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NEAR EXACT QUANTILE ESTIMATES OF THE BETA DISTRIBUTION

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Abstract

We propose a near exact equation that links the quantile function and its quartiles of beta distribution (BD). This is done with the use of quantile mechanics approach whereby the probability density function of beta distribution is transformed into second order nonlinear ordinary differential equations whose solutions give the required inverse cumulative distribution function of the distribution. The median can easily be obtained from the proposed equations. Further efforts are required to refine and simplify the results. The results from this paper will greatly affect the way, beta distribution is used in engineering, in general, and wireless communications, in particular. Furthermore, the results obtained are very close the machine values.

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1. Introduction

Beta distribution is one of the classes of continuous distributions termed as interval bounded probability distributions. Other interval bounded/distributions are as follows: Kumaraswamy, Kumaraswamy Kumaraswamy, arcsine, logit-normal, continuous uniform, Irwin-Hall, Bates, Kent, continuous logarithmic, Marchenko Pasteur, PERT, raised cosine, reciprocal, triangular, trapezoidal, *u*-quadratic, von Mises-Fisher and Wigner semicircle distributions. The distributions are so, because their support is oriented towards 0 and 1 [1] or bounded in a definite interval [2]. Beta distribution is characterized by two non-negative shape parameters which determine the nature of the shape of the distribution. The distribution does not possess rate, scale and location parameters. The near flexible nature of the probability density function (PDF) of the distribution makes it an easy candidate for compounded or convoluting with other distributions such as normal, Gumbel, log-logistics, Chen, Kumaraswamy, Laplace and Nakagami. The distribution is applied widely in Bayesian analysis as prior distribution and also used in modeling random behavior of proportions, ratios and percentages.

Engineering and wireless communications have benefited from modeling and simulation using the beta distribution. Some complex systems are now better understood with the help of the distribution. Some of the recent applications of the beta distribution in modeling different aspects in wireless data analysis are provided in Table 1.

Table 1. Beta distribution in wireless communication modeling and simulation

References	Contributions
[3, 4]	Stochastic modeling of spectrum usage in IEEE 802.11-based wireless local area networks.
[6]	Evaluation of the trustworthiness of nodes of wireless sensor networks.
[6]	Modeling of packet generation of some realistic internet of things applications.
[7]	Outage probability estimation in finite wireless networks.
[8]	Optimization of target node operations (location, rotation, intersection and translation) in wireless sensor networks.
[9]	A trust-based management scheme based on beta distribution was proposed for defending wireless sensor networks against reputation time varying attacks.
[10]	Calculation of node trust value of intrusion detection in wireless sensor networks.
[11]	Simulation and prediction of malicious attacks in wireless sensor networks.
[12]	Calculation of the throughput and different probabilities of success for machine to machine devices used in wireless communications.

Most at times, researchers hugely depend on the machine values of statistical software for the probability functions of the distribution. Prominent among the probability functions is the quantile function (QF) or the inverse cumulative distribution function (CDF). Exact values of the quantile function of the distribution are not available and the implication is that approximate values are used for simulation and modeling. The absence of the closed form of the distribution is traced to the intractable nature of the CDF and consequently, the inversion method cannot be applied in transforming the CDF into the QF of the distribution. Beta is not only the distribution that falls into this category, others are but not limited to Nakagami, normal, gamma, Erlang, chi-squared, student's, *F* distributions.

Approximations are the only viable options when the closed forms are not available. The approximations can take the form of the use of series, numerical algorithm, functional or proxy methods. For beta distribution, readers are referred to the works of [13] and [14]. This paper applied the quantile mechanics approach to the PDF of the beta distribution and the end result is the required approximation (near exact forms) of the QF.

The next Section 2 provides the characteristics of the beta distribution, while Section 3 formulates the model based on the adopted methodology. Section 4 gives the result, Section 5 gives the median analysis and finally, Section 6 concludes the paper.

2. Beta Distribution

This section contains the basic properties of beta distribution. The PDF of the distribution is given as:

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a, b)}. \tag{1}$$

For $a, b > 0$ and $x \in (0, 1)$, $B(a, b)$ is a constant of normalization to force the integral of the PDF become one which is given as:

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{2}$$

The CDF is given as:

$$F(x) = I_x(a, b). \tag{3}$$

The CDF is equal to the regularized incomplete beta function of the parameters a and b, which is given as:

$$I_x(a, b) = \frac{B(x, a, b)}{B(a, b)}, \tag{4}$$

where $B(x, a, b)$ is the incomplete beta function of parameters a and b around the support x that defined the distribution.

The expected value or the mean is given as:

$$E[X] = \frac{a}{a+b}. \tag{5}$$

The geometric mean is given as:

$$G[X] = E[\ln X] = \psi(a) - \psi(a+b), \tag{6}$$

where ψ is a digamma function. The harmonic mean is given as:

$$H[X] = \frac{a-1}{a+b-1}. \tag{7}$$

This is valid for $a > 1$ and $b > 0$.

The median is given as:

$$M = I_{\frac{1}{2}}^{-1}(a, b). \tag{8}$$

The median can be approximated using different equations, one of which is given as:

$$M \approx \frac{a - \frac{1}{3}}{a + b - \frac{2}{3}} = \frac{3a-1}{3(a+b)-2}. \tag{9}$$

This is valid for $a, b > 1$.

The mode of the distribution is given as:

$$M_d = \frac{a-1}{a+b-2}. \tag{10}$$

This is valid for $a, b > 1$.

The variance of the distribution is given as:

$$Var[X] = \frac{ab}{(a+b+1)(a+b)^2}. \tag{11}$$

The variance of the geometric mean is given as:

$$VarG[X] = Var[\ln X] = \psi_T(a) - \psi_T(a+b), \tag{12}$$

where ψ_T is a digamma function.

The mean absolute deviation of the distribution is given as:

$$MAD = \frac{2a^a b^b}{B(a, b)(a+b)^{a+b+1}}. \tag{13}$$

The mean absolute difference of the distribution is given as:

$$MD = \frac{4(B(a+b, a+b))}{(a+b)(B(a, a)B(b, b))}. \tag{14}$$

The Gini coefficient of the distribution is given as:

$$Gc = \frac{2(B(a+b, a+b))}{(a)(B(a, a)B(b, b))}. \tag{15}$$

The skewness of the distribution is given as:

$$Sk = \frac{2(b-a)(a+b+1)^{\frac{1}{2}}}{(a+b+2)(a+b)^{\frac{1}{2}}}. \tag{16}$$

The excess kurtosis of the distribution is given as:

$$Ek = \frac{6(a-b)^2(a+b+1) - ab(a+b+2)}{ab(a+b+2)(a+b+3)}. \tag{17}$$

The moment (MGF) and characteristic (CF) generating functions of the distribution are given as:

$$MGF = 1 + \sum_{n=1}^{\infty}\left(\prod_{r=0}^{n-1}\frac{a+r}{a+b+r}\right)\frac{t^n}{n!}, \tag{18}$$

$$CF = 1 + \sum_{n=1}^{\infty}\left(\prod_{r=0}^{n-1}\frac{a+r}{a+b+r}\right)\frac{(it)^n}{n!}. \tag{19}$$

The higher moments that can be used to generate all the moments of the distribution are given as:

$$E[X^n] = \prod_{r=0}^{n-1}\frac{a+r}{a+b+r}. \tag{20}$$

The shape parameters can be estimated using the methods of moments or the maximum likelihood estimation.

3. Model Formulation

This work adopted quantile mechanics (QM) introduced by Steinbrecher and Shaw [15]. Generally, it has in existence that the reciprocal of the PDF of continuous distribution is the first order ordinary differential equations (ODEs) of the quantile function of the distribution known as the quantile differential equation [16]. Solution of the first order ODE is computationally expensive and complex. In other to reduce the complexity, Steinbrecher and Shaw [15] obtained the second order nonlinear ODE from the first order, which according to them, reduces the complexity involved in the solution of the respective ODEs. They demonstrated the method using the normal, gamma, beta and student distributions. The method and similar techniques have been applied in [16-22].

The adoption of this approach becomes unarguably a better option since the distributions with shape parameters are often difficult to handle, of which, beta distribution is one of them. Applying the QM approach, equation (1) becomes:

$$\frac{dQ}{dp} = B(a, b)Q^{1-a}(1-Q)^{1-b}. \tag{21}$$

Differentiating equation (21), we have

$$\frac{d^2Q}{dp^2} = B(a, b)[-(1-b)Q^{1-a}(1-Q)^{-b} + (1-a)Q^{-a}(1-Q)^{1-b}]\frac{dQ}{dp}. \tag{22}$$

By some algebraic manipulations, we have

$$\frac{d^2Q}{dp^2} = B(a, b)\left[(1-a)Q^{-a}\frac{Q^{1-a}}{Q^{1-a}}(1-Q)^{1-b} - (1-b)Q^{1-a}(1-Q)^{-b}\frac{(1-Q)^{1-b}}{(1-Q)^{1-b}}\right]\frac{dQ}{dp}. \tag{23}$$

We simplify to obtain the beta distribution quantile differential equation

$$\frac{d^2Q}{dp^2} = \left[\frac{1-a}{Q} - \frac{1-b}{I-Q}\right]\left(\frac{dQ}{dp}\right)^2, \tag{24}$$

with initial conditions

$$Q(0) = 0, \quad Q'(0) = 1.$$

4. Quantile Function of Beta Distribution

Solution to equation (24) is restricted to particular values of the shape parameters. The solution is very complex and can be functional or series in nature. This work treads on the path of functional. The result obtained from this work is unique because of finding the solution of equation (24) in functional form against widely used and readily available series solution.

Three cases are considered. Readers can simply replicate this to other values of the shape parameters *a* and *b*. In all the cases, the quantile function $Q(p) \in (0, 1)$.

Case I. This is where for the shape parameter $b = 1$, equation (24) becomes:

$$\frac{d^2Q}{dp^2} = \left(\frac{1-a}{Q}\right)\left(\frac{dQ}{dp}\right)^2. \tag{25}$$

Different cases are now considered under this case:

A. When $a = 2$ and $b = 1$.

When $a = 2$ and $b = 1$, equation (25) becomes:

$$\frac{d^2Q}{dp^2} = -\frac{1}{Q}\left(\frac{dQ}{dp}\right)^2. \tag{26}$$

The solution of equation (26) is given as:

$$Q(p) = \frac{\sqrt{2p}}{\sqrt{2}} = \sqrt{p}. \tag{27}$$

B. When $a = 3$ and $b = 1$.

When $a = 3$ and $b = 1$, equation (25) becomes:

$$\frac{d^2Q}{dp^2} = -\frac{2}{Q}\left(\frac{dQ}{dp}\right)^2. \tag{28}$$

The solution of equation (28) is given as:

$$Q(p) = 0.793700526(3p)^{\frac{1}{3}}. \tag{29}$$

C. When $a = 4$ and $b = 1$.

When $a = 4$ and $b = 1$, equation (25) becomes:

$$\frac{d^2Q}{dp^2} = -\frac{3}{Q}\left(\frac{dQ}{dp}\right)^2. \tag{30}$$

The solution of equation (30) is given as:

$$Q(p) = \frac{(4p)^{\frac{1}{4}}}{\sqrt{2}}. \tag{31}$$

Comparison made with R software values is presented in Tables 2, 3 and 4, respectively.

Table 2. The comparison of the exact and approximate quantile values for $a = 2$ and $b = 1$

p	Exact	Approximate	Error
0.1	0.316228	0.316228	0
0.2	0.447214	0.447214	0
0.3	0.547723	0.547723	0
0.4	0.632456	0.632456	0
0.5	0.707107	0.707107	0
0.6	0.774597	0.774597	0
0.7	0.83666	0.83666	0
0.8	0.894427	0.894427	0
0.9	0.948683	0.948683	0

Table 3. The comparison of the exact and approximate quantile values for $a = 3$ and $b = 1$

p	Exact	Approximate	Error
0.1	0.464159	0.464159	5.55E-17
0.2	0.584804	0.584804	0
0.3	0.669433	0.669433	0
0.4	0.736806	0.736806	0
0.5	0.793701	0.793701	0
0.6	0.843433	0.843433	0
0.7	0.887904	0.887904	0
0.8	0.928318	0.928318	0
0.9	0.965489	0.965489	0

Table 4. The comparison of the exact and approximate quantile values for $a = 4$ and $b = 1$

p	Exact	Approximate	Error
0.1	0.562341	0.562341	0
0.2	0.66874	0.66874	0
0.3	0.740083	0.740083	0
0.4	0.795271	0.795271	0
0.5	0.840896	0.840896	1.11E-16
0.6	0.880112	0.880112	0
0.7	0.914691	0.914691	0
0.8	0.945742	0.945742	0
0.9	0.974004	0.974004	0

Case II. This is where for the shape parameter $a = 1$, equation (24) becomes:

$$(1 - Q)\frac{d^2Q}{dp^2} = (b - 1)\left(\frac{dQ}{dp}\right)^2. \tag{32}$$

Different cases are now considered under this case.

It should be noted that the solution of ODEs under this case is obtained without the initial values because of the high nonlinearity of the ODEs; consequently, initial values are assumed.

A. When $a = 1$ and $b = 2$.

When $a = 1$ and $b = 2$, equation (32) becomes:

$$(1 - Q)\frac{d^2Q}{dp^2} = \left(\frac{dQ}{dp}\right)^2. \tag{33}$$

The solution of equation (33) of the ODE is given as:

$$Q(p) = 1 - \sqrt{2c_1 p + c_1 c_2 + 1}. \tag{34}$$

Assuming c_1 to be 1, equation (34) becomes:

$$Q(p) = 1 - \sqrt{2p + c_2 + 1}. \tag{35}$$

The values c_2 for each quartile are obtained from equation (35). The values of c_2 and comparison of the approximates with the exact values are presented in Table 5.

Table 5. The comparison of the exact and approximate quantile values for $a = 1$ and $b = 2$

p	Exact	c_2	Approximates	Error
0.1	0.051317	–0.3	0.051317	2.78E-17
0.2	0.105573	–0.6	0.105573	2.78E-17
0.3	0.16334	–0.9	0.16334	0
0.4	0.225403	–1.2	0.225403	8.33E-17
0.5	0.292893	–1.5	0.292893	5.55E-17
0.6	0.367544	–1.8	0.367544	1.11E-16
0.7	0.452277	–2.1	0.452277	3.33E-16
0.8	0.552786	–2.4	0.552786	2.22E-16
0.9	0.683772	–2.7	0.683772	1.11E-16

B. When $a = 1$ and $b = 3$.

When $a = 1$ and $b = 3$, equation (32) becomes:

$$(1 - Q)\frac{d^2Q}{dp^2} = 2\left(\frac{dQ}{dp}\right)^2. \tag{36}$$

The solution of equation (36) of the ODE is given as:

$$Q(p) = 1 + (3)^{\frac{1}{3}}(c_1 p + c_2 c_2)^{\frac{1}{3}}. \tag{37}$$

Assuming c_1 to be 1, equation (37) becomes:

$$Q(p) = 1 + (3)^{\frac{1}{3}}(p + c_2)^{\frac{1}{3}}. \tag{38}$$

The values c_2 for each quartile are obtained from equation (38). The values of c_2 and comparison of the approximates with the exact values are presented in Table 6.

Table 6. The comparison of the exact and approximate quantile values for $a = 1$ and $b = 3$

p	Exact	c_2	Approximates	Error
0.1	0.034511	–0.4	0.034511	0
0.2	0.071682	–0.46667	0.071682	0
0.3	0.112096	–0.53333	0.112096	0
0.4	0.156567	–0.6	0.156567	0
0.5	0.206299	–0.66667	0.206299	0
0.6	0.263194	0.73333	0.263194	0
0.7	0.330567	–0.8	0.330567	0
0.8	0.415196	–0.86667	0.415196	0
0.9	0.535841	–0.93333	0.535841	0

C. When $a = 1$ and $b = 4$.

When $a = 1$ and $b = 4$, equation (32) becomes:

$$(1 - Q)\frac{d^2Q}{dp^2} = 3\left(\frac{dQ}{dp}\right)^2. \tag{39}$$

The solution of equation (39) of the ODE is given as:

$$Q(p) = 1 - (2)^{\frac{1}{2}}(c_1)^{\frac{1}{4}}(c_2 + p)^{\frac{1}{4}}. \qquad (40)$$

Assuming c_1 to be 1, equation (40) becomes:

$$Q(p) = 1 - \sqrt{2}(c_2 + p)^{\frac{1}{4}}. \qquad (41)$$

The values c_2 for each quartile are obtained from equation (41). The values of c_2 and comparison of the approximates with the exact values are presented in Table 7.

Table 7. The comparison of the exact and approximate quantile values for $a = 1$ and $b = 4$

p	Exact	c_2	Approximates	Error
0.1	0.025996254	0.125	0.025996254	9.36751E-17
0.2	0.054258391	0	0.054258391	2.08167E-17
0.3	0.085308781	–0.125	0.085308781	1.249E-16
0.4	0.119888263	–0.25	0.119888263	1.52656E-16
0.5	0.159103585	–0.375	0.159103585	1.11022E-16
0.6	0.204729271	–0.5	0.204729271	1.11022E-16
0.7	0.259917196	–0.625	0.259917196	1.11022E-16
0.8	0.331259695	–0.75	0.331259695	2.22045E-16
0.9	0.437658675	–0.875	0.437658675	2.22045E-16

Case III. The analysis now goes for all other cases when any of the two shape parameters are greater than one. It can be observed that for the shape parameters greater than one, the resulting beta distribution quantile differential equation becomes so complex and computationally expensive. Originally, the distributions are related, as they are from the same CDF. The relationship now extends to their quantile function and approximation.

This paper now proposes an innovative way of obtaining the closed form for the quantile function of beta distribution based on the use of the closed form of the newly obtained ones in Cases I and II, respectively. This is the act of leveraging on the established relationship between the target and the already obtained QF. From this moment, the results obtained from Cases I and II are to be applied to obtain the QF of any given parameters of the beta distribution without the need for solving the associated ODE. It should be noted that the readers are at liberty to use any of the results in Cases I and II, which as shown later, are easy, less complex and computationally efficient. This is entirely a new result that has not been reported in literature to the best of the knowledge of the authors. Some examples are given.

Example 1. This is a case where $a = 2$ and $b = 2.$ The closed form of QF for $a = 2$ and $b = 1$ is adopted. The exact QF values of beta $(2, 1)$ (equation (27)) and the target, beta $(2, 2)$ are linked by dividing the later by the former to obtain the correcting factor called c. The correcting factor is multiplied with the closed form obtained for B $(2, 1)$ to get the desired result. The QF for the B $(2, 2)$ is given as:

$$Q(p) = c\sqrt{p}. \tag{42}$$

The correcting factor c and the comparison with the exact (R software) values are presented in Table 8.

Table 8. The correcting factor and comparison of the exact and approximate quantile values for $a = 2$ and $b = 2$

p	Exact	c	Approximates	Error
0.1	0.195800106	0.6191743	0.195800106	0
0.2	0.287140725	0.642066181	0.287140725	0
0.3	0.363257491	0.663214407	0.363257491	1.11022E-16
0.4	0.432931077	0.684524137	0.432931077	5.55112E-17
0.5	0.5	0.707106781	0.5	0
0.6	0.567068923	0.732082831	0.567068923	0
0.7	0.636742509	0.761052863	0.636742509	0
0.8	0.712859275	0.797000898	0.712859275	0
0.9	0.804199894	0.84770112	0.804199894	0

Example 2. This is a case where $a = 4$ and $b = 3$. The closed form of QF for $a = 4$ and $b = 1$ is adopted. The exact QF values of beta $(4, 1)$ (equation (31)) and the target, beta $(4, 3)$ are linked by dividing the later by the former to obtain the correcting factor called *c*. The correcting factor is multiplied with the closed form obtained for B $(4, 1)$ to get the desired result. The QF for the B $(4, 3)$ is given as:

$$Q(p) = c\frac{(4p)^{\frac{1}{4}}}{\sqrt{2}}. \tag{43}$$

The correcting factor *c* and the comparison with the exact (R software) values are presented in Table 9.

Table 9. The correcting factor and comparison of the exact and approximate quantile values for $a = 4$ and $b = 3$

p	Exact	*c*	Approximates	Error
0.1	0.333194387	0.592512717	0.333194387	0
0.2	0.414605765	0.619980225	0.414605765	0
0.3	0.476058199	0.643249912	0.476058199	5.55112E-17
0.4	0.529215781	0.665453614	0.529215781	0
0.5	0.578592809	0.688066686	0.578592809	0
0.6	0.626920268	0.71231895	0.626920268	0
0.7	0.676676115	0.73978639	0.676676115	1.11022E-16
0.8	0.731350846	0.773309368	0.731350846	0
0.9	0.799091121	0.82041894	0.799091121	1.11022E-16

5. Median Analysis

One of the key contributions of this research is to obtain the closed form expressions for the median of the beta distribution. The result obtained from this paper is compared with the one commonly used in literature and the exact values. The extent of the comparison contributes greatly to the advancement of knowledge in this area. This is because the median is needed in engineering analysis, in general, and in wireless communications, in particular. The quantile function is compared with the one shown in equation (9). The analysis is presented in Table 10. Error A is the absolute value of the difference between the exact and the approximates from this paper while Error B is the absolute value of the difference between the exact and the one from equation (9). It can be seen that the result from this paper is better than the one commonly used in the literature. The strength of equation (9) is that it is simple and only one formula is needed. Precision and reliability are the strength of the closed form of the median proposed in this paper.

Table 10. Median analysis and comparison with the exact

a	b	Exact	Approximates	Equation 9	Error A	Error B
2	1	0.707107	0.707107	0.714286	0	0.007179
3	1	0.793701	0.793701	0.8	0	0.006299
4	1	0.840896	0.840896	0.846154	1.11E-16	0.005258
1	2	0.292893	0.292893	0.285714	5.55E-17	0.007179
1	3	0.206299	0.206299	0.2	0	0.006299
1	4	0.159104	0.159104	0.153846	1.11E-16	0.005257
2	2	0.5	0.5	0.5	0	0
4	3	0.578593	0.578593	0.578947	0	0.000355
2	1	0.707107	0.707107	0.714286	0	0.007179

6. Conclusion

The research led to the following contributions:

- Closed form of the quantile function of the beta distribution has been obtained.
- A close look at the result showed that it performed very well at the extremes (tails) of the distribution.
- A methodology has been developed to obtain the quantile function of any given shape parameter of beta distribution.
- The result will improve greatly the extent of which the distribution is applied in wireless communications and engineering.
- The quantiles obtained in this paper can be transformed to obtain the closed form of the CDF of the distribution.
- The median(s) obtained from this paper is/are closer to the exact. A closed form for the median of the beta distribution has been obtained.
- The precision of the median implies that it can be used in simulation and modeling of the probability density function of the beta distribution used in wireless communications and other areas.

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