Parallel Solver for Oscillatory Stiff Systems of ODEs

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Abstract – The aim of this study will be to design Parallel solver (PS) for oscillatory stiff systems of ordinary differential equations (ODEs). PS will be constructed via a type of specially transformed exponentially fitted multinomial approximant in accordance with the behaviour of the solution. The method of interpolation and collocation will be utilized. The principal local truncation errors of PS will be used to derive a suitable step size and decide the error tolerance criteria for establishing the convergence of PS. Some examples of stiff ODEs will be examined and compared with existing methods to show case the efficiency and accuracy of the scheme. Parallel solver will be seen as a unique model for solving stiff ODEs without dependent on absolute stability as required by backward differentiation formula.

Keywords- parallel solver; exponentially fitted method; stiff ODEs; error tolerance criteria; suitable variable step size

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1 Introduction

The model of a large number of technological and applied science problems result to systems of ODEs. The computational solution of such problems via numerical integration, interpolation/collocation method and many more demands time, large space, softcode especially when the ODEs systems is large or the rating of the right hand side function is very costly. Thus, there is a need for effective parallel solver method to provide a quicker solution of such stiff systems. The numerical solution of ODEs by Parallelism can be split up into three families, viz the parallelism throughout the system, parallelism throughout the method and parallelism throughout the time. In this study, we will solely look at parallelism throughout the method. It can be declared clearly that parallelism throughout the method for the solution of ODEs takes it foundation in a family of proficiencies referred to as block methods [6, 27-28].

Stiff derivative equations are described as those whose precise solution possess a condition of the class e^{-ct}, where c is named as a large prescribe constant. This is commonly one component of the solution, named the transient solution; the more essential part of the solution is named the steady-state solution. A transient component of a stiff equation possesses magnitude $c^n e^{-ct}$, the differential will not decompose as rapidly. Stiff systems frequently constitute more than one element or depend on two or more elements connected together for effective functioning. For instances; electric circuit, mechanical and chemical flux systems, and traffic electronic network. Such systems need two or more dependant variable quantity for modelling the behaviour or function of the systems. This can be reported in terminal figure of a set of coupled first order differential equations. We seek for a solution with the example of a pair of coupled stiff systems with two or more variable quantities

$$\frac{dy_1}{dt} = A_{11}y_1 + A_{12}y_2 + A_{13}y_3$$
$$\frac{dy_2}{dt} = A_{21}y_1 + A_{22}y_2 + A_{23}y_3$$
$$\frac{dy_3}{dt} = A_{31}y_1 + A_{32}y_2 + A_{33}y_3. \qquad (1)$$

Equation (1) can be composed as a matrix equation

$$\frac{d\bar{y}}{dt} = \begin{bmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \\ \frac{dy_3}{dt} \end{bmatrix} = \begin{bmatrix} A_1 & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{23} & A_{33} \end{bmatrix} = A\bar{y},$$
$$\bar{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad (2)$$

The set of differential equations can be extracted as

$$\frac{d\bar{y}}{dt} = \begin{bmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \\ \frac{dy_3}{dt} \end{bmatrix} = A\bar{y} + g(t), \qquad \bar{g}(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \\ g_3(t) \end{bmatrix}. \qquad (3)$$

In equation (3), $g_1(t)$, $g_2(t)$ and $g_3(t)$ are the driving mathematical functions [9, 17].

Definition 1: The initial value problem (1) is ordered to be stiff if it meets $u_i < 0,, i = 1(1)m$, $\underset{1 \le i \le m}{\text{Max}} |u_i| > \underset{1 \le i \le m}{\min} |u_i|$, ; i.e., whensoever

(i) $re(\lambda_i) < 0, i = 1(1)m$, and
(ii) the stiffness ratio $s > 1$.

In addition, it should be mentioned that, this is a quite a general resolution with respect to mathematics. Stiffness takes place whensoever the step length is restrained by stability, rather than order onditions [10].

Definition 2: The initial value problem (1)-(3) is stiff oscillatory or having periodic vibrations whensoever the eigenvalues $\lambda_i = u_i + jv_i$, $i = 1(1)m$ of the Jacobian $J = \left(\frac{\partial f}{\partial dy}\right)$ have the succeeding attributes:

$$u_i < 0,, i = 1(1)m,$$
$$\underset{1 \le i \le m}{\text{Max}} |u_i| > \underset{1 \le i \le m}{\min} |u_i|,$$

or whensoever the stiffness ratio meets

$$\max_{1 \le i \le m} \left| \frac{u_i}{u_j} \right| > 1$$

and

$$|u_i| < |v_j|$$

For at least single pair of $i \in 1 \le i \le m$ [10].

Theorem 1: Let f: R → R be continuous and 2π −periodic. Then for each $\varepsilon > 0$, there exists a trigonometric polynomial $\mathrm{P(x)} = \sum_{\mathrm{j=-n}}^{\mathrm{k}} \mathrm{c_j}\, \mathrm{e^{ijx}}$ such that for all x, $|\mathrm{f(x)} - \mathrm{P(x)}| < \varepsilon$. Tantamountly, as for any such f, there must exist a successive polynomial such that $\mathrm{P_n} \to \mathrm{f}$ in a uniform manner on R [7].

The parallel solver of (1) can be constituted as the computational scheme in form of explicit and implicit methods.

$$A^{(0)}Y_n = \sum_{i=1}^{k} A^{(i)}Y_{n-i} + h\sum_{i=0}^{k} A^{(i)}F_{n-i}$$
$$A^{(0)}Y_n = \sum_{i=2}^{k} A^{(i)}Y_{n-i} + h\sum_{i=1}^{k} A^{(i)}F_{n+i} \quad (4)$$

where $Y_n = \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ \cdot \\ \cdot \\ \cdot \\ y_{n+r} \end{bmatrix}$, $F_n = \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ \cdot \\ \cdot \\ \cdot \\ f_{n+r} \end{bmatrix}$ (for $n = nr$, $n = 0,1, \ldots$), $A^{(i)}$ and $B^{(i)}$ are $r \times r$ matrices.

The parallel solver is called an explicit scheme if and only if the constant matrix $B^{(0)}$ is a zero matrix or otherwise referred as an implicit scheme [27].

Theorem 2: The A-stable multi-step scheme

(i) must exist as an implicit, and
(ii) the almost precise A-stable multi-step scheme is $y_{n+1} = y_n + \frac{h}{2}(f_n + f_{n+1})$ of order p = 2 and error coefficient $c_3 = -\frac{1}{12}$.

Dahlquist suggested several methods for outwitting the above theorem 2. Among them are the exponentially fitting and extrapolation processes. This study will explore the combination of expanded exponentially fitted and extrapolation processes to bring about parallel solver for stiff ODEs. Nevertheless, the exponentially fitted method agrees with behaviour of the stiff solution and the components of the extrapolation processes will be to implement the suited variable step size and error tolerance criteria to enhance the convergence of every loop. See [10, 14-15].

In literature, bookmen suggested the hybrid multi-step and hybrid implicit Runge-Kutta to solve (1). These methods comes with the vantage of not demanding for initiating values and own great stability regions. Majority of the backward differentiation formula and block hybrid backward differentiation formula has auto-initiating values with good region of absolute constancy. Other methods like block solver, parallel block backward differentiation formula and variable step block backward differentiation formula possesses some vantages like auto-initiating values, parallel execution, varying step size algorithm, step size modification with great regions of absolute stability. The major difficulties and challenges of these methods is evident on the inability to use exponentially fitted method in line with behaviour of the system. Backward differential formulas are over dependent on region of absolute stability without finding a suitable step size to ensure convergence. Again, the idea of auto initiating is gear towards time saving but there are single methods of order four that consumes time with better initiating results. On the other hand, parallel solver will be regarded as an alternative method to outwit the Dahlquist theorem and backward differentiation formula by introducing exponentially fitted method to approximate in accordance with the behaviour of the stiff solution. Again, parallel solver will introduce the extrapolation processes to overcome the great region of absolute stability by bringing about the suitable variable step size and error tolerance criteria which possesses the capability to change the step size, modify the order, vary the step and decide convergence. Parallel solver is a tedious computation and timing consuming procedure with a unique capacity to utilize the principal local truncations to find a suited variable step size and derive the error tolerance criteria [1-4, 10-11, 15-29].

The motivation of this research emanates from the fact that backward differentiation formulas are considered the ideal solver for stiff ODEs. This is due to the strong region of absolute stability. Parallel solver is of Adams family which is specially designed to bypass this condition by introducing the exponentially fitted method with the combination of variable step, variable order and variable step size together with error tolerance criteria. The contribution of this research study will be using the exponentially fitted method in line with behaviour of the exponential solution to build the model. Again, the idea of variable step, variable order with variable step is introduced to overcome the barriers.

2 Developing a Parallel Solver

The parallel solver of the explicit and implicit block methods will be developed as a combination of variable step and variable order techniques. This technique utilizes the $j-step\ of\ order\ k+1$ for the explicit block method with y_{n-1} as the point of interpolation and f_{n-1}, f_{n-2}, f_{n-3} as the points of collocation. On the other hand, $j-1-step\ of\ order\ k$ is employed for the implicit block method with y_{n-2} as the point of interpolation as well as f_{n+1}, f_{n+2}, f_{n+3} as the points of collocation. Furthermore, this technique will employ the expanded exponentially fitted as the multinomial approximant defined as

$$y(x)=\sum_{i=0}^{k}\bar{a}_i\left(\frac{x-x_n}{h}\right)^i+\sum_{i=0}^{1}\frac{e^{wx}}{i!}. \qquad (5)$$

The expansion of (5) will give birth to the expanded exponentially fitted method as

$$y(x)=a_0+a_1\frac{w(x-x_n)}{h}+a_2\frac{w^2(x-x_n)^2}{h^2}+a_3\frac{w^3(x-x_n)^3}{h^3}+a_4\frac{w^4(x-x_n)^4}{24h^4}, \qquad (6)$$

where a_0, a_1, a_2, a_3 and a_4 for k=4 are constant quantity to be specified in a peculiar way. Accepting that (6) gratifies the Weierstrass approximation theorem and correspond precisely to the solution at some selected points of interval x_{n-1}, x_{n-2} to generate the approximation as

$$y(x_{n-i})\approx y_{n-i}, \quad y(x_{n-2})\approx y_{n-2}. \qquad (7)$$

Interpolating and collocating (7) to gratifies (1) at the level where $x_{n+i}, i=0,1,2,3$ will generate the approximation of the succeeding approximation as

$$y'(x_{n-i})\approx f_{n-i},\ i=0,1,2,3.\ \ y'(x_{n+i})\approx f_{n+i},\ i=1,2,3. \qquad (8)$$

Bringing together the approximations of (7) and (8) will lead to fivefold and fourfold systems of equation for both explicit and implicit block methods as

$$\begin{aligned}
a_0&=y_{n-1}+\tfrac{3}{8}f_n+\tfrac{19}{24}f_{n-1}-\tfrac{5}{24}f_{n-2}+\tfrac{1}{8}f_{n-3},\\
a_1&=\tfrac{1}{w}f_n,\\
a_2&=\tfrac{11}{6w^2}f_n-\tfrac{3}{w^2}f_{n-1}+\tfrac{3}{2w^2}f_{n-2}-\tfrac{1}{3w^2}f_{n-3},\\
a_3&=\tfrac{2}{w^3}f_n-\tfrac{5}{w^3}f_{n-1}+\tfrac{4}{w^3}f_{n-2}-\tfrac{1}{w^3}f_{n-3},\\
a_4&=\tfrac{1}{w^4}f_n-\tfrac{3}{w^4}f_{n-1}+\tfrac{3}{w^4}f_{n-2}-\tfrac{1}{w^4}f_{n-3}. \qquad (9)
\end{aligned}$$

$$\begin{aligned}
a_0&=y_{n-2}+\tfrac{37}{3}f_{n+1}-\tfrac{50}{3}f_{n+2}+\tfrac{19}{3}f_{n+3},\\
a_1&=\tfrac{3}{w}f_{n+1}-\tfrac{3}{w}f_{n+2}+\tfrac{1}{w}f_{n+3},\\
a_2&=-\tfrac{5}{2w^2}f_{n+1}+\tfrac{4}{w^2}f_{n+2}-\tfrac{3}{2w^2}f_{n+3},\\
a_3&=\tfrac{1}{w^3}f_{n+1}-\tfrac{2}{w^3}f_{n+2}+\tfrac{1}{w^3}f_{n+3}. \qquad (10)
\end{aligned}$$

Equations (9) and (10) are the unknown physical quantities for developing the explicit and implicit block methods to be determined. The unknown physical quantities of equations (9) and (10) will be substituted into equation (6) to get the continuous explicit and implicit block methods. This continuous explicit and implicit block method will be evaluated at some selected points to achieve the parallel solver for the explicit and implicit block methods as

$$y(x)=y_{n-1}+h(\beta_0(w,x)f_n+\beta_1(w,x)f_{n-1}+\beta_2(w,x)f_{n-2})+\beta_3(w,x)f_{n-3} \qquad (11)$$

$$y(x)=y_{n-2}+h(\beta_0(w,x)f_{n+1}+\beta_1(w,x)f_{n+2}+\beta_2(w,x)f_{n+3}), \qquad (12)$$

where w is the frequency, $\beta_0(w,x), \beta_1(w,x)$ and $\beta_2(w,x)$ are fixed constants [7, 13-14, 20-26].

2.1 Developing the Error Tolerance Criteria for Implementing Parallel Solver

To launch this process, the collection of the principal local truncation errors of $k-step$ explicit block method with order $p+1$ method and $k-1-step\ with\ order\ p$ for the implicit block method is utilized. Parallel solver will be used to execute the approximation of the error tolerance criteria of the k-step explicit block and implicit block methods in the absence of estimating higher differential coefficients of $y(x)$. Accepting that $p+1=\bar{p}$, where $p+1\ and\ \bar{p}$ represents the order of the explicit and implicit block methods. Right away, for a method of order p, the analysis of $k-step$ explicit with order $p+1$ of the explicit block method will yield the principal local truncation errors as

$$\begin{aligned}
&C_{p+4}^{[1]}\mathrm{h}^{p+4}y^{(p+4)}(\mathrm{x_n})=\mathrm{y}(\mathrm{x}_{n+1})-\bar{y}_{n+1}^{[\bar{\mathrm{I}}_1]}+\left(-\frac{13}{720}+\frac{11}{2w^4}+\frac{25}{4w^3}+\frac{9}{4w^2}\right)+O(\mathrm{h}^{p+5})\\
&C_{p+4}^{[2]}\mathrm{h}^{p+4}y^{(p+4)}(\mathrm{x_n})=\mathrm{y}(\mathrm{x}_{n+2})-\bar{y}_{n+2}^{[\bar{\mathrm{I}}_1]}+\left(\frac{173}{720}+\frac{88}{w^4}+\frac{50}{w^3}+\frac{9}{w^2}\right)+O(\mathrm{h}^{p+5}),\qquad(13)\\
&C_{p+4}^{[3]}\mathrm{h}^{p+4}\mathrm{y}^{(p+4)}(x_n)=\mathrm{y}(\mathrm{x}_{n+3})-\bar{y}_{n+3}^{[\bar{\mathrm{I}}_3]}+\left(\frac{1439}{720}+\frac{891}{2w^4}+\frac{675}{4w^3}+\frac{81}{4w^2}\right)+O(\mathrm{h}^{p+5})
\end{aligned}$$

Similarly, inquiring into the $\mathrm{k}-1-\mathrm{step}$ implicit block method with $order\ p$ will generate the principal local truncation errors as

$$\begin{aligned}
&\bar{C}_{\bar{p}+3}^{[1]}\mathrm{h}^{\bar{p}+3}y^{(\bar{p}+3)}(\mathrm{x_n})=\mathrm{y}(\mathrm{x}_{n+1})-\bar{y}_{n+1}^{[\bar{\mathrm{q}}_1]}-\frac{(176-300w+216w^2+215w^3)}{24w^3}+O(\mathrm{h}^{\bar{p}+4})\\
&\bar{C}_{\bar{p}+3}^{[2]}\mathrm{h}^{\bar{p}+3}y^{(\bar{p}+3)}(\mathrm{x_n})=\mathrm{y}(\mathrm{x}_{n+2})-\bar{y}_{n+2}^{[\bar{\mathrm{q}}_1]}-\frac{(176-150w+54w^2+25w^3)}{3w^3}+O(\mathrm{h}^{\bar{p}+4}),\qquad(14)\\
&\bar{C}_{\bar{p}+3}^{[3]}\mathrm{h}^{\bar{p}+3}\mathrm{y}^{(\bar{p}+3)}(x_n)=\mathrm{y}(\mathrm{x}_{n+3})-\bar{y}_{n+3}^{[\bar{\mathrm{q}}_3]}-\frac{9(176-100w+24w^2+5w^3)}{8w^3}+O(\mathrm{h}^{\bar{p}+4})
\end{aligned}$$

$\mathrm{C}_{p+5}^{[1]}, C_{p+5}^{[2]}, \mathrm{C}_{p+5}^{[3]}$, $\bar{C}_{\bar{\bar{p}}+4}^{[1]}, \bar{C}_{\bar{p}+4}^{[2]}$ and $\bar{C}_{\bar{\bar{p}}+4}^{[3]}$ are in existence as distinguish entity of step size $\bar{\mathrm{h}}$ and $\mathrm{y}(\mathrm{x})$ will work as the accurate solution to differential coefficient gratifying the initial presumption $\bar{\mathrm{y}}(\mathrm{x_n})\approx\bar{\mathrm{y}}_\mathrm{n}$.

Continuing to build the presumption that for small measures of h,

$$\mathrm{y}^{(5)}(\mathrm{x_n})\approx\bar{y}^{(4)}(\mathrm{x_n})\qquad(15)$$

and the efficiency of the error control procedure banks on this presumption (15).

On deduction of (14) from (13) and disregarding terms of degree $O(\mathrm{h}^{\bar{\bar{p}}+4})$ as well as presume (15) will result to the error tolerance criteria for the principal local truncation errors as

$$\begin{aligned}
&\bar{C}_{\bar{\bar{p}}+4}^{[1]}\mathrm{h}^{\bar{\bar{p}}+4}\mathrm{y}^{(\bar{\bar{p}}+4)}(\mathrm{x_n})\approx\frac{6450}{6437}\left|\mathrm{y}_{n+1}^{[\bar{\mathrm{I}}_1]}-\bar{y}_{n+1}^{[\bar{\mathrm{q}}_1]}\right|<\varepsilon_1\\
&\bar{C}_{\bar{\bar{p}}+4}^{[2]}\mathrm{h}^{\bar{\bar{p}}+4}\mathrm{y}^{(\bar{\bar{p}}+4)}(\mathrm{x_n})\approx\frac{6000}{6173}\left|\mathrm{y}_{n+2}^{[\bar{\mathrm{I}}_2]}-\bar{y}_{n+2}^{[\bar{\mathrm{q}}_2]}\right|<\varepsilon_1,\qquad(16)\\
&\bar{C}_{\bar{\bar{p}}+4}^{[3]}\mathrm{h}^{\bar{\bar{p}}+4}\mathrm{y}^{(\bar{\bar{p}}+4)}(\mathrm{x_n})\approx\frac{4050}{5489}\left|\mathrm{y}_{n+2}^{[\bar{\mathrm{I}}_3]}-\bar{y}_{n+2}^{[\bar{\mathrm{q}}_3]}\right|<\varepsilon_1.
\end{aligned}$$

Stating the arguments that $\mathrm{y}_{n+1}^{[\bar{\mathrm{I}}_1]}\neq\bar{y}_{n+1}^{[\bar{\mathrm{q}}_1]}$, $\mathrm{y}_{n+2}^{[\bar{\mathrm{I}}_2]}\neq\bar{y}_{n+2}^{[\bar{\mathrm{q}}_2]}$ and $\mathrm{y}_{n+3}^{[\bar{\mathrm{I}}_3]}\neq\bar{y}_{n+3}^{[\bar{\mathrm{q}}_3]}$ are referred to as the predicting and correcting approximant of the principal local truncation errors for the explicit block and implicit block methods. $\bar{C}_{\bar{\bar{p}}+4}^{[1]}\mathrm{h}^{\bar{\bar{p}}+4}\mathrm{y}^{(\bar{\bar{p}}+4)}(\mathrm{x_n})$, $\bar{C}_{\bar{\bar{p}}+4}^{[2]}\mathrm{h}^{\bar{\bar{p}}+4}\mathrm{y}^{(\bar{\bar{p}}+4)}(\mathrm{x_n})$ and $\bar{C}_{\bar{\bar{p}}+4}^{[3]}\mathrm{h}^{\bar{\bar{p}}+4}\mathrm{y}^{(\bar{\bar{p}}+4)}(\mathrm{x_n})$ are distinctly addressed as the principal local truncation errors and ε_1 are the boundaries of the error tolerance criteria for implementing parallel algorithm

In addition, the approximants of the principal local truncation errors (16) will be utilize to resolve whether to allow the effect of the loop with the current step size or repeat the loop with a reduce step size. This procedure is verified based on the trial run carried out by (16) [5, 8-9, 15-16, 20-26].

2.2 Step Size Adjustment and Error Control Procedures for Parallel Solver

The global error of (16) can be approximated by

$$\begin{aligned}
&\left|\mathrm{y}(\mathrm{x}_{n+1})-\bar{y}_{n+1}^{[\bar{\mathrm{I}}_1]}\right|\approx\frac{\left|z(\mathrm{x}_{n+1})-\bar{y}_{n+1}^{[\bar{\mathrm{q}}_1]}\right|}{\mathrm{h}}\approx\frac{6450}{6437\mathrm{h}}\left|\mathrm{y}_{n+1}^{[\bar{\mathrm{I}}_1]}-\bar{y}_{n+1}^{[\bar{\mathrm{q}}_1]}\right|\\
&\left|\mathrm{y}(\mathrm{x}_{n+2})-\bar{y}_{n+2}^{[\bar{\mathrm{I}}_2]}\right|\approx\frac{\left|z(\mathrm{x}_{n+2})-\bar{y}_{n+2}^{[\bar{\mathrm{q}}_2]}\right|}{\mathrm{h}}\approx\frac{6000}{6173\mathrm{h}}\left|\mathrm{y}_{n+2}^{[\bar{\mathrm{I}}_2]}-\bar{y}_{n+2}^{[\bar{\mathrm{q}}_2]}\right|\qquad(17)\\
&\left|\mathrm{y}(\mathrm{x}_{n+3})-\bar{y}_{n+3}^{[\bar{\mathrm{I}}_3]}\right|\approx\frac{\left|z(\mathrm{x}_{n+3})-\bar{y}_{n+3}^{[\bar{\mathrm{q}}_3]}\right|}{\mathrm{h}}\approx\frac{4050}{5489\mathrm{h}}\left|\mathrm{y}_{n+3}^{[\bar{\mathrm{I}}_3]}-\bar{y}_{n+3}^{[\bar{\mathrm{q}}_3]}\right|,
\end{aligned}$$

where z constitute the solution to the first derivative equation gratifying the initial condition $z(x_n)=\mathrm{y}(\mathrm{x_n})$.

Imagine if we immediately reconstruct the situation with a new step size qh producing new approximants $\mathrm{y}_{n+1}^{[\bar{\mathrm{I}}_1]},\bar{y}_{n+1}^{[\bar{\mathrm{q}}_1]}$, $\mathrm{y}_{n+2}^{[\bar{\mathrm{I}}_2]},\bar{y}_{n+2}^{[\bar{\mathrm{q}}_2]}$ and $\mathrm{y}_{n+3}^{[\bar{\mathrm{I}}_3]}$ and $\bar{y}_{n+3}^{[\bar{\mathrm{q}}_3]}$. To check and control the global error to inside ε_1, we select q such that

$$\begin{aligned}
&\frac{\left|z(x_n+qh)-\bar{y}_{n+1}^{[\bar{\mathrm{q}}_1]}(utilizing\ the\ step\ size\ qh)\right|}{qh}<\varepsilon_1\\
&\frac{\left|z(x_n+qh)-\bar{y}_{n+2}^{[\bar{\mathrm{q}}_2]}(utilizing\ the\ step\ size\ qh)\right|}{qh}<\varepsilon_1\qquad(18)\\
&\frac{\left|z(x_n+qh)-\bar{y}_{n+3}^{[\bar{\mathrm{q}}_3]}(utilizing\ the\ step\ size\ qh)\right|}{qh}<\varepsilon_1
\end{aligned}$$

Employing (14), we get

$$\frac{\left|z(x_n+qh)-\bar{y}_{n+1}^{[\bar{q}_1]}(utilizing\ qh)\right|}{qh}=\frac{215}{24}=$$

$$\left|z^{(4)}(\bar{x}_n)\right|q^4h^4\approx\frac{215}{24}\left[\frac{720}{6437}\left|y_{n+1}^{[\bar{l}_1]}-\bar{y}_{n+1}^{[\bar{q}_1]}\right|\right]q^4h^4,$$

$$\frac{\left|z(x_n+qh)-\bar{y}_{n+2}^{[\bar{q}_2]}(utilizing\ qh)\right|}{qh}=\frac{215}{24}=$$

$$\left|z^{(4)}(\bar{x}_n)\right|q^4h^4\approx\frac{25}{3}\left[\frac{720}{6173}\left|y_{n+2}^{[\bar{l}_2]}-\bar{y}_{n+2}^{[\bar{q}_2]}\right|\right]q^4h^4, \quad (19)$$

$$\frac{\left|z(x_n+qh)-\bar{y}_{n+3}^{[\bar{q}_3]}(utilizing\ qh)\right|}{qh}=\frac{215}{24}=$$

$$\left|z^{(4)}(\bar{x}_n)\right|q^4h^4\approx\frac{45}{8}\left[\frac{720}{5489}\left|y_{n+3}^{[\bar{l}_3]}-\bar{y}_{n+3}^{[\bar{q}_3]}\right|\right]q^4h^4,$$

Therefore, we require to select q with

$$\frac{215}{24}\left[\frac{720}{6437}\left|y_{n+1}^{[\bar{l}_1]}-\bar{y}_{n+1}^{[\bar{q}_1]}\right|\right]q^4h^4=\frac{6450}{6437}\frac{\left|y_{n+1}^{[\bar{l}_1]}-\bar{y}_{n+1}^{[\bar{q}_1]}\right|}{h}/h<\varepsilon_1,$$

$$\frac{25}{3}\left[\frac{720}{6173}\left|y_{n+2}^{[\bar{l}_2]}-\bar{y}_{n+2}^{[\bar{q}_2]}\right|\right]q^4h^4=\frac{6000}{6173}\frac{\left|y_{n+2}^{[\bar{l}_2]}-\bar{y}_{n+2}^{[\bar{q}_2]}\right|}{h}<\varepsilon_1, \quad (20)$$

$$\frac{45}{8}\left[\frac{720}{5489}\left|y_{n+3}^{[\bar{l}_3]}-\bar{y}_{n+3}^{[\bar{q}_3]}\right|\right]q^4h^4=\frac{4050}{5489}\frac{\left|y_{n+3}^{[\bar{l}_3]}-\bar{y}_{n+3}^{[\bar{q}_3]}\right|}{h}<\varepsilon_1.$$

Accordingly, this will require the change in step size from $h\ to\ qh$, where q gratifies

$$q<\left(\left(\frac{6437}{6450}\right)\frac{h\varepsilon_1}{\left|y_{n+1}^{[\bar{l}_1]}-\bar{y}_{n+1}^{[\bar{q}_1]}\right|}\right)^{\frac{1}{4}}\approx 0.999496\left(\frac{h\varepsilon_1}{\left|y_{n+1}^{[\bar{l}_1]}-\bar{y}_{n+1}^{[\bar{q}_1]}\right|}\right),$$

$$q<\left(\left(\frac{6173}{6000}\right)\frac{h\varepsilon_1}{\left|y_{n+1}^{[\bar{l}_1]}-\bar{y}_{n+1}^{[\bar{q}_1]}\right|}\right)^{\frac{1}{4}}\approx 1.00713\left(\frac{h\varepsilon_1}{\left|y_{n+2}^{[\bar{l}_2]}-\bar{y}_{n+2}^{[\bar{q}_2]}\right|}\right), \quad (21)$$

$$q<\left(\left(\frac{5489}{4050}\right)\frac{h\varepsilon_1}{\left|y_{n+1}^{[\bar{l}_1]}-\bar{y}_{n+1}^{[\bar{q}_1]}\right|}\right)^{\frac{1}{4}}\approx 1.07897\left(\frac{h\varepsilon_1}{\left|y_{n+3}^{[\bar{l}_3]}-\bar{y}_{n+3}^{[\bar{q}_3]}\right|}\right).$$

Thus, a number of estimate suppositions have been established in this development, hence in practical applications the new step size q is selected in a conservative manner. A step size change for parallel algorithm is more pricey and tedious in terms of functional valuations than for a multi-step method [9].

According to [15-16], the broad computing experience that has been compiled throughout the years suggest that the primal to greater efficiency and error control in explicit block and implicit block methods is the capability to change automatically not just the step size, simply also the order (and step number of the methods utilized).

3 Numerical Examples of Stiff ODEs

The numerical examples of stiff ODEs consider for this research study are those with stiff oscillatory and vibrating solutions.

System Problem 1

The first example is a virtually sinusoidal problem defined in the interval $0 \leq t \leq 10$.

$y_1'(x) = -2y_1 + y_2 + 2Sinx, \quad y_1(0) = 2$

$y_2'(x) = 998y_1 - 999y_2 + 999Sinx, \quad y_2(0) = 3$

with analytical solution

$y_1 = 2e^{-x} + Sinx$

$y_2 = 2e^{-x} + Cosx$

Author: [4, 20].

System Problem 2

$y_1' = 198y_1 + 199y_2, \quad y_1(0) = 1, \quad 0 \leq x \leq 10$

$y_2' = -398y_1 - 399y_2, \quad y_2(0) = -1$

with analytical solution: $y_1(x) = e^{-x}, \quad y_2(x) = -e^{-x}$

For System 2, $\bar{\lambda} = -1\ and\ \underline{\lambda} = -200$.

Author: [12, 19].

System Problem 3

$y_1' = -20y_1 - 19y_2, \quad y_1(0) = 2, \quad 0 \leq x \leq 20$

$y_2' = -19y_1 - 20y_2, \quad y_2(0) = 0$

with analytical solution: $y_1(x) = e^{-39x} + e^{-x}$, $y_2(x) = e^{-39x} - e^{-x}$.

Author: [19]

4 Results and Discussion

The numerical examples of system1, system 2 and system 3 are all oscillatory stiff systems of ordinary differential equations which must gratifies the condition of definition 1 and 2 with respect to the oscillating behaviour or periodic vibrations. Most block backward differentiation formula derivations are carried out employing the multinomial approximant, Lagrange multinomial, backward difference multinomial and Taylor series expansion of the linear operator. The PS in accordance with definition 1 and 2 is formulated using exponentially fitted which is one of the principal justification to outwit the Dahlquist theorem and backward differentiation formula. These aspects yield a very good precision and efficiency with the introduction of the suitable variable step size and error tolerance criteria. The strength of the parallel solver lies in the ability to find a suitable step size and generate the error tolerance criteria to foster the convergence of the loop. The PS performs better compare to $1BDF, 2BDF, 3BDF,\ BDF(5),$ $HBSDBDF\ and\ 3NBBDF$ due to the task involved in designing a suitable variable step size for each loop to ensure the convergence at every error tolerance criteria. Again, PS implements an expanded exponentially fitted multinomial approximant based on the oscillatory behaviour of the solution as seen in the numerical examples. This strategy will ensure faster convergence of the loop with very good precision and efficiency. On the other hand, $HBSDBDF$ is auto-initiating with good region of absolute stability and fixed large step length h to ensure the implementation. Therefore, $HBSDBDF$ shows well stable properties with precision and efficiency while the $1BDF, 2BDF, 3BDF, BDF(5)\ and\ 3NBBDF$ are all centred on trimming the entire number of paces, computing time utilized and possessing good stability region [4, 12, 19-20].

Table 1. Numerical Results for System Problem 1

MU	MAXE	ETC
$BBDF(5)$	1.02772×10^{-4}	10^{-4}
$PS(y_1)$	3.83169×10^{-5}	10^{-4}
$PS(y_2)$	3.8188×10^{-5}	10^{-4}
$BBDF(5)$	1.02861×10^{-6}	10^{-6}
$PS(y_1)$	3.98×10^{-7}	10^{-6}
$PS(y_2)$	4.10127×10^{-7}	10^{-6}
$HBSDBDF$	8.9924×10^{-7}	10^{-7}
$PS(y_1)$	6.29699×10^{-8}	10^{-7}
$PS(y_2)$	3.71197×10^{-8}	10^{-7}
$PS(y_1)$	3.99507×10^{-9}	10^{-8}
$PS(y_2)$	4.13105×10^{-9}	10^{-8}
$BBDF(5)$	6.77840×10^{-9}	10^{-9}
$HBSDBDF$	5.9042×10^{-9}	10^{-9}
$PS(y_1)$	6.30645×10^{-10}	10^{-9}
$PS(y_2)$	3.72004×10^{-10}	10^{-9}
$PS(y_1)$	6.29699×10^{-11}	10^{-10}
$PS(y_2)$	3.83059×10^{-11}	10^{-10}
$HBSDBDF$	4.5695×10^{-11}	10^{-11}
$PS(y_1)$	6.3074×10^{-12}	10^{-11}
$PS(y_2)$	3.72058×10^{-12}	10^{-11}
$HBSDBDF$	2.9376×10^{-13}	10^{-13}
$PS(y_1)$	6.30607×10^{-14}	10^{-13}
$PS(y_2)$	3.73035×10^{-14}	10^{-13}

Table 2. Numerical Results for System Problem 2

MU	MAXE	ETC
$3BDF$	1.07308×10^{-2}	10^{-2}
$PS(y_1)$	3.95037×10^{-3}	10^{-2}
$1BDF$	3.61405×10^{-3}	10^{-3}
$2BDF$	7.18323×10^{-3}	10^{-3}
$3BDF$	1.10060×10^{-3}	10^{-3}
$PS(y_1)$	3.76613×10^{-4}	10^{-3}
$1BDF$	3.67235×10^{-4}	10^{-4}
$2BDF$	7.34012×10^{-4}	10^{-4}
$3BDF$	1.10333×10^{-4}	10^{-4}
3NBBDF	1.94447×10^{-4}	10^{-4}
$PS(y_1)$	3.97367×10^{-5}	10^{-4}
$1BDF$	3.67815×10^{-5}	10^{-5}
$2BDF$	7.35584×10^{-5}	10^{-5}
$3BDF$	1.10361×10^{-5}	10^{-5}
$PS(y_1)$	4.20444×10^{-6}	10^{-5}
$1BDF$	3.67873×10^{-6}	10^{-6}
$2BDF$	7.35741×10^{-6}	10^{-6}
$3BDF$	1.10363×10^{-6}	10^{-6}
3NBBDF	2.07993×10^{-6}	10^{-6}
$PS(y_1)$	4.51844×10^{-7}	10^{-6}
$1BDF$	3.67839×10^{-7}	10^{-7}
$2BDF$	7.35747×10^{-7}	10^{-7}
$PS(y_1)$	4.25334×10^{-8}	10^{-7}
$3NBBDF$	2.09995×10^{-8}	10^{-8}
$PS(y_1)$	4.53559×10^{-9}	10^{-8}
$3NBBDF$	2.10257×10^{-10}	10^{-10}
$PS(y_1)$	4.53733×10^{-11}	10^{-10}
$3NBBDF$	1.41029×10^{-11}	10^{-11}
$PS(y_1)$	4.2587×10^{-12}	10^{-11}

Table 3. Numerical Results for System Problem3

MU	MAXE	ETC
$3NBBDF$	6.98707×10^{-2}	10^{-2}
$PS(y_1)$	4.048×10^{-3}	10^{-2}
$PS(y_2)$	4.04018×10^{-3}	10^{-2}
$3NBBDF$	5.40956×10^{-3}	10^{-3}
$PS(y_1)$	4.06886×10^{-4}	10^{-3}
$PS(y_2)$	4.0628×10^{-4}	10^{-3}
$3NBBDF$	3.08942×10^{-5}	10^{-5}

$PS(y_1)$	4.56163×10^{-6}	10^{-5}
$PS(y_2)$	4.55556×10^{-6}	10^{-5}
$3NBBDF$	3.18534×10^{-7}	10^{-7}
$PS(y_1)$	4.61687×10^{-8}	10^{-7}
$PS(y_2)$	4.61079×10^{-8}	10^{-7}
$3NBBDF$	3.19872×10^{-9}	10^{-9}
$PS(y_1)$	4.62245×10^{-10}	10^{-9}
$PS(y_2)$	4.61638×10^{-10}	10^{-9}

4.1 Nomenclature

The nomenclatures utilized in the tables represent the following meaning.
$PS(y_1)$: parallel solver of solution y_1
$PS(y_2)$: parallel solver of solution y_2
MU : method used
$MAXE$: maximum errors
ETC : error tolerance criteria
$HBSDBDF$: hybrid block second derivative backward differentiation formula [4].
$1BDF$: $r = 1 - point$ BDF method [12].
$2BDF$: $r = 2 - point$ BBDF method [12].
$3BDF$: $r = 3 - point$ BBDF method [12].
$3NBBDF$: $3 - point$ fifth order new BBDF method [19].
$BBDF(5)$: fifth order Block Backward Differentiation Formulas [20].

5 Conclusion

Parallel solver for oscillatory stiff systems of ODEs has been suggested. Parallel solver is a fusion of the explicit and implicit blocks method developed via interpolation and collocation methods with the help of the exponentially fitted method as the polynomial. The exponentially fitted method and the components of the extrapolation processes such as variable step, variable order and suitable variable step size were used to outwit the Dahlquist obstacle and backward differentiation formulas. This combination is geared to foster error control with an improve accuracy, greater efficiency and maximize errors. Three problems were examined under the following error tolerance criteria; $10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}, 10^{-7}, 10^{-8}, 10^{-9}$, $10^{-10}, 10^{-11}, 10^{-13}$and compared with PS. The convergence of PS is made possible with the help of deciding a suitable step size to meet the error tolerance criteria which in turn lead to achieving lesser maximum error. The following methods of $1BBDF, 2BBDF, 3BBDF, BBDF(5), 3NBBDF$, $HBSDBDF$ has good stability properties which is implemented using large step size compare to $PS(y_1), PS(y_2)$ that requires the determination of a suitable variable step size and error tolerance criteria during implementation. Table 1, Table 2 and Table 3 presents the end result of the PS compare with other subsisting methods of $1BBDF, 2BBDF, 3BBDF, BBDF(5), 3NBBDF$, $HBSDBDF$ .Thus, PS which involves tedious computing strategies of implementing variable step, variable order and finding suitable variable step size has the advantage of high precision, high efficiency with more preferred maximum errors compare to subsisting methods of $1BBDF, 2BBDF, 3BBDF, BBDF(5), 3NBBDF$, $HBSDBDF$.

Further Study:
The further study will be to implement parallel solver in higher order of ODEs.

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Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this article and among authors.

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Olasunmbo Olaoluwa Agoola carried out the editing and supervision.
Jimevwo Godwin Oghonyon devised the idea, method used and implemented the code using Mathematica.
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