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On Non-Linear Non-Gaussian Autoregressive Model with Application to Daily Exchange Rate

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Abstract. The most often used distribution in statistical modeling follows Gaussian distribution. But many real-life time series data do not follow normal distribution and assumptions; therefore, inference from such a model could be misleading. Thus, a re-parameterized non-Gaussian Autoregressive (NGAR) model that has the capabilities of handling non-Gaussian time series was proposed, while Anderson Darling statistics was used to identify the distribution embedded in the time series. In order to determine the performance of the proposed model, the Nigerian monthly exchange rate (Dollar-Naira Selling Rate) was analyzed using proposed and classical autoregressive models. The proposed model was used to determine the joint distribution of the observed series by separating the marginal distribution from the serial dependence. The maximum Likelihood (MLE) estimation method was used to obtain an optimal solution in estimating the generalized gamma distribution of the proposed model. The selection criteria used in this study were Akaike Information Criterion (AIC). The result revealed through the value of the Anderson Darling statistics that the data set were not normally distributed. The best model was selected using the minimum values of AIC value. The study concluded that the proposed model clearly shows that the non-Gaussian Autoregressive model is a very good alternative for analyzing time series data that deviate from the assumptions of normality and, in particular, for the estimation of its parameters.

Keywords: Non-Gaussian, Autoregressive model, Generalized Gamma, Anderson Darling

1. Introduction

Modelling plays a vital role in various areas of research. It provides a way to learn from a given set of data. It is important that an appropriate model is applied in fitting a given set of data having observed so that promising results can be obtained, [1-3] mentioned how the choice of model applies to stochastic process.

According to [4], a time series consists of statistical observation made sequentially and usually in an equal space of time. One of the main goals of time series data analysis is to identify a model within a given class of flexible models which can express a time structured relationship of the process that generates the data. Therefore, a time series model for the observed data $\{x_t\}$ is a joint distribution of a sequence of random variables $\{X_t\}$ of realization $\{x_t\}$.

Time series data can be Gaussian and non-Gaussian (N-G) in nature. Non-Gaussian time series data were observed in the field of physical and social sciences, counts, or non-negative observations.

Diverse models have been proposed in modeling time series data that are non-Gaussian. As mentioned by [5], Gaussian distribution is a popular probability distribution in statistics; because the distribution has a well-defined probability density function, and analysis can easily be assessed. However, not all the data that are modeled are Gaussian distributed in nature [6].

While the Gaussian process has remarkable advantages in modeling, the assumption of a Gaussian distribution is questionable for truly non-Gaussian time series data [7]. This problem can be solved by transforming the non-Gaussian data with an elementary transformation and assuming that the transformed data were observations from an underlying Gaussian process [8]. However, this technique has its downsides because an elementary transformation will only lessen the skewness and/or kurtosis without altering the marginal distribution to be Gaussian-shaped [7]. An alternative to the transformation method is to specify the joint distribution of the non-Gaussian process directly; however, unlike Gaussian process models, the specification of a non-Gaussian process's joint distribution goes well beyond specifying a mean vector and covariance matrix (if they exist). Hence, this study seeks to find a method that preserves useful modeling techniques developed for traditional Gaussian process models and effectively captures the non-Gaussian distribution.

Various methods have been proposed and adopted in the literature for solving the problem of non-Gaussian time series data. This method includes the transformation method, transition method, and Generalized Linear Mixed Models (GLMMs) among others [9-11]. But these methods all have their various drawbacks. Therefore, this study seeks to use the reparameterization method to bridge the gap of non-flexibility of model and their estimation problem. This study aims to develop a re-parameterized Autoregressive model that appropriately analyzes time series data that follows non-normal distribution [2].

Bishop carried out a comparative study of Gaussian and non-Gaussian distributions in time series analysis, and the outcome of the study is that the Gaussian distribution can easily be applied to data because its likelihood is non-tractable and analysis based on it can be derived in an explicit form [5]. However, [6] opined that all data could not necessarily be modeled with Gaussian distribution because of the nature inherent in such datasets. The Gaussian distribution has an unbounded support, whereas some data are semi-bounded or bounded support. Studies by [10]; [12] mentioned that non-Gaussian statistical models would be appreciated to model data that is not normally distributed.

The literature's techniques for modeling non-Gaussian time series data fall into four main categories [13]. The frequently used method is the empirical transformations, including the log transformation, the square root transformation, and many more. The Box-Cox transformation function is commonly applied with the log transformation as a special case [14]. Rasmussen [15] provided a systematic review of the Box-Cox transformation and extended the Box-Cox transformation to handle the responses with incomplete observations using an MLE method. Lipsitz, Ibrahim, and Molenberghs [16] extended the Box-Cox transformation to the linear mixed model, discussing the impact of choice in the transformation parameter on the estimated coefficients and their standard errors. They pointed out that an adjustment is needed to correct the bias in the variance of the estimated coefficients induced by using the estimated transformation parameter. Gurka, Edwards, Muller, Kupper, and Lawrence [17] presented univariate and multivariate time series models for processes with non-Gaussian marginal distributions. These models include bivariate autoregressive models for processes with bivariate exponential marginal. Examples of applications to real data sets were given for some of the models discussed. When applicable, the theory of positive dependence is used to establish the association of the processes.

Block, Langberg, and Stoffer [18] studied the statistical properties of the daily North Atlantic Oscillation (NAO) index. It was discovered that previous NAO modeling efforts simply considered Gaussian noise, which can be termed inconsistent in relation to the complexity of the system. They went ahead to establish an autoregressive model with non-Gaussian noise, and from their results, the established model gave a better fit to the time series for the four seasons separately. Also, the usefulness of the suggested model was also appraised by means of an investigation of its forecast skill. The remaining part of this paper is sectionalized as follows; section 2 is materials and method, section 3 is results obtained, and section 4 is the summary and conclusion, while section 5 is the recommendations.

2. Material and Methods

2.1 Transformed or Generalized Gamma Distribution

The probability density function (pdf) provides the numerical distribution features of random variables and is a crucial foundation for understanding the non-Gaussian traits of the autoregressive model.

Let $G(x,\alpha\ ,\gamma,k)$ be the cdf of Generalized Gamma (GG) distribution given by

$$G(t,\alpha\ ,\gamma,k)=\frac{\gamma\left(k,\left(\frac{t}{\alpha}\right)\right)^{\gamma}}{\Gamma(\mathrm{k})} \tag{1}$$

where $\alpha>0,\gamma>0,k>0,\gamma(k,x)=\int_0^x \omega^{k-1}e^{-\omega}d\omega$ is the incomplete gamma function and $\Gamma(\cdot)$ is the gamma function.

Is the distribution below the pdf for the above CDF?

$$f(t)=\frac{\lambda\rho\gamma}{\alpha\Gamma(k)}\left(\frac{t}{\alpha}\right)^{\gamma k-1} exp\left[-\left(\frac{t}{\alpha}\right)^{\gamma}\right]\left\{\gamma,\left[k,\left(\frac{t}{\alpha}\right)\right]\right\}^{\lambda-1}\left(1-\left\{\gamma_1,\left[k,\left(\frac{t}{\alpha}\right)^{\gamma}\right]\right\}^{\lambda}\right)^{\rho-1} \tag{2}$$

Here, $\gamma_1(\cdot,\cdot)$ is the incomplete gamma ratio function denoted and defined by $\gamma_1(k,x)/\Gamma(k)$, that is the cdf of the standard gamma distribution with parameter k .

α is the scale parameter, and the other parameters $\gamma, k\ and\ \lambda$ are the shape parameters

If $\rho=1$, the GG distribution reduces to

$$f(t)=\frac{\lambda\gamma}{\alpha\Gamma(k)}\left(\frac{t}{\alpha}\right)^{\gamma k-1} exp\left[-\left(\frac{t}{\alpha}\right)^{\gamma}\right]\left\{\gamma,\left[k,\left(\frac{t}{\alpha}\right)\right]\right\}^{\lambda-1} \tag{3}$$

where α is the scale parameter, and the other parameters γ, k and λ are the shape parameters. The expected value and the variance are given as follows:

$$\left.\begin{aligned} E(X)&=\int_o^{\infty} tf(t)\,dt \\ &=\frac{k}{t^{\gamma-1}} \end{aligned}\right\} \tag{4}$$

$$Var(X)=\frac{\alpha^2\Gamma\left(k+\frac{1}{\gamma}\right)}{\Gamma(k)}-\left(\frac{\alpha\Gamma\left(k+\frac{1}{\gamma}\right)}{\Gamma(k)}\right)^2 . \tag{5}$$

The maximum likelihood (ML) estimates of the parameters are as follows:

$$\left.\begin{aligned}
\hat{\alpha} &= \frac{\sum\left(x_i-\bar{x}\right)^2 - n\left(\bar{x}-\hat{\beta}\right)^2 - n\bar{x}}{n\left(\bar{x}-\hat{\beta}\right)^2} \\
\hat{\beta} &= \frac{\sum\left(x_i-\bar{x}\right)^3 - \left[\sum\left(x_i-\bar{x}\right)^2\right]\left[\sum x_i + n\bar{x}\right] + 2n\bar{x}\left[n\bar{x}^3 - \bar{x}^2\right]}{\sum x_i^2\left[4n\bar{x}-3n-3\right] + n\bar{x}^2\left[8n\bar{x}+13n\right] + n^2\bar{x}} \\
\hat{\delta} &= \frac{\left(\hat{\beta}-\bar{x}\right)}{\hat{\alpha}+\bar{x}-\hat{\beta}}
\end{aligned}\right\} \tag{6}$$

2.2 Wold's Representation

An $AR(1)$ process, is given as

$$X_t = \delta + \phi X_{t-1} + U_t \tag{7}$$

where the inhomogeneous part defined by $\delta + U_t$ contains a constant term δ and a pure random process ut. Generalizing (7), this gives the $\left(AR(2)\right)$ and it is written as

$$X_t = \delta + \phi_1 X_{t-1} + \phi_2 X_{t-2} + u_t \tag{8}$$

where u_t denote a pure random process having a variance σ^2, and $\phi_2 \neq 0$.

$$X_t = \delta + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + \varepsilon_t \tag{9}$$

Equation (9) denote the $AR(p)$ process which is described with the stochastic difference equation, with $\alpha_p = 0$, where ε_t is again a pure random process. By means of the lag operator, this can be written as:

$$\left(1-\phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p\right) X_t = \delta + \varepsilon_t \tag{10}$$

$AR(p)$ process in (9) is stationary, then it gives

$$\lambda^p - \phi_1 \lambda^{p-1} - \phi_2 \lambda^{p-2} - \ldots - \phi_p \tag{11}$$

$$1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p = 0 \tag{12}$$

If the stability conditions are satisfied, $AR(p)$ becomes

$$X_t = \frac{\delta}{1-\phi_1-\phi_2-\ldots-\phi_p} + \sum_{j=o}^{\infty} \psi_1 \varepsilon_{t-j} \tag{13}$$

2.3 Generalized Gamma Autoregressive model (GGMA)

From (4), resolve $\frac{\Gamma\left(k+1/\gamma\right)}{\Gamma(k)}$ using Stirling's formula,

$$\frac{\Gamma\left(k+1/\gamma\right)}{\Gamma(k)} \approx k^{\frac{1}{\gamma}} \tag{14}$$

Therefore,

$$E(X) = ak^{\frac{1}{\gamma}} \tag{15}$$

Equating (11) and (14) gives,

$$\frac{\delta}{1-\phi_1-\phi_2-\cdots-\phi_p} = ak^{\frac{1}{\gamma}} \tag{16}$$

$$1-\phi_1-\phi_2-\cdots-\phi_p = \delta ak^{1/\gamma} \tag{17}$$

$$1-\delta ak^{1/\gamma} = \sum_{i=1}^{p}\phi_i \qquad i = 1,2,\cdots p \tag{18}$$

Recall equation (7) where,

$$X_t = \phi_1 X_{t-1} + \varepsilon_t$$

Substitute for equation (18)

$$X_t = \left(1-\delta ak^{1/\gamma}\right) X_{t-1} + \varepsilon_t \tag{19}$$

$$X_t = X_{t-1} - \delta ak^{1/\gamma} X_{t-1} + \varepsilon_t \tag{20}$$

2.4 Data Description

In order to apply the underlying models in solving the problem, data of daily exchange rate of Nigeria Naira was obtained from the central bank of Nigeria official website https://www.cbn.gov.ng/rates/exchratebycurrency.asp, between March 27, 1981, to November 30, 2017. The return was obtained and tabulated in Table 1, following the transformation in the equation:

$$r_t = \log\left(P_t\right) - \log\left(P_{t-1}\right) \tag{21}$$

where, P_t is the exchange rate (Dollar-Naira) during the time t, and P_{t-1} is exchange rate during the time $t-1$.

The descriptive analysis is used to summarize the characteristics of the variables considered in this research with the aim of showing the important features of each of the variables through the use of a time plot.

3. Results

Table 1: Descriptive statistics of logarithmic returns

Currency	**Mean**	**Median**	**Minimum**	**Maximum**	**St Dev**	**Skewness**	**Kurtosis**
USD	-0.0007	-0.00022	0.03723	0.02668	0.00722	0.88692	7.52121

Table 1 shows the descriptive statistics of the logarithmic returns of the USD exchange price against NGN. The result showed that the mean of the variable is negative at -0.0007. In the case of skewness, the variable is positively skewed with a value of 0.88692. It has a kurtosis value of 7.52121 which means that the kurtosis is leptokurtic in nature; its value is greater than 3. The results showed that the USD deviates from the Gaussian distribution.

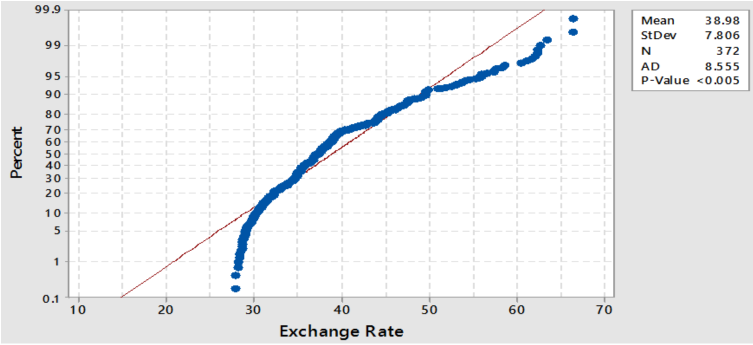

Figure 1: Probability Plot of Exchange Rate

The time plot of the USD-NGN Exchange Rate (1981-2017) is presented in the Figures below.

The Anderson - Darling test shows that the exchange rate of NGN-USD for the period observed was not normally distributed at ($p<0.05$). This may be due to the presence of extreme values inherent in the data. This is further shown in Figures 1 and Figure 2, respectively.

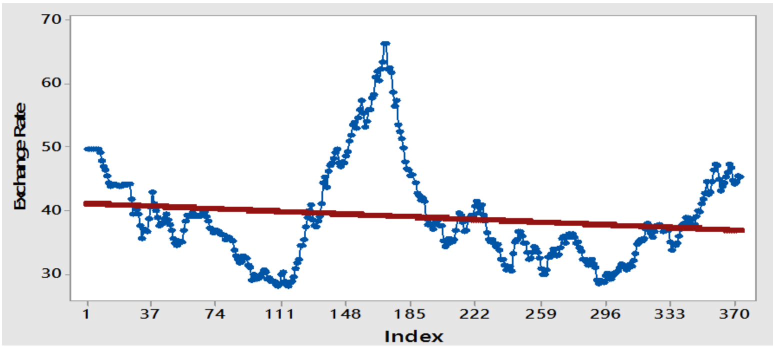

Figure 2: Time Plot of Exchange Rate

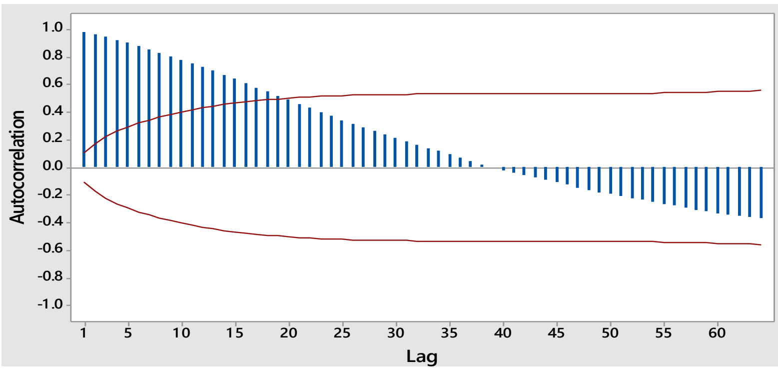

Figure 3: Correlogram Plot of Exchange Rate Rate

From Figure 2, the autocorrelation coefficient, partial autocorrelation coefficient, the result shows positive at lag 1 (AC = 0.1, $p<0.005$), and it concluded that USD returns may be considered independent in time. See Figure 3 for the Correlation plot of the exchange rate.

3.1 Generalized Gamma Autoregressive model (GGAR)

Table 2: Anderson-Darling Statistics for Exchange Rate

Real data	**Gamma**	**Poisson**	**Chi-Square**	**Geometric**	**Exponential**	**Normal**
Exchange Rate	0.793	0.498	0.743	0.543	**0.478**	0.877

From Table 2, the Anderson Darling statistics for the data are presented as follows. For the exchange rate, the Anderson Darling statistics were obtained for the following non-Gaussian distributions; Gamma, Chi-square, Geometric, Exponential, and Anderson Darling statistics for Normal distribution. The AD statistics obtained are 0.973, 0.498, 0.543, 0.478 and 0.877 for Gamma, Chi-square, Geometric, Exponential, and normal distributions respectively. It is discovered that the exchange rate data follows an Exponential distribution; this is so because it has the least Anderson Darling statistics.

Table 3: Parameters of Fitted Distributions

	Distribution	**Parameters**	**Estimate**
Exchange Rate	Exponential	Shape	10.41
		Rate	-

Table 3 summarizes the parameters of the fitted distributions. It can be deduced that the distribution fitted is an exponential distribution with an estimated parameter for the exchange rate.

Thus, from a pool of models, the model with the least AIC obtained is Exponential $AR(1)$; this invariably means that $EAR(1)$ is the best model for the dataset. This model can be written as:

Table 4: EAR Model Estimation for Exchange Rate

	Coefficient	*Std. Error*	*Z*	*p-value*	
Const	0.926162	0.0695078	13.3246	<0.0001	***
phi_1	0.089453	0.0316722	0.9676	0.3332	

Mean dependent (MD) var	0.960552		S.D. dependent var	0.985980
Mean of innovations	0.000000		S.D. of innovations	0.986501
Log-likelihood	−1402.440		Akaike criterion	0.407663
Schwarz criterion	0.561784		Hannan-Quinn	0.504762

In the modelling of the exchange rate data (see Table 4), $EAR(1)$ is the selected model with the corresponding AIC value of 0.407663. the model is given as

$$X_t = 0.0894534 X_{t-1} + \varepsilon_t$$

From the proposed model, $X_t = X_{t-1} - \ldots - \sigma\alpha k^{1/\gamma} X_{t-1} + \varepsilon_t$

Recall that $1-\varphi_1 = \sigma\alpha k^{1/\gamma} = 1-0.0894534 = 0.9105466$

thus becoming, $X_t = X_{t-1} - 0.9105466 X_{t-1} + \varepsilon_t$

This indicates that the slope coefficient is considerably different from zero (0), indicating that the lag one (1) variable is a useful predictor

.

4. Summary and Conclusion

This research centers on modelling non-Gaussian time-correlated data. One major contribution of this research is to develop a class of model that is capable of capturing a large group of non-Gaussian time series or longitudinal data rather than restricting the method to a specific type of non-Gaussian distributed data. Predominantly, the study allows the non-Gaussian time series or longitudinal data to be arbitrarily shaped. For instance, skewed, heavy-tailed, or multi-modal): the very loose assumptions on the model permit it to outperform its Gaussian competitors in model fitting and prediction when analyzing truly non-Gaussian time-correlated data.

The NGAR model proposed is a non-linear time series model. This is particularly interesting as many real-life phenomena are normally non-linear in either parameter or even non-stationary. The results clearly showed that it is a good alternative for fitting time-series data, and estimation of its parameters shows promising results. The results obtained in this study partly agree with the work of [19], who developed a hierarchical Gaussian process model for inference and forecasting of functional time series data. In a similar vein, the work of [1], [2], and [18] agreed with the results of this research that non-Gaussian autoregressive models and their parametric estimation methods are better and most efficient techniques to discuss time-series data that deviate from the assumptions of normality.

5. Recommendations

Due to the ever-changing situation in our immediate environments, this has caused a lot of changes in the distributional pattern of time series data over the years, making it often non-Gaussian in nature. Therefore, analyzing this requires the construction of efficient and better statistical analysis methods, which will give room for proper and correct statistical inference that leads to a valid decision and conclusion. Based on the results obtained from this study, the following are recommended:

i. That Empirical Distribution Function (EDF) should be applied before the parameters of autoregressive models are estimated.
ii. Generalized Gamma distribution based on exponential distribution should be employed in fitting non-Gaussian time series data as it shows promising results relative to other models.
iii. Gaussian and non-Gaussian autoregressive models should be applied to other phenomena like elections, electricity consumption, to mention but a few.

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References

[1] Ma Z. (2014). BG-NMF: A Variational Bayesian NMF model for bounded support data. *IEEE Transactions on Pattern Analysis and Machine Intelligence*. Submitted.
[2] Adedotun A. F., Taiwo A. I and Olatayo T. O. (2020). A Re-parametrised Autoregressive Model for Modelling Gross Domestic Product. *ANNALS of Faculty Engineering Hunedoara-International Journal of Engineering*, xvii **2**, 184-188
[3] Adesanya K. K., Taiwo A. I, Adedotun A. F. and Olatayo T. O. (2018). Modeling Continuous Non-Linear Data with Lagged Fractional Polynomial Regression. *Asian Journal of Applied Sciences*, **5**, 315-320.
[4] Wei W. W. S. (2006). Time Series Analysis Univariate and Multivariate Methods, *Second Edition. Addison Welsley: Pearson Press*.
[5] Bishop C. M. (2006). Pattern Recognition and Machine Learning. *Journal of Information Science and Statistics. Springer*.
[6] Banfield J. D. and Raftery A. E. (2004). Model-based Gaussian and Non-Gaussian clustering, *Biometrics*, **3**, 803–821.
[7] Xu, Z. (2014). Modeling Non-Gaussian Time-correlated Data Using Nonparametric Bayesian Method. *Ph.D Thesis Ohio State University*.
[8] Sun J., Frees E. W. and Rosenberg M. A. (2008). Heavy-tailed longitudinal data modeling using copulas. Insurance: *Mathematics and Economics*, **2**, 817–830.
[9] Bishop S.A., Okagbue H.I., Odukoya J. A. (2020). Statistical analysis of childhood and early adolescent externalizing behaviors in a middle low-income country. *Heliyon*, **2**, 03377.

[10] Blei D. M. (2004). Probabilistic Models of Text and Images, Ph.D. dissertation, *University of California, Berkeley*, 2004.

[11] Z. Ma, A. Leijon, and W. B. Kleijn, "Vector quantization of LSF parameters with a mixture of Dirichlet distributions," *IEEE Trans. Audio, Speech, Language Process.*, **9**, 1777–1790, Sep. 2013

[12] Fitzmaurice G., Davidian M., Verbeke G., and Geert M. (2008). Longitudinal Data Analysis. Chapter and Hall/CRC.

[13] Oguntunde P.E, Adejumo O.A, Odetunmibi O.A, Okagbue H.I, and Adejumo A.O: (2018): "Data analysis on physical and mechanical properties of cassava pellets", Data in Brief, **3**, 286-302.

[14] Stoica P. and Moses R. (1997). Introduction to spectral analysis. *Prentice Hall*.

[15] Rasmussen, C. E. (2006). Gaussian processes for machine learning. *MIT Press*.

[16] Lipsitz S. R., Ibrahim J., and Molenberghs G. (2000). Using a box-cox transformation in the analysis of longitudinal data with incomplete responses. *Journal of the Royal Statistical Society Series* C, **3**, 287–296.

[17] Gurka M. J., Edwards L. J., Muller K. E., Kupper and Lawrence L. (2006). Extending the Box-Cox transformation to the linear mixed model. *Journal of the Royal Statistical Society*: Series A (Statistics in Society), **2**, 273–288.

[18] Block H. W., Langberg N. A., and Stoffer D. S. (1980). Time Series Models for Non-Gaussian Processes, *AMS* 1980 subject classifications.

[19] Thomas N., Franzke C. L. E. and Vannacvi A. (2017). Predictability and Non-Gaussian Characteristics of the North Atlantic Oscillation. *American Meteorological Society*, **9,** 589-602.