differential Equations and Differential-algebraic Equations, SIAM, USA, 1998.

[2] M. Bond, Convolutions and the Weierstrass Approximation Theorem, 2009. https://bondmatt.files.wordpress.com/2009/06/weierstrass.pdf.

[3] W. Cheney and D. Kincaid, Numerical Mathematics and Computing, Thomson Brooks/Cole, USA, 2008.

[4] J. R. Dormand, Numerical Methods for Differential Equations, CRC Press, New York, 1996.

[5] J. D. Faires and R. L. Burden, Numerical Methods, Brooks/Cole, USA, 2002.

[6] J. D. Faires and R. L. Burden, Numerical Analysis, Brooks/Cole, USA, 2010.

[7] S. O. Fatunla, Numerical Methods for Initial Value Problems in Ordinary Differential Equations, Academic Press Inc., New York, 1988.

[8] C. W. Gear, Numerical Value Problems in ODEs, Prentice-Hall, Inc., New Jersey, USA, 1971.

[9] E. Hairer, S. P. Norsett and G. Wanner, Solving Ordinary Differential Equations I, Springer Heidelberg Dordrecht, New York, 2009.

[10] M. I. Hazizah and B. I. Zarina, Diagonally implicit block backward differentiation formula with optimal stability properties for stiff ordinary differential equations, Symmetry 11 (2019), 1-18.

[11] M. Ibrahim, M. Suleman, K. Iskandar and Z. Majid, Block method for generalized multistep Adams and backward differentiation formulae in solving first order ODEs, Matematika 21 (2005), 25-33.

[12] Z. B. Ibrahim, N. A. A. Mohd Nasir, K. I. Othman and N. Zainuddin, Adaptive order of block backward differentiation formulas for stiff ODEs, Numerical Algebra Control and Optimization 7 (2017), 95-106.

[13] J. D. Lambert, Computational Methods in Ordinary Differential Equations, John Wiley and Sons, New York, 1973.

[14] J. D. Lambert, Numerical Methods for Ordinary Differential Systems, John Wiley and Sons, New York, 1991.

[15] H. Musa, M. B. Suleiman, F. Ismail, N. Senu and Z. B. Ibrahim, An Accurate block solver for stiff initial value problems, ISRN Appl. Math. 2013 (2013), Art. ID 567451, 10 pp.

[16] F. F. Ngwane and S. N. Jator, Solving oscillatory problems using a block hybrid trigonometrically fitted method with two off-step points, Electron. J. Differential Equations 20 (2013), 119-132.

[17] F. F. Ngwane and S. N. Jator, Block hybrid method using trigonometric basis for initial value problems with oscillating problems, Numer. Algor. 63 (2013), 713-725.

[18] F. F. Ngwane and S. N. Jator, Trigonometrically-fitted second derivative method for oscillatory problems, SpringerPlus 3 (2014), 1-11.

[19] J. G. Oghonyon, S. A. Okunuga, N. A. Omoregbe and O. O. Agboola, A computational approach in estimating the amount of pond and determining the long time behavioural representation of pond pollution, Glob. J. Pure Appl. Math. 11 (2015), 2773-2786.

[20] J. G. Oghonyon, J. Ehigie and S. K. Eke, Investigating the convergence of some selected properties on block predictor-corrector methods and it's applications, Journal of Engineering and Applied Sciences 11 (2017), 2402-2408.

[21] J. G. Oghonyon, O. A. Adesanya, H. Akewe and H. I. Okagbue, Softcode of multi-processing Milne's device for estimating first-order ordinary differential equations, Asian Journal of Scientific Research 11 (2018), 553-559.

[22] J. G. Oghonyon, O. F. Imaga and P. O. Ogunniyi, The reversed estimation of variable step size implementation for solving nonstiff ordinary differential equations, International Journal of Civil Engineering and Technology 9 (2018), 332-340.

[23] J. G. Oghonyon, S. A. Okunuga and H. I. Okagbue, Expanded trigonometrically matched block variable-step-size techniques for computing oscillating vibrations, Lecture Notes in Engineering and Computer Science 2239 (2019), 552-557.

[24] L. K. Yap, I. Fudziah and M. Suleiman, Block Methods for Special Second Order Ordinary Differential Equations, Universiti Putra, Malaysia, 2011.