PAPER • OPEN ACCESS

Iterative methods for approximate solution of the Ornstein-Uhlenbeck process with normalised brownian motion

To cite this article: O. P. Ogundile and S. O. Edeki 2021 *J. Phys.: Conf. Ser.* **1734** 012015

Iterative methods for approximate solution of the Ornstein-Uhlenbeck process with normalised brownian motion

O. P. Ogundile[1] and S. O. Edeki[2*]
[1,2]Department of Mathematics, Covenant University Ota, Nigeria
[*]Corresponding Author Email: soedeki@yahoo.com

Abstract. This work considers the concept of the Normalised Brownian motion for the solutions of the Ornstein-Uhlenbeck process using the Daftardar-Jafari Method (DJM) and Picard Iterative Method (PIM) as the approximate-analytical methods of solutions. The results obtained from DJM are compared with those of the PIM. The obtained results, therefore, show the effectiveness of the proposed methods.

1. Introduction

Stochastic differential equations (SDEs) are particular types of differential equation possessing a term or more that is a stochastic process, whereby the resultant solution gives a random process. In general, it has its application in applied sciences and Engineering. In most cases, obtaining their analytical solutions appears intricate and challenging [1-4]. In this work, a particular class of SDE known as the Ornstein-Ulenbeck process (O-U process), will be considered. The O-U process is a stationary Gaussian (stochastic) process with its applications in financial mathematics and other physical sciences [5-6].

The general Stochastic differential equation can be expressed as:

$$dS(t) = a\left(S(t), t\right)dt + b\left(S(t), t\right)dB(t), t \in [0,1], \tag{1.1}$$

where a, and b denote the drift and volatility coefficients respectively, and $S(t)$ represents the Stock price in the financial mathematics domain. $B(t)$, is the standard Brownian motion with its differential equivalence as $dB(t)$, which is the noise term.

In integral form, (1.1) can be re-written as:

$$S(t) = S_0 + \int_0^t a\left(S(\tau), \tau\right) + \int_0^t b\left(S(\tau), \tau\right)dB_\tau. \tag{1.2}$$

The first integral in (1.2) is known as the Riemann-Stiltjes, and the second is a stochastic integral driven by the Brownian motion $B(t)$. Researchers have used different numerical methods to investigate the approximate solutions of related models [7-16].

Here, two approximate methods: Daftardar-Gejji Jafari method (DJM) and Picards Iterative method (PIM) are employed for the approximate solutions using the Normalised induced Brownian motion transform.

2. Methods of Solution

Here we consider the notions of DJM, PIM, and the normalized Induced Brownian Motion.

2.1 DJM Structure

To analyse the DJM, we consider the general functional equation [17, 18]:

$$h = g + L(h) + N[h], \tag{2.1}$$

where g is a given or a known function, $N[\cdot]$ and $L[\cdot]$ are the nonlinear and linear operators, respectively. Suppose we define $M[h]$ as:

$$M[h] = L[h] + N[h], \tag{2.2}$$

so (2.1) becomes:

$$h = g + M[h]. \tag{2.3}$$

By considering the solution, h of (2.2) with the series form:

$$\begin{cases} h = \sum_{i=0}^{\infty} h_i, \\ M[h] = \bar{N}\left[\sum_{i=0}^{\infty} h_i\right], \end{cases} \tag{2.4}$$

the nonlinear operator M can easily be decomposed as

$$M\left(\sum_{i=0}^{\infty} h_i\right) = M[h_0] + \sum_{i=1}^{\infty}\left[M\left(\sum_{i=0}^{n} h_i\right) - M\left(\sum_{i=0}^{n-1} h_i\right)\right], n = 1,\ 2,\dots \tag{2.5}$$

Therefore, putting (2.4) and (2.5) into (2.3), we obtain

$$\sum_{i=0}^{\infty} h_i = g + M[s_0] + \sum_{i=1}^{\infty}\left[M\left(\sum_{i=0}^{n} h_i\right) - M\left(\sum_{i=0}^{n-1} h_i\right)\right]\ , n = 1,\ 2,\dots\ , \tag{2.6}$$

So, the recurrence relation is:

$$\begin{cases} h_0 = h \\ s_1 = M(h_0) \\ h_{n+1} = M\left[\sum_{i=0}^{n} h_i\right] - M\left[\sum_{i=0}^{n-1} h_i\right],\ n = 1,\ 2,\dots \end{cases} \tag{2.7}$$

such that:

$$\left.\begin{aligned} h &= g + \sum_{i=1}^{\infty} h_i \\ &= \sum_{i=0}^{\infty} h_i. \end{aligned}\right\} \tag{2.8}$$

The series converges absolutely and uniformly to the solution of (2.1).

2.2 Picard Iterative Method [19, 20]

Consider the differential equation of this form:

$$\begin{cases} s' = f(t, s), \\ s(0) = s_0. \end{cases} \tag{2.9}$$

First order differential equations fall under this type of equation, and PIM is one of the suitable method for handling this type of differential equation. Now, by integrating both sides of equation (2.8), we get

$$\int_0^t s'(\tau)d\tau = \int_0^t f\left(\tau, s(\tau)\right)d\tau, \tag{2.10}$$

therefore, following the basic concept of calculus (2.9) becomes:

$$s(t) - s(0) = \int_0^t f\left(\tau, s(\tau)\right)d\tau,$$

$$s(t) = s(0) + \int_0^t f\left(\tau, s(\tau)\right)d\tau. \tag{2.11}$$

Since $s(t)$ is appearing on both sides of equation (2.10) for arbitrary t, we therefore adopt this iterative process by choosing an initial condition

$$s(0) = s_0,\ \ for\ n \geq 1,\ n \in \mathbb{Z}^+:$$

$$s_{n+1} = s_0 + \int_0^t f\left(\tau, s_n(\tau)\right)d\tau. \tag{2.12}$$

therefore, the approximation of (2.9) yields:

$$s(t) = \lim s_{n+1}(t),\ as\ n \to \infty.$$

We refer the readers to see references [21-24], for other dimensions of applications, numerical or exact solution methods for functional differential equations, including SDEs.

2.3 Normalized Brownian Motion [25]

We will take $Z_0,\ Z_1,...$ to be mutually and independent random variables, such that the distribution are Gaussian and identically independent with $N(0,1)$. The random process,

$$B(t) = \frac{Z_0}{\sqrt{2\pi}}t + \sum_{k=1}^{\infty}\sqrt{\frac{2}{\pi}}\frac{Z_k}{k}\sin(kt),\ for\ t \in [0,\pi], \tag{2.13}$$

is therefore referred to as normalized Brownian Motion on the interval $[0,\pi]$.

Then by differentiating (2.12) , it gives:

$$dB(t) = \frac{Z_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty}\sqrt{\frac{2}{\pi}}Z_k\cos(kt). \tag{2.14}$$

In finite form, (2.13) is expressed as:

$$dB(t) = \frac{Z_0}{\sqrt{2\pi}} + \sum_{k=1}^{5}\sqrt{\frac{2}{\pi}}Z_k\cos(kt), \tag{2.15}$$

then replacing the normalized Brownian Motion in (1.1) with (2.14), gives::

$$\begin{cases} dS(t) = a(S,t)dt + b(S,t)\dfrac{Z_0}{\sqrt{2\pi}} + \displaystyle\sum_{k=1}^{5}\sqrt{\frac{2}{\pi}}Z_k\cos(kt), \\ \mathrm{S}(0)=\mathrm{S}_0, \end{cases} \tag{2.16}$$

where $L = 5$ and Z_k are randomly generated random variables.

3. Numerical Example

Consider the Stochastic Differential Equation:

$$\begin{cases} dS_t = -aS_t dt + bdW_t, \\ S(0) = 1, \end{cases} \tag{3.1}$$

where $a, b > 0$, then choose $a = 1, b = 1$.
The linear SDE (3.1) can be expressed in the integral form as:

$$S_t = S_0 - \int_0^t S_u du + \int_0^t dWu \tag{3.2}$$

here, the second integral is the Ito integral. This type of system is called the Ornstein-Uhlenbeck (O-U) process [26-29].

Solution (DJM):
In integral form, (4.1) becomes:

$$S_t = 1 - a\int_0^t S_u du + b\int_0^t dW_u \ . \tag{3.3}$$

Therefore, with the following definitions,

$$M(S_t) = -\int_0^t S_u du + \int_0^t dW_u \ , \tag{3.4}$$

$$dW_u = \frac{Z_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} \sqrt{\frac{2}{\pi}} Z_k \cos(kt) \ , \tag{3.5}$$

and

$$M(S_t) = -\int_0^t S_u du + \int_0^t \left(\frac{Z_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} \sqrt{\frac{2}{\pi}} Z_k \cos(kt) \right), \tag{3.6}$$

we obtain the following iteration via DJM :

$$S_0 = 1,$$

$$S_1 = M(S_0)$$

$$= -\int_0^t S_0 du + \int_0^t (dW_u) du$$

$$= -\int_0^t (S_0) du + \int_0^t \left(\left[\frac{Z_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} \sqrt{\frac{2}{\pi}} Z_k \cos(kt) \right] \right) du,$$

$$S_2 = M(S_0 + S_1) - M(S_0)$$

$$= -\left\{ \int_0^t (S_0 + S_1) du - \int_0^t (dW_u) du \right\} - \left\{ \int_0^t (S_0) du + \int_0^t (dW_u) du \right\}$$

$$= -\left\{ \int_0^t ((S_0 + S_1)) du - \int_0^t \left(\left(\frac{Z_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} \sqrt{\frac{2}{\pi}} Z_k \cos(kt) \right) \right) du \right\} - \{S_1\},$$

Journal of Physics: Conference Series **1734** (2021) 012015 doi:10.1088/1742-6596/1734/1/012015

$$S_3 = M\left(S_0+S_1+S_2\right)-M\left(S_0+S_1\right)$$
$$=-\left\{\int_0^t\left(S_0+S_1+S_2\right)du-\int_0^t\left(dW_u\right)du\right\}+\left\{\int_0^t\left(S_0+S_1\right)du-\int_0^t\left(dW_u\right)du\right\}$$
$$=-\left\{\int_0^t\left(S_0+S_1+S_2\right)du-\int_0^t\left(\left(\frac{Z_0}{\sqrt{2\pi}}+\sum_{k=1}^{\infty}\sqrt{\frac{2}{\pi}}Z_k\cos\left(kt\right)\right)\right)du\right\}$$
$$+\left\{\int_0^t\left(S_0+Z_1\right)du-\int_0^t\left(\left(\frac{Z_0}{\sqrt{2\pi}}+\sum_{k=1}^{\infty}\sqrt{\frac{2}{\pi}}Z_k\cos\left(kt\right)\right)\right)du\right\},$$
$$\vdots$$
$$S_6=\bar{N}\left(S_0+S_1+S_2+S_3+S_4+S_5\right)-\bar{N}\left(S_0+S_1+S_2+S_3+S_4\right)$$
$$=-\left\{\int_0^t\left(S_0+S_1+S_2+S_3+S_4+S_5\right)du-\int_0^t\left(dW_u\right)du\right\}$$
$$+\left\{\int_0^t\left(S_0+S_1+S_2+S_3+S_4\right)du-\int_0^t\left(dW_u\right)du\right\}$$
$$=-\left\{\int_0^t\left(S_0+S_1+S_2+S_3+S_4+S_5\right)du-\int_0^t\left(\left(\frac{Z_0}{\sqrt{2\pi}}+\sum_{k=1}^{\infty}\sqrt{\frac{2}{\pi}}Z_k\cos\left(kt\right)\right)\right)du\right\}$$
$$+\left\{\int_0^t\left(S_0+S_1+S_2+S_3+S_4\right)du-\int_0^t\left(\frac{Z_0}{\sqrt{2\pi}}+\sum_{k=1}^{\infty}\sqrt{\frac{2}{\pi}}Z_k\cos\left(kt\right)\right)du\right\},$$
$$S=\sum_{i=0}^{6}S_i.$$

Solution (PIM):

Next, we re-express (3.1) in an integral form as in (2.11):

$$S_{n+1}=S_0+\int_0^t f\left(u,S_n\left(u\right)\right)du \tag{3.7}$$
$$S_0=1$$
$$S_1=1-\int_0^t S_0du+\int_0^t\left(dW_u\right)du,$$
$$=1-\int_0^t S_0du+\int_0^t\left(\frac{Z_0}{\sqrt{2\pi}}+\sum_{k=1}^{\infty}\sqrt{\frac{2}{\pi}}Z_k\cos\left(kt\right)\right)du.$$
$$S_2=1-\int_0^t S_1du+\int_0^t\left(dW_u\right)du,$$
$$=1-\int_0^t S_1du+\int_0^t\left(\frac{Z_0}{\sqrt{2\pi}}+\sum_{k=1}^{\infty}\sqrt{\frac{2}{\pi}}Z_k\cos\left(kt\right)\right)du.$$
$$S_3=1-\int_0^t\left(S_2\right)du+\int_0^t\left(dW_u\right)du,$$
$$=1-\int_0^t\left(S_2\right)du+\int_0^t\left(\frac{Z_0}{\sqrt{2\pi}}+\sum_{k=1}^{\infty}\sqrt{\frac{2}{\pi}}Z_k\cos\left(kt\right)\right)du.$$

$$S_4 = 1 - \frac{1}{2}\int_0^t (S_3)\,du + \int_0^t (dW_u)du,$$

$$= 1 - \int_0^t (S_3)du + \int_0^t \left(\frac{Z_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} \sqrt{\frac{2}{\pi}} Z_k \cos(kt) \right) du.$$

$$S_5 = 1 - \int_0^t (S_4)\,du + \int_0^t (dW_u)du,$$

$$= 1 - \int_0^t (S_4)du + \int_0^t \left(\frac{Z_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} \sqrt{\frac{2}{\pi}} Z_k \cos(kt) \right) du.$$

$$S_6 = 1 - \int_0^t (S_5)\,du + \int_0^t (dW_u)du,$$

$$= 1 - \int_0^t (S_5)du + \int_0^t \left(\frac{Z_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} \sqrt{\frac{2}{\pi}} Z_k \cos(kt) \right) du.$$

$$\vdots$$

$$S_{n+1} = S_0 + \int_0^t ((S_n)\,dW_u)du.$$

Table 1: Error analysis for the O-U Process

t	$Z_6^{DJM}(t)$	$Z_6^{PIM}(t)$	$\lvert Z_{DJM} - Z_{PIM} \rvert$
0.0	1.000000000000000	1.000000000000000	0.000E+00
0.1	1.094638480760513	1.094638563724062	8.296E-08
0.2	1.179085738282314	1.179088391665922	2.653E-06
0.3	1.250846853088319	1.250866976984742	2.012E-05
0.4	1.303968602180530	1.304053197396319	8.460E-05
0.5	1.329414276295206	1.329671348303800	2.571E-04
0.6	1.317032687467557	1.317668074502840	6.354E-04
0.7	1.258651282423057	1.260011018982764	1.360E-03
0.8	1.151443473767513	1.154058021873473	2.612E-03
0.9	1.000562230460032	1.005187751214804	4.626e-03
1.0	8.201461236883663	8.277972780380032	7.651E-03

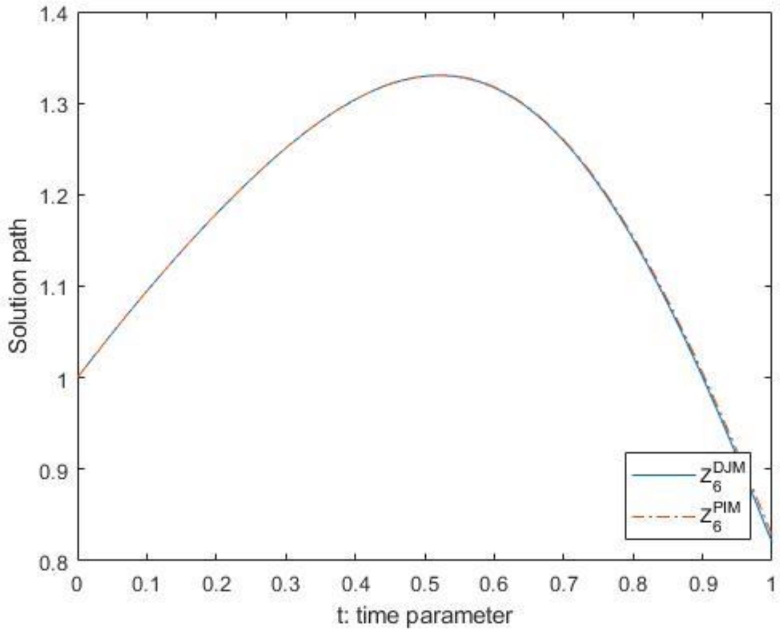

Fig 1: Graph of the considered problem

4. Conclusion

The induced normalized Brownian motion approach for approximate solution of the Ornstein-Uhlenbeck process has been successfully considered in the present work. The solutions were obtained easily by the proposed methods: DJM and PIM, even with less computing time. Therefore, it is noted for effectiveness and consequently suggested for other well-known stochastic models.

Acknowledgment

The authors sincerely appreciate CUCRID for sponsoring this research.

Conflict of Interests

The authors declare that there is no conflict of interest.

References

[1] Bayram M., Automatic analysis of the control of metabolic networks, Computers in Biology and Medicine, 26 (5), (1996): 401-408.
[2] Guzel, N. and Bayram, M., Numerical solution of differential-algebraic equations with index2. Applied Mathematics and Computation, 174 (2), (2006): 1279-1289.
[3] Nouri, K. Study on efficiency of Adomian decomposition method for stochastic differential equations, International Journal of Nonlinear Analysis and Applications, 8 (2017): 61-68.
[4] Ogundile, O.P. and Edeki, S. O., Approximate Analytical Solution of Linear Stochastic Differential Equations on the basis of Karhunen-Loéve Expansion Finite Series, Communication in Mathematical Biology and Neurosciences, 2020 (2020).
[5] Bibbona, E.; Panfilo, G.; Tavella, P., The Ornstein-Uhlenbeck process as a model of a low pass filtered white noise. Metrologia. 45(6) (2008):S117–S126. Bibcode:2008Metro..45S.117B. doi:10.1088/0026-1394/45/6/S17.

[6] Tim L., Xin, L., Optimal Mean Reversion Trading with Transaction Costs and Stop-Loss Exit, International Journal of Theoretical & Applied Finance. 18 (3) (2015): 1550020. arXiv:1411.5062. doi:10.1142/S021902491550020X.

[7] Sauer T., Numerical solution of stochastic differential equations in finance. In Duan J-C, Hardle W, Gentle J, eds. Handbook of Computational Finance. Springer, Berlin-Heidelberg; (2012): 529–550.

[8] Bayram, M., Partal, T. and Orucova Buyukoz, G., Numerical methods for simulation of stochastic differential equations, Advances in Difference Equations, 17(2018). https://doi.org/10.1186/s13662-018-1466-5M

[9] Asgari, E., Hasheminejad, M. Khodabin and Maleknejad, K., Numerical solution of nonlinear stochastic integral equation by stochastic operational matrix based on Bernstein polynomials, Bulletin Mathématique de laSociété des SciencesMathématiques de Roumanie. 57 (2014): 3-12.

[10] Cortes, J. C., Jodar, L. and L. Villafuerte, Mean square numerical solution of random differential equations: facts and possibilities, Computers & Mathematics with Applications. 53 (2007): 1098-1106.

[11] Jankovic, S. and Ilic, D., One linear analytic approximation for stochastic integro-differential equations, Acta Mathematica Scientia 30 (2010): 1073-1085.

[12] Mohammadi, F., A wavelet-based computational method for solving stochastic Ito-Volterra equations, Journal of Computational Mathematics 298 (2015): 254-265.

[13] Mohammadi, F., Second kind Chebyshev wavelet Galerkin method for stochastic Itô-Volterra integral equations, Mediterranean Journal of Mathematics 13 (2016):2613-2631.

[14] Murge, M. G. and Pachpatte, B. G., Successive approximations for solutions of second order stochastic integro-differential equations of Ito type, Indian Journal of Pure and Applied Mathematics. 21 (1990): 260-274.

[15] Heydari, M. H., Hooshmandasl, M. R., Maalek, F. M. and Cattani, C., A computational method for solving stochastic Ito-Volterra integral equations based on stochastic operational matrix for generalized hat basis functions, Journal of Computational Physics. 270 (2014): 402-415.

[16] Nouri, K., Study on efficiency of Adomian decomposition method for stochastic differential equations, International Journal of Nonlinear Analysis and Applications, 8 (2017): 61-68.

[17] Daftardar-Gejji, V. and Jafari, H., An iterative method for solving nonlinear functional equations, Journal of Mathematical Analysis 316 (2006): 753–763.

[18] Bhalekar, S. and Daftardar-Gejji, V., Convergence of the new iterative method, International Journal. of Differential Equations, 2011 (2011): 1-10.

[19] Edeki, S.O., Opanuga, A. A and Okagbue, H .I, On the iterative techniques for numerical solutions of linear and nonlinear differential equations. Journal of Mathematical and Computer. Science, 4 (4), (2014): 716-727.

[20] Ogundile O.P. and Edeki S.O. 2020 Karhunen-Loéve expansion of Brownian motion for approximate solutions of linear stochastic differential models using Picard iteration, Journal of Mathematics and Computational Science. **10**, 1712-1723

[21] Akinlabi, G.O., Adeniyi, R.B. and Owoloko, E. A., The solution of boundary value problems with mixed boundary conditions via boundary value methods, International Journal of Circuits, Systems and Signal Processing, 12, (2018): 1-6.

[22] Akinlabi, G.O., and Adeniyi, R.B., Sixth-order and fourth-order hybrid boundary value methods for systems of boundary value problems, WSEAS Transactions on Mathematics. 17 (2018): 258-264.

[23] Biazar J Ebrahimi H and Ayati Z 2009 An approximation to the solution of telegraph equation by variational iteration method, Numer. Methods Partial Differential Eq. 25(2009) 797-801.

[24] Khodabin, M., Maleknejad K. and Hosseini Shekarabi, F., Application of triangular functions to numerical solution of stochastic Volterra integral equations, International Journal of Applied Mathematics. 43 (2011), 1-9.
[25] Bibi, A. and Merahi, F., Adomian decomposition method applied to linear stochastic differential equations, International Journal of Pure and Applied Mathematics, 118(3), (2018): 501-510.
[26] Charles, W. M. and Van der Weide, J. A. M., Stochastic Differential Equations, Introduction to Stochastic Models for Pollutants Dispersion, Epidemic and Finance. (2011):1-156.
[27] Heseltine, J. and Kim, E. J. (2019) Comparing information metrics for a coupled Ornstein-Uhlenbeck process, Entropy, 21, (8), Article number 775.
[28] Zheng, J. , Tong, C.Q and Zhang, G.-J. (2018) Modeling stochastic mortality with O-U type processes, 33(1), 48-58.
[29] Chumpong, K. Mekchay, K and Rujivan, S. (2020) A simple closed-form formula for the conditional moments of the Ornstein-Uhlenbeck process, Songklanakarin Journal of Science and Technology,42,(4),836-845.