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Solution of a barrier option Black-Scholes model based on projected differential transformation method

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Abstract. In this article, the solution of the linear variant of a Barrier Option Black-Scholes Model (BOBSM) is considered via a semi-analytical approach referred to as the Projected Differential Transformation Method (PDTM). Similar to the traditional Differential Transformation Method, this new approach demonstrates feasible progress and efficiency of operation. For simplicity of illustrative, the BOBSM is converted to an equivalent heat-like form, and a series-form of the solution (root) is successfully obtained. Hence the PDTM is suggested for both pure and functional sciences for strongly nonlinear differential models with financial applications.

Keywords: Option pricing; Black-Scholes equation; Differential model, Barrier option; closed-form solution

1. Introduction

In financial practices, the role of the classical Black-Scholes model (BSM) cannot be overemphasized [1]. This is based on some basic assumptions under which this classical arbitrage pricing theory is built. Nonetheless, some conditions and constraints tell the choice of options to determine the financial transactions. Options trading is a matter of specific relevance and concern owing to its role in the financial system [2-5]. Standard option contracts are exchanged on options markets, and their rates are commonly published. Nonetheless, there is also a need for more tailored option contracts such as exotic options that are structured to adapt to more sophisticated approaches for risk reduction. Among other financial options, the Barrier option is termed contingent upon hitting some stock price, named the barrier, before its expiration [6-9]. The barrier option is either knock-out or knock-in. Here, the barrier path is of great interest since a knock-down becomes worthless in value once the stock price reaches the barrier at any time prior to the specified expiration. On the other hand, a knock-in option only yields a payoff once the barrier is crossed by the stock price. Barrier options are becoming increasingly common, mostly due to the reduced expense of keeping a barrier option as opposed to holding a regular call/put option. Still, exotic options are hard to price as payoff functions rely on the whole direction of the underlying operation, rather than its valuation at a particular time moment [4, 10-13]. It is a path-dependent option, that implies that the payoff depends on the path followed by the



underlying asset price, which means that the price of the barrier options is extremely sensitive to volatility [5-7]. Barrier options are commonly used in risk management by retail investors, banks, and businesses, and early exercise versatility offers their holders more American-style rights. Since a broad range of exchanged options is of American type, the problem of valuing American options has become an important topic of financial economics [5, 8, 12].

This paper will consider a barrier option pricing model in the Black-Scholes framework with an emphasis on down-and-out call options on a certain barrier term, as follows.

Suppose at time $t < T$, with $S(t)$ as the stock price at time t such that K and B are the strike price and barrier option respectively, then the corresponding payoff of a down-and-out call option is defined as:

$$f_0^d(S) = (S_T - K)^+, t < T \quad (1.1)$$

with B being above $S(t)$. The barrier is remarked to be below the initial stock price; otherwise, the option is worthless. The notion of the down-and-out option was nurtured since such goes past the known barrier. The option cannot pay –off anything unless it is guaranteed that S_t crosses the barrier at some $t < T$.

2. The Dynamics of the Barrier Option

Let $S(t)$, K , and B be as defined above for $t < T$. Then, a barrier option is referred to as a traditional option with an additional constraint involving B , such that the following partial Black-Scholes Model (BSM) is satisfied.

$$\begin{cases} M_t + rSM_s + \frac{1}{2}S^2\sigma^2M_{ss} - rM = 0 \\ M(S, T) = (S - K)^+, S \in (0, \infty), \end{cases} \quad (2.1)$$

where $M_{(\cdot)}$ denotes partial derivative operator w.r.t. a subscripted variable, $M(S, t)$ is the option value, $\sigma > 0$, the volatility parameter, r , the risk-free interest rate, and $M(B, t) = 0$ is the extra (additional) condition. For simplicity, we intend to reduce (2.1) to its equivalent heat form based on the following change of variables:

$$\begin{cases} w = \ln(SB^{-1}) \Rightarrow S = Be^w \\ \tau = T - t. \end{cases} \quad (2.2)$$

Thus,

$$\begin{cases} M_t = \frac{\partial M}{\partial t} = -\frac{\partial M}{\partial \tau} \\ M_w = \frac{\partial M}{\partial w} = \frac{\partial M}{\partial S} \frac{\partial S}{\partial w} = S \frac{\partial M}{\partial S} \\ M_{ww} = \frac{\partial^2 M}{\partial w^2} = \frac{\partial}{\partial w} \left(\frac{\partial M}{\partial w} \right) = S \frac{\partial M}{\partial S} + S^2 \frac{\partial^2 M}{\partial S^2} \end{cases} \quad (2.3)$$

Putting (2.2) and (2.3) in (2.1), with little algebra, we have:

$$\begin{cases} M_\tau = \left(r - \frac{1}{2}\sigma^2\right)M_w + \frac{1}{2}\sigma^2M_{ww} - rM \\ M(w, 0) = (Be^w - K)^+, M(0, \tau) = 0. \end{cases} \quad (2.4)$$

Suppose we further define:

$$\begin{cases} M(w, \tau) = Be^{\alpha w + \beta \tau} v(w, \tau), \\ \alpha, \beta \geq 0, \end{cases} \quad (2.5)$$

then, (2.4) yields the heat-like equation of the form:

$$\begin{cases} v_\tau = v_{ww}, \\ v(w, 0) = \begin{cases} \left(e^{w-\alpha w} - \frac{K}{B}e^{-\alpha w}\right), w > 0, \\ -\left(e^{\alpha w - w} - \frac{K}{B}e^{\alpha w}\right), w < 0. \end{cases} \end{cases} \quad (2.6)$$

Note that the classical BSM for European call option is retrievable from (2.4-2.6) for $B = 1$ and $\alpha = 0$.

Financial models and the likes are, in most cases, in the form of ordinary or partial differential equations [14-16]. Few of these differential models have known exact solutions. However, obtaining the solutions of some of these seems tedious and time-consuming; this is even when the existence of the solutions is guaranteed [17-25]. Thus, a lot of numerical approaches have been proposed and adopted; notwithstanding, better approaches are anticipated [26-37]. In this regard, the Barrier option model built on the classical BSM is reduced to an equivalent heat-like equation, and a fast and efficient semi-analytical method is proposed [38].

2.1 Remarks on the PDTM

In this section, the basic concepts and procedures regarding the proposed method (PDTM) are presented [38].

Let $q(x, t)$ defined on a given domain, G , be an analytic function, at a specified point (x_0, t_0) , such that the Taylor series expansion of $q(x, t)$, is ascertained. Then, the projected differential transform of $q(x, t)$ and its inverse projected differential transform are defined and represented respectively as:

$$\left. \begin{aligned} Q(x, l) &= \frac{1}{l!} \left[\frac{\partial^l q(x, t)}{\partial t^l} \right]_{t=t_0} \\ q(x, t) &= \sum_{l=0}^{\infty} Q(x, l) (t - t_0)^l \end{aligned} \right\} \quad (2.7)$$

The following properties (P1-P5) and theorems associated with the method of solution are noted as follows in Table 1:

Table 1: Some Basic Properties of the PDTM

Property	Original function form	Projected Transform form
<i>P1</i>	$q(x, t) = \alpha q_a(x, t) + \beta q_b(x, t)$	$Q(x, \hbar) = \alpha Q_a(x, l) + \beta Q_b(x, l)$
<i>P2</i>	$q(x, t) = \alpha \frac{\partial^n q_*(x, t)}{\partial t^n}$,	$l!z(x, l) = \alpha(l+n)!Q_*(x, l+n)$
<i>P3</i>	$q(x, t) = \alpha \frac{\partial q_*(x, t)}{\partial t}$	$l!Q(x, l) = \alpha(l+1)!Q_*(x, l+1)$
<i>P4</i>	$q(x, t) = f(x) \frac{\partial^n q_*(x, t)}{\partial x^n}$	$Q(x, l) = f(x) \frac{\partial^n Q_*(x, l)}{\partial x^n}$
<i>P5</i>	$q(x, t) = f(x) q_*^2(x, t)$,	$Q(x, l) = f(x) \sum_{i=0}^l Q_*(x, i) Q_*(x, l-i)$.

3. Applications

In this section, the proposed method is applied to the derived model in (2.6) for $w > 0$, hence, we have:

$$\begin{cases} v_\tau = v_{ww}, \\ v(w, 0) = \left(e^{w-\alpha w} - \frac{K}{B} e^{-\alpha w} \right), w > 0. \end{cases} \quad (3.1)$$

As such, taking the PDT of (3.1) gives:

$$\begin{aligned} & PDT(v_\tau = v_{ww}) \\ & \Rightarrow \begin{cases} \left(V(w, l+1) = \frac{1}{l+1} V_{ww}(w, l) \right), l = 0, 1, 2, \dots \\ V(w, 0) = \left(e^{(1-\alpha)w} - \frac{K}{B} e^{-\alpha w} \right) \end{cases} \end{aligned} \quad (3.2)$$

such that:

$$v(w, \tau) = \sum_{l=0}^{\infty} V(w, l) \tau^l. \quad (3.3)$$

From (3.2), for $l = 0, 1, 2, 3, \dots$, the following are respectively obtained:

$$\begin{aligned} V(w, l+1) &= \frac{1}{l+1} V_{ww}(w, l) \\ V(w, 1) &= V_{ww}(w, 0) = \left((1-\alpha)^2 e^{(1-\alpha)w} - \frac{K}{B} \alpha^2 e^{-\alpha w} \right) \\ V(w, 2) &= \frac{1}{2} V_{ww}(w, 1) \\ &= \frac{1}{2} \left((1-\alpha)^4 e^{(1-\alpha)w} - \frac{K}{B} \alpha^4 e^{-\alpha w} \right) \end{aligned}$$

$$\begin{aligned}
V(w, 3) &= \frac{1}{3} V_{ww}(w, 2) \\
&= \frac{1}{3} \left((1-\alpha)^6 e^{(1-\alpha)w} - \frac{K}{B} \alpha^6 e^{-\alpha w} \right) \\
V(w, 4) &= \frac{1}{4} V_{ww}(w, 3) \\
&= \frac{1}{4} \left((1-\alpha)^8 e^{(1-\alpha)w} - \frac{K}{B} \alpha^8 e^{-\alpha w} \right) \\
&\vdots \\
V(w, j) &= \frac{1}{j} \left((1-\alpha)^{2j} e^{(1-\alpha)w} - \frac{K}{B} \alpha^{2j} e^{-\alpha w} \right), j = 1, 2, 3, \dots \\
\therefore v(w, \tau) &= \sum_{l=0}^{\infty} V(w, l) \tau^l \\
&= v(w, 0) + \sum_{j=1}^{\infty} V(w, j) \tau^j \\
&= \left(e^{(1-\alpha)w} - \frac{K}{B} e^{-\alpha w} \right) + \sum_{j=1}^{\infty} \left[\frac{1}{j} \left((1-\alpha)^{2j} e^{(1-\alpha)w} - \frac{K}{B} \alpha^{2j} e^{-\alpha w} \right) \right] \tau^j.
\end{aligned} \tag{3.4}$$

Thus,

$$\begin{cases} M(w, \tau) = B e^{\alpha w + \beta \tau} v(w, \tau) \\ \quad = B e^{\alpha w + \beta \tau} \left\{ \left(e^{(1-\alpha)w} - \frac{K}{B} e^{-\alpha w} \right) + \sum_{j=1}^{\infty} \left[\frac{1}{j} \left((1-\alpha)^{2j} e^{(1-\alpha)w} - \frac{K}{B} \alpha^{2j} e^{-\alpha w} \right) \right] \tau^j \right\}. \end{cases} \tag{3.5}$$

Hence, for $\tau = T - t$, and $w = \ln(SB^{-1})$ the option value, $M(S, t)$, is obtained.

We remarked here that (3.5) would serve as a benchmark for comparison using alternative methods for further researches.

4. Conclusions

This research initiated the formulation of the Projected Differential Transformation Method (PDTM) for approximate-analytical approaches to the linear form of the Barrier Option Pricing Model in the framework of the classical Black-Scholes equation. The problem has been solved without a variable-discretizing call. The result obtained suggested that the PDTM was efficient and accurate. The results were presented in a series form with fewer interventions in the computational period. Therefore the method is recommended for application in applied sciences for strongly nonlinear differential and other associated financial models.

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