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The Inverted Gompertz-Fréchet Distribution with Applications

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Abstract

The probability distribution is an important aspect of probability theory because of its vast relevance in almost all human disciplines. Its applicability in finance, medicine, agriculture, actuarial science, demography, and econometrics, to mention but a few, is highly commendable. The use of probability distributions to model real-life data is an age-old practice, but most of the standard distributions are not flexible enough to model emergent real-life occurrences. This shortcoming gave birth to diverse extensions of the standard distribution. In this work, the Inverted Gompertz-Fréchet (IGoFre) distribution is developed by transforming the independent variable of Gompertz-Fréchet, and one great feature of this model is its capacity to model both positively and negatively skewed datasets with increasing and decreasing hazard rates. It is a distribution whose random variable follows the reciprocal of the Gompertz-Fréchet distribution. The proposed probability distribution is developed with the aim of modeling real data with non-monotonic failure rates. The proposed distribution does not involve the addition of extra parameters, thereby removing the difficulties encountered in deriving its properties. The statistical and mathematical properties of the new distribution, such as the survival function, hazard function, distributions of the minimum and maximum order statistics, moments, mean, median, variance, skewness, and kurtosis, and Renyi entropy are derived. The method of maximum likelihood was used to estimate the model's parameters. Tables of percentage points, which could be of immense benefit for the test of the hypothesis, were generated for different values of the parameters. The proposed distribution was applied to two real-life data sets. The results revealed that the IGoFre distribution performs better than the Gompertz-Weibull, Gompertz-Fréchet, Gompertz Bur XII, and Gompertz-Lomax distributions for the datasets. Also, it was discovered that the proposed model is suitable for modeling both positively skewed and negatively skewed datasets.

Keywords: IGoFre distribution, Gompertz-Fréchet distribution, maximum likelihood estimation, Quantile function, Percentage points, moments, linear representation.

I. Introduction

The Gompertz distribution is an exponentially increasing, continuous probability distribution that is mainly applied in human mortality, Biology, and Demography. Several authors have extended the Gompertz distribution with the sole aim of providing a better fit to real-life data (Thomas et al. [1])

because it is believed that extended distributions perform better than the parent distributions. Karamikabir et al. [2]. The Fréchet distribution, also referred to as the Inverse Weibull distribution, is a special case of the generalized extreme value distribution, which has been applied to modeling interfacial damage in microelectronic packages and the material properties of constituent particles in an aluminum alloy by Harlow [3]. Pelumi et al. [4] developed the Gompertz Fréchet distribution, which is a four-parameter compound distribution derived by using the Gompertz generalized family of distributions proposed by Alizadeh, M. et al. [5]. The Inverse Gompertz distribution which is the distribution of the reciprocal of the random variables which follow the Gompertz distribution was introduced by M.S. Eliwa et al [6]. Extending the work of Eliwa [6], Pelumi Oguntunde et al. [7] derived Gompertz inverse exponential distribution; and its inverse, christened as Zech distribution was derived by Sunday Adeyeye et al. [8]. Gompertz-Flexible Weibull distribution was introduced by Khaleel et al. [9]. The Weibull log-logistic exponential {distribution} was studied by Job and Ogunsanya [10] as an extension of the Flexible Weibull distribution introduced by Ahmad and Hussain [11]. Ogunsanya et al. [12] studied both the classical and Bayesian properties of the Weibull inverse Rayleigh distribution. Fréchet distribution is mainly applied to extreme events such as annual maximum rainfalls and river discharges in hydrology. Many generalizations of Fréchet distribution have been made in the literature. Afify et al. [13] defined and studied a new four-parameter lifetime model called the Weibull-Fréchet distribution. Mahmoud et al. [14] proposed and studied a new five-parameter distribution called the Kumaraswamy Exponentiated Fréchet distribution, in which twenty-seven other distributions are embedded. Afify et al. [15] proposed Kumaraswamy's Marshall-Olkin Fréchet distribution, which generalizes the Marshall-Olkin Fréchet distribution and about seventeen other models. Nadarajah and Gupta [16] introduced the beta Fréchet distribution generated from the logit of the beta random variable. Krishna et al. [17] discussed the applications of the Marshall-Olkin Fréchet distribution in stress strength reliability analysis, acceptance sampling, and time series modeling. Sindhu et al. [18] proposed a novel three-parameter distribution called Exponentiated Transformation of Gumbel Type II Distribution for modeling COVID-19 data. Also, Ogunsanya and Job [19] developed a new generator called Half Lapalce to model heavy-tail datasets with missing data. [20] suggested a three-parameter modified Kies-Fréchet model and derived its statistical properties. Sindhu et al. [21] used the foundation of the new power function distribution to propose the NHPP-NPF (non-homogeneous Poisson method, new power function) distribution. Shafiq et al. [22] proposed a new three-parameter lifetime model based on the type-1 half-logistic-G family and the unit-Gompertz model, named the half-logistic unit-Gompertz type-I distribution, by adding a new turning parameter to the unit-Gompertz model to increase its flexibility. Sindhu and Hussain [23] used the Bayesian framework to derive the distributions of future responses for the left-censored sample, assuming an informative and uninformative prior distribution for the parameter. Sindhu et al. [24] selected a suitable prior for the Bayesian analysis of the parameters of the two-component mixture of the Burr Type X distribution. Sindhu et al. [25] derived the probability density function of a mixture for an inverse Maxwell density and studied its main distributional properties and reliability characteristics. Ogunsanya et al. [26] introduced and studied Rayleigh Cauchy distribution as another method of determining the moment of Cauchy distribution. Sindhu and Atangana.[27] suggested a model of reliability formed on inverse power law and generalized inverse Weibull model. Adewara et al. [28] studied the properties and applications of the Gompertz distribution. Adewara et al. [29] introduced Exponentiated Gompertz Exponential distribution and applied it to real data sets. Inverted Kumaraswamy distribution was introduced by Al-Fattah et al. [30] by using the method of transformation. Also, Hassan et. al. [31] proposed inverted Topp-Leone distribution and derived its statistical properties.

2. Methodology

The concept of Inverted distribution has not gained enough popularity as expected simply because researchers in the field of statistical sciences have not exploited substantially the gains of "Variable transformation" which this paper tends to demonstrate.

Let a random variable Y follow the Gompertz Fréchet distribution, then a random variable X=$\frac{1}{Y}$ follows an inverted–Gompertz Fréchet distribution. To the best of our knowledge, no work has been done on deriving a distribution whose random variables follow the inverse of the Gompertz-Fréchet (GF) distribution, hence the motivation for this work. This work aims to derive the inverted Gompertz–Fréchet (IGoFre) distribution and obtain statistical and mathematical properties of the new model as well as its application to real data sets.

The Gompertz–Fréchet distribution proposed by Pelumi et al. [4] is a four–parameter continuous distribution having the cumulative distribution function (cdf) given in Equation (1).

If a random variable Y~GF $(\alpha, \beta, \gamma, \theta)$, then

$$G(y; \alpha, \beta, \gamma, \theta) = 1 - e^{\left[\frac{\theta}{\gamma}\left(1-\left(1-e^{-\left(\frac{\alpha}{y}\right)^{\beta}}\right)^{-\gamma}\right)\right]}, y > 0, \alpha > 0, \beta > 0, \gamma > 0, \theta > 0 \quad (1)$$

2.1 IGoFre Distribution

The IGoFre distribution is a distribution of the random variable $X = Y^{-1}$.

The cdf of the IGoFre distribution is given by equation (2)

$$F(x; \alpha, \beta, \gamma, \theta) = e^{\left[\frac{\theta}{\gamma}\left(1-\left\{1-e^{-(\alpha x)^{\beta}}\right\}^{-\gamma}\right)\right]}, \ x > 0, \alpha > 0, \beta > 0, \gamma > 0, \theta > 0 \quad (2)$$

For more explanations on the derivation of the cumulative distribution function of the IGoFre distribution, readers are referred to Adeyeye et al. [8].

Differentiating (2) with respect to x, we have the probability density function (pdf) of the IGoFre distribution given in Equation (3).

$$f(x; \alpha, \beta, \gamma, \theta) = \theta\beta\alpha^{\beta}x^{\beta-1}e^{-(\alpha x)^{\beta}}\left\{1-e^{-(\alpha x)^{\beta}}\right\}^{-\gamma-1} e^{\left[\frac{\theta}{\gamma}\left(1-\left\{1-e^{-(\alpha x)^{\beta}}\right\}^{-\gamma}\right)\right]}, x > 0, \alpha > 0, \beta > 0, \gamma > 0, \theta > 0 \quad (3)$$

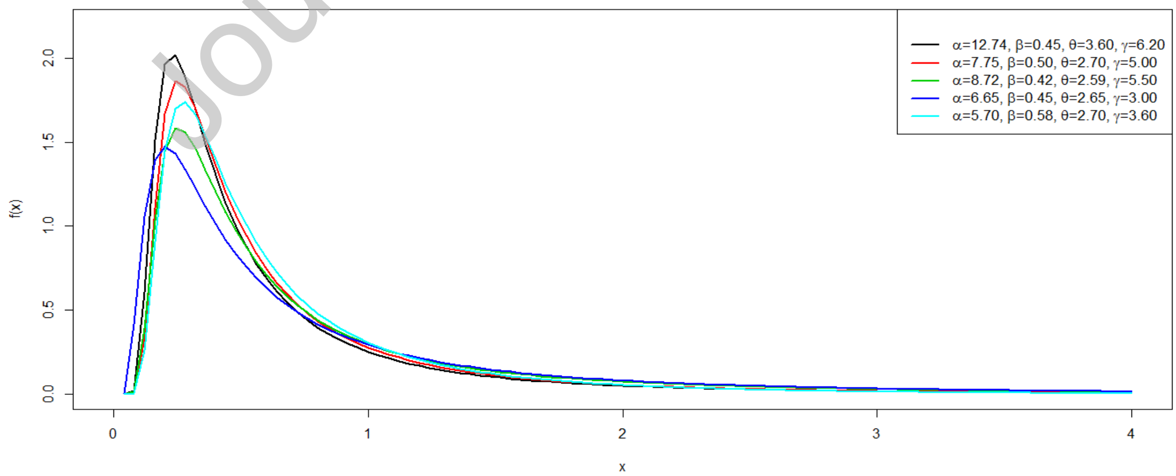

Figure 1: The pdf plots of IGoFre distributions at different parameter values.

The plots reveal that the distribution is positively skewed with high kurtosis.

Reliability Analysis

The survival function of IGoFre distribution can be derived from

$$S(x) = 1 - F(x) = 1 - e^{\left[\frac{\theta}{\gamma}\left(1-\left\{1-e^{-(\alpha x)^{\beta}}\right\}^{-\gamma}\right)\right]}$$

The hazard function of IGoFre distribution is derived from

$$h(x) = \frac{f(x)}{S(x)} = \frac{\theta\beta\alpha^{\beta}x^{\beta-1}e^{-(\alpha x)^{\beta}}\left(1-e^{-(\alpha x)^{\beta}}\right)^{-\gamma-1}e^{\left[\frac{\theta}{\gamma}\left(1-\left\{1-e^{-(\alpha x)^{\beta}}\right\}^{-\gamma}\right)\right]}}{1-e^{\left[\frac{\theta}{\gamma}\left(1-\left\{1-e^{-(\alpha x)^{\beta}}\right\}^{-\gamma}\right)\right]}}$$

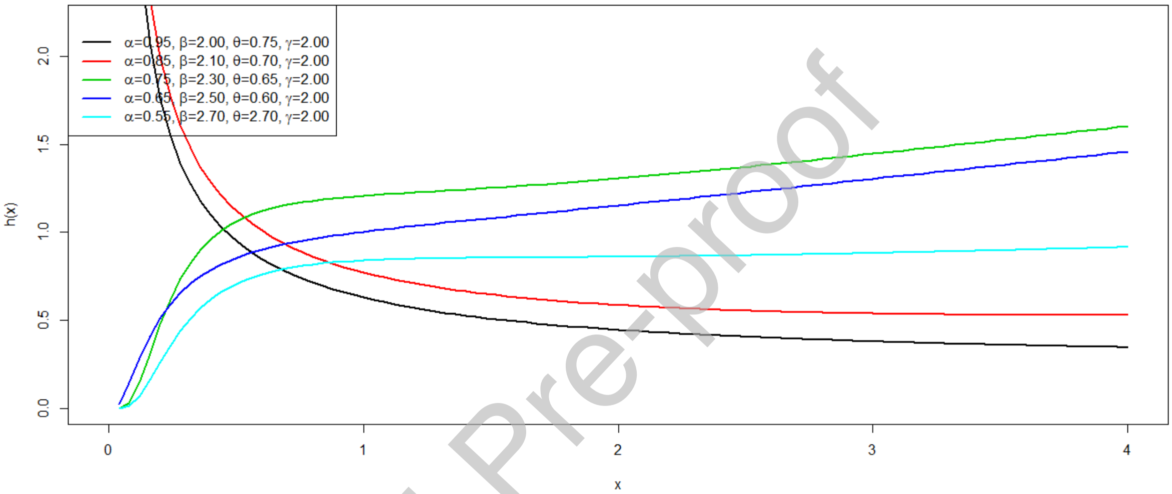

Figure 2: The hazard plots of IGoFre distribution at different parameters. The plots reveal that its hazard function could be increasing or increasing depending on the parameter values.

Useful Expansions

$$f(x;\alpha,\beta,\gamma,\theta) = \theta\beta\alpha^{\beta}x^{\beta-1}e^{-(\alpha x)^{\beta}}\left\{1-e^{-(\alpha x)^{\beta}}\right\}^{-\gamma-1}e^{\left[\frac{\theta}{\gamma}\left(1-\left\{1-e^{-(\alpha x)^{\beta}}\right\}^{-\gamma}\right)\right]}$$

Using the exponential expansion in Job and Ogunsanya [10]

$$e^{x} = \sum_{k=0}^{\infty}\frac{x^{k}}{k!}$$

$$e^{\left[\frac{\theta}{\gamma}\left(1-\left\{1-e^{-(\alpha x)^{\beta}}\right\}^{-\gamma}\right)\right]} = \sum_{k=0}^{\infty}\left(1-\left\{1-e^{-(\alpha x)^{\beta}}\right\}^{-\gamma}\right)^{k}\left(\frac{\theta}{\gamma}\right)^{k}\frac{1}{k!}$$

So that the pdf in (3) becomes,

$$f(x) = \theta\beta\alpha^{\beta}\left(\frac{\theta}{\gamma}\right)^{k}\frac{1}{k!}x^{\beta-1}e^{-(\alpha x)^{\beta}}\left\{1-e^{-(\alpha x)^{\beta}}\right\}^{-\gamma-1}\sum_{k=0}^{\infty}\left(1-\left\{1-e^{-(\alpha x)^{\beta}}\right\}^{-\gamma}\right)^{k}$$

From the mixture representation

$$(1-z)^k=\sum_{j=0}^{\infty}\frac{\Gamma(\mathrm{k}+\mathrm{j})}{\Gamma(k)j!}z^j(-1)^j$$

$$\left(1-\left\{1-e^{-(\alpha x)^{\beta}}\right\}^{-\gamma}\right)^k=\sum_{j=0}^{\infty}(-1)^j\frac{\Gamma(\mathrm{k}+\mathrm{j})}{\Gamma(k)j!}\left\{1-e^{-(\alpha x)^{\beta}}\right\}^{-\gamma j}$$

$$f(x)=\frac{\theta^{k+1}\beta\alpha^{\beta}}{\gamma^k k!}x^{\beta-1}e^{-(\alpha x)^{\beta}}\left\{1-e^{-(\alpha x)^{\beta}}\right\}^{-\gamma-1}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(\mathrm{k}+\mathrm{j})}{\Gamma(k)j!}(-1)^j\left\{1-e^{-(\alpha x)^{\beta}}\right\}^{-\gamma j}$$

$$f(x)=\frac{\theta^{k+1}\beta\alpha^{\beta}}{\gamma^k k!}x^{\beta-1}e^{-(\alpha x)^{\beta}}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(\mathrm{k}+\mathrm{j})}{\Gamma(k)j!}(-1)^j\left\{1-e^{-(\alpha x)^{\beta}}\right\}^{-\gamma j}\left\{1-e^{-(\alpha x)^{\beta}}\right\}^{-\gamma-1}$$

$$f(x)=\frac{\theta^{k+1}\beta\alpha^{\beta}}{\gamma^k k!}x^{\beta-1}e^{-(\alpha x)^{\beta}}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(\mathrm{k}+\mathrm{j})}{\Gamma(k)j!}(-1)^j\left\{1-e^{-(\alpha x)^{\beta}}\right\}^{-[\gamma(J+1)+1]}$$

By using a mixture representation

$$(1-z)^{-k}=\sum_{i=0}^{\infty}\frac{\Gamma(\mathrm{k}+\mathrm{i})}{\Gamma(k)i!}z^i$$

$$\left\{1-e^{-(\alpha x)^{\beta}}\right\}^{-[\gamma(J+1)+1]}=\sum_{i=0}^{\infty}\frac{\Gamma\{-[\gamma(\mathrm{j}+1)+1]+i\}}{\Gamma\{-[\gamma(\mathrm{j}+1)+1]\}i!}e^{-(\alpha x)^{\beta i}}$$

$$f(x)=\frac{\theta^{k+1}\beta\alpha^{\beta}}{\gamma^k k!}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{\Gamma(\mathrm{k}+\mathrm{j})}{\Gamma(k)j!}(-1)^j\sum_{i=0}^{\infty}\frac{\Gamma\{-[\gamma(\mathrm{j}+1)+1]+i\}}{\Gamma\{-[\gamma(\mathrm{j}+1)+1]\}i!}e^{-(\alpha x)^{\beta i}}e^{-(\alpha x)^{\beta}}x^{\beta-1}$$

$$f(x)=\frac{\theta^{k+1}\beta\alpha^{\beta}}{\gamma^k k!}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{\Gamma(\mathrm{k}+\mathrm{j})}{\Gamma(k)j!}(-1)^j\sum_{i=0}^{\infty}\frac{\Gamma\{-[\gamma(\mathrm{j}+1)+1]+i\}}{\Gamma\{-[\gamma(\mathrm{j}+1)+1]\}i!}e^{-(\alpha x)^{\beta(i+1)}}x^{\beta-1}$$

Let $(-1)^j\frac{\Gamma(\mathrm{k}+\mathrm{j})}{\Gamma(k)j!}\frac{\Gamma\{-[\gamma(\mathrm{j}+1)+1]+i\}}{\Gamma\{-[\gamma(\mathrm{j}+1)+1]\}i!}=c_{i,j,k}$

$$f(x)=\frac{\theta^{k+1}\beta\alpha^{\beta}}{\gamma^k k!}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}c_{i,j,k}\,x^{\beta-1}e^{-(\alpha x)^{\beta(i+1)}} \tag{4}$$

3. Estimation of Parameters

The method of maximum likelihood is used to estimate the model's parameters. Assuming each of the random samples $x_1,x_2,\dots,x_n$ has the pdf of IGoFre distribution, the likelihood function is given by

$$L(x_1,x_2,\ldots,x_n;\alpha,\beta,\theta,\gamma)=\prod_{i=1}^{n}\left\{\theta\beta\alpha^{\beta}x^{\beta-1}e^{-(\alpha x)^{\beta}}\left\{1-e^{-(\alpha x)^{\beta}}\right\}^{-\gamma-1}e^{\left[\frac{\theta}{\gamma}\left(1-\left\{1-e^{-(\alpha x)^{\beta}}\right\}^{-\gamma}\right)\right]}\right\}. \tag{5}$$

Denoting the log-likelihood function by l, i.e. let $l=\log L(x_1,x_2,\ldots,x_n;\alpha,\beta,\theta,\gamma)$ (6)

$$L = n\log\theta + n\beta\log\alpha + n\log\beta + (\beta-1)\sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}(\alpha x_i)^{\beta}$$
$$-(\gamma+1)\sum_{i=1}^{n}\log\left\{1-e^{-(\alpha x_i)^{\beta}}\right\} \quad + \frac{\theta}{\gamma}\sum_{i=1}^{n}\left[\left(1-\left\{1-e^{-(\alpha x_i)^{\beta}}\right\}^{-\gamma}\right)\right].$$

The maximum likelihood estimates of the parameters α, β, θ and γ are the solutions of the simultaneous equations $\frac{dl}{d\alpha} = 0, \frac{dl}{d\beta} = 0, \frac{dl}{d\theta} = 0$ and $\frac{dl}{d\gamma} = 0$

$$\frac{dl}{d\theta} = \frac{n}{\theta} + \frac{1}{\gamma}\sum_{i=1}^{n}\left[\left(1-\left\{1-e^{-(\alpha x_i)^{\beta}}\right\}^{-\gamma}\right)\right],$$

$$\frac{dl}{d\alpha} = \frac{n\beta}{\alpha} - \frac{\beta}{\alpha}\sum_{i=1}^{n}(\alpha x_i)^{\beta} - \frac{\frac{\beta(\gamma+1)}{\alpha}\sum_{i=1}^{n}(\alpha x_i)^{\beta}e^{-(\alpha x_i)^{\beta}}}{n+\sum_{i=1}^{n}-e^{-(\alpha x_i)^{\beta}}} + \frac{\theta\beta}{\alpha}\sum_{i=1}^{n}\frac{\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{-\gamma}(\alpha x_i)^{\beta}e^{-(\alpha x_i)^{\beta}}}{\left[\left(1-e^{-(\alpha x_i)^{\beta}}\right)\right]},$$

$$\frac{dl}{d\beta} = n\log\alpha + \frac{n}{\beta} + \sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}(\alpha x_i)^{\beta}\log(\alpha x_i) - (\gamma+1)\frac{\left[(\alpha x_i)^{\beta}\log(\alpha x_i)\,e^{-(\alpha x_i)^{\beta}}\right]}{n+\sum_{i=1}^{n}\left(-e^{-(\alpha x_i)^{\beta}}\right)} +$$

$$\theta\sum_{i=1}^{n}\frac{\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{-\gamma}(\alpha x_i)^{\beta}\log(\alpha x_i)\,e^{-(\alpha x_i)^{\beta}}}{1-e^{-(\alpha x_i)^{\beta}}}$$

and

$$\frac{dl}{d\gamma} = -\log\left(n+\sum_{i=1}^{n}\left[\left(-e^{-(\alpha x_i)^{\beta}}\right)\right]\right) - \frac{\theta}{\gamma^2}\left[n+\sum_{i=1}^{n}-\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{-\gamma}\right] +$$
$$\frac{\theta}{\gamma}\left[\sum_{i=1}^{n}\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{-\gamma}\log\left(1-e^{-(\alpha x_i)^{\beta}}\right)\right].$$

Setting $\frac{dl}{d\theta} = 0$, the maximum likelihood estimate of parameter θ is given as

$$\hat{\theta} = -\frac{n}{\sum_{i=1}^{n}\left[\frac{1}{\gamma}\left(1-\left\{1-e^{-(\alpha x_i)^{\beta}}\right\}^{-\gamma}\right)\right]} \tag{7}$$

The estimates of the remaining parameters can be found numerically with the availability of datasets.

Distribution of Order Statistics

Assuming that $x_1, x_2, \dots, x_n$ are random samples from a cdf and pdf of IGoFre distribution already defined in (2) and (3) respectively. The pdf of jth order statistics of any random variable X is given by:

$$f_{j:n}(x) = \frac{n!}{(j-1)!\,(n-j)!}f(x)F(x)^{j-1}[1-F(x)]^{n-j}. \tag{8}$$

Putting the pdf and cdf of the jth order statistics of IGoFre distribution in (8),

$$f_{j:n}(x)=\frac{n!}{(j-1)!(n-j)!}\theta\beta\alpha^{\beta}x^{\beta-1}e^{-(\alpha x)^{\beta}}\left(1-e^{-(\alpha x)^{\beta}}\right)^{-\gamma-1}e^{\left[\frac{\theta}{\gamma}\left(1-\left\{1-e^{-(\alpha x)^{\beta}}\right\}^{-\gamma}\right)\right]}.e^{\left[\frac{\theta}{\gamma}\left(1-\left\{1-e^{-(\alpha x)^{\beta}}\right\}^{-\gamma}\right)\right]^{j-1}}\left\{1-\left(e^{\left[\frac{\theta}{\gamma}\left(1-\left\{1-e^{-(\alpha x)^{\beta}}\right\}^{-\gamma}\right)\right]}\right)\right\}^{n-j}.$$

Simplifying Equation (17),

$$f_{j:n}(x)=\frac{n!}{(j-1)!(n-j)!}\theta\beta\alpha^{\beta}x^{\beta-1}e^{-(\alpha x)^{\beta}}\left(1-e^{-(\alpha x)^{\beta}}\right)^{-\gamma-1}.\left\{e^{\left[\frac{\theta}{\gamma}\left(1-\left\{1-e^{-(\alpha x)^{\beta}}\right\}^{-\gamma}\right)\right]}\right\}^{j}\left\{1-\left(e^{\left[\frac{\theta}{\gamma}\left(1-\left\{1-e^{-(\alpha x)^{\beta}}\right\}^{-\gamma}\right)\right]}\right)\right\}^{n-j}.$$

Therefore, the distributions of the minimum and maximum order statistics for the IGoFre distribution are respectively given by (9) and (10)

$$f_{1:n}(x)=n\theta\beta\alpha^{\beta}x^{\beta-1}e^{-(\alpha x)^{\beta}}\left(1-e^{-(\alpha x)^{\beta}}\right)^{-\gamma-1}.\left\{e^{\left[\frac{\theta}{\gamma}\left(1-\left\{1-e^{-(\alpha x)^{\beta}}\right\}^{-\gamma}\right)\right]}\right\}\left\{1-\left(e^{\left[\frac{\theta}{\gamma}\left(1-\left\{1-e^{-(\alpha x)^{\beta}}\right\}^{-\gamma}\right)\right]}\right)\right\}^{n-1} \quad (9)$$

and

$$f_{n:n}(x)=n\theta\beta\alpha^{\beta}x^{\beta-1}e^{-(\alpha x)^{\beta}}\left(1-e^{-(\alpha x)^{\beta}}\right)^{-\gamma-1}.\left\{e^{\left[\frac{\theta}{\gamma}\left(1-\left\{1-e^{-(\alpha x)^{\beta}}\right\}^{-\gamma}\right)\right]}\right\}^{n} \quad (10)$$

4. Quantile Function and Median of IGoFre Distribution

The quantile function is derived by inverting the cdf

$$Q(u)=G^{-1}(u) \quad (11)$$

$$Q(u)=\frac{1}{\alpha}\left\{\ln\left[1-\left(1-\frac{\gamma}{\theta}ln\,u\right)^{-\frac{1}{\gamma}}\right]\right\}^{\frac{1}{\beta}}, \quad (12)$$

where $u\sim Uniform$ (0,1).

To generate random numbers from IGoFre distribution, it is sufficient that

$$x=\frac{1}{\alpha}\left\{\ln\left[1-\left(1-\frac{\gamma}{\theta}ln\,u\right)^{-\frac{1}{\gamma}}\right]\right\}^{\frac{1}{\beta}} \quad (13)$$

The median of the IGoFre distribution can be obtained by substituting $u=0.5$ in equation (13) as follows:

$$Median=\frac{1}{\alpha}\left\{\ln\left[1-\left(1-\frac{\gamma}{\theta}ln\,0.5\right)^{-\frac{1}{\gamma}}\right]\right\}^{\frac{1}{\beta}} \quad (14)$$

We can derive other quantiles from (12) by substituting appropriate values of $''u''$.

4.1 Measures of Skewness and Kurtosis Based on Quantile Function

Galton [32] and Moors [33] defined the measures of skewness (S) and kurtosis (K) as

$$S = \frac{Q\left(\frac{6}{8}\right) - 2Q\left(\frac{4}{8}\right) + Q\left(\frac{2}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)} \tag{15}$$

$$K = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) + Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)} \tag{16}$$

To obtain $Q\left(\frac{1}{8}\right), Q\left(\frac{2}{8}\right), Q\left(\frac{3}{8}\right), Q\left(\frac{4}{8}\right), Q\left(\frac{5}{8}\right), Q\left(\frac{6}{8}\right)$ and $Q\left(\frac{7}{8}\right)$, substitute $\frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}, \frac{5}{8}, \frac{6}{8}, and\ \frac{7}{8}$ respectively for u in equation (12).

$$Q\left(\frac{1}{8}\right) = \frac{1}{\alpha}\left\{\ln\left[1 - \left(1 - \frac{\gamma}{\theta} ln\left(\frac{1}{8}\right)\right)^{-\frac{1}{\gamma}}\right]\right\}^{\frac{1}{\beta}}, \quad Q\left(\frac{2}{8}\right) = \frac{1}{\alpha}\left\{\ln\left[1 - \left(1 - \frac{\gamma}{\theta} ln\left(\frac{2}{8}\right)\right)^{-\frac{1}{\gamma}}\right]\right\}^{\frac{1}{\beta}}$$

$$Q\left(\frac{3}{8}\right) = \frac{1}{\alpha}\left\{\ln\left[1 - \left(1 - \frac{\gamma}{\theta} ln\left(\frac{3}{8}\right)\right)^{-\frac{1}{\gamma}}\right]\right\}^{\frac{1}{\beta}}, \quad Q\left(\frac{4}{8}\right) = \frac{1}{\alpha}\left\{\ln\left[1 - \left(1 - \frac{\gamma}{\theta} ln\left(\frac{4}{8}\right)\right)^{-\frac{1}{\gamma}}\right]\right\}^{\frac{1}{\beta}}$$

$$Q\left(\frac{5}{8}\right) = \frac{1}{\alpha}\left\{\ln\left[1 - \left(1 - \frac{\gamma}{\theta} ln\left(\frac{5}{8}\right)\right)^{-\frac{1}{\gamma}}\right]\right\}^{\frac{1}{\beta}}, \quad Q\left(\frac{6}{8}\right) = \frac{1}{\alpha}\left\{\ln\left[1 - \left(1 - \frac{\gamma}{\theta} ln\left(\frac{6}{8}\right)\right)^{-\frac{1}{\gamma}}\right]\right\}^{\frac{1}{\beta}}$$

$$Q\left(\frac{7}{8}\right) = \frac{1}{\alpha}\left\{\ln\left[1 - \left(1 - \frac{\gamma}{\theta} ln\left(\frac{7}{8}\right)\right)^{-\frac{1}{\gamma}}\right]\right\}^{\frac{1}{\beta}}$$

The skewness of IGoFre distribution is obtained by substituting appropriate quantiles into (15).

$$S_{IGF} = \frac{\left\{\ln\left[1-\left(1-\frac{\gamma}{\theta} ln\left(\frac{6}{8}\right)\right)^{-\frac{1}{\gamma}}\right]\right\}^{\frac{1}{\beta}} - 2\left\{\ln\left[1-\left(1-\frac{\gamma}{\theta} ln\left(\frac{4}{8}\right)\right)^{-\frac{1}{\gamma}}\right]\right\}^{\frac{1}{\beta}} + \left\{\ln\left[1-\left(1-\frac{\gamma}{\theta} ln\left(\frac{2}{8}\right)\right)^{-\frac{1}{\gamma}}\right]\right\}^{\frac{1}{\beta}}}{\left\{\ln\left[1-\left(1-\frac{\gamma}{\theta} ln\left(\frac{6}{8}\right)\right)^{-\frac{1}{\gamma}}\right]\right\}^{\frac{1}{\beta}} - \left\{\ln\left[1-\left(1-\frac{\gamma}{\theta} ln\left(\frac{2}{8}\right)\right)^{-\frac{1}{\gamma}}\right]\right\}^{\frac{1}{\beta}}} \tag{17}$$

It can be inferred from (17) that the values of parameter α do not affect the shape of the distribution.

The kurtosis of IGoFre distribution is derived by substituting appropriate quantiles into equation (16).

$$K_{IGF} = \frac{\left\{\ln\left[1-\left(1-\frac{\gamma}{\theta} ln\left(\frac{7}{8}\right)\right)^{-\frac{1}{\gamma}}\right]\right\}^{\frac{1}{\beta}} - \left\{\ln\left[1-\left(1-\frac{\gamma}{\theta} ln\left(\frac{5}{8}\right)\right)^{-\frac{1}{\gamma}}\right]\right\}^{\frac{1}{\beta}} + \left\{\ln\left[1-\left(1-\frac{\gamma}{\theta} ln\left(\frac{3}{8}\right)\right)^{-\frac{1}{\gamma}}\right]\right\}^{\frac{1}{\beta}} - \left\{\ln\left[1-\left(1-\frac{\gamma}{\theta} ln\left(\frac{1}{8}\right)\right)^{-\frac{1}{\gamma}}\right]\right\}^{\frac{1}{\beta}}}{\left\{\ln\left[1-\left(1-\frac{\gamma}{\theta} ln\left(\frac{6}{8}\right)\right)^{-\frac{1}{\gamma}}\right]\right\}^{\frac{1}{\beta}} - \left\{\ln\left[1-\left(1-\frac{\gamma}{\theta} ln\left(\frac{2}{8}\right)\right)^{-\frac{1}{\gamma}}\right]\right\}^{\frac{1}{\beta}}} \tag{18}$$

The peakedness or otherwise of IGoFre distribution is independent of parameter α.

4.2 The moment of IGoFre Distribution

Given a random variable X, the rth moment about the origin is given as

$$E(X^r) = \int x^r f(x) dx$$

The moment about the origin of IGoFre distribution is derived as follows:

$$E(X^r) = \int_0^\infty x^r \frac{\theta^{k+1}\beta\alpha^\beta}{\gamma^k k!}\sum_{k=0}^\infty\sum_{j=0}^\infty\sum_{i=0}^\infty c_{i,j,k}\, x^{\beta-1}e^{-(\alpha x)^{\beta(i+1)}}\, dx$$

$$E(X^r) = \frac{\theta^{k+1}\beta\alpha^\beta}{\gamma^k k!}\sum_{k=0}^\infty\sum_{j=0}^\infty\sum_{i=0}^\infty c_{i,j,k}\int_0^\infty x^r\, x^{\beta-1}e^{-(\alpha x)^{\beta(i+1)}}dx$$

$$E(X^r) = \frac{\theta^{k+1}\beta\alpha^\beta}{\gamma^k k!}\sum_{k=0}^\infty\sum_{j=0}^\infty\sum_{i=0}^\infty c_{i,j,k}\int_0^\infty x^{r+\beta-1}\, e^{-(\alpha x)^{\beta(i+1)}}dx$$

Let $y = (\alpha x)^{\beta(i+1)}$

$x = \frac{1}{\alpha}y^{\frac{1}{\beta(i+1)}}$

$$dx = \frac{1}{\alpha}\,\frac{1}{\beta(i+1)}\,y^{\frac{1}{\beta(i+1)}-1}dy$$

Let

$$I = \int_0^\infty x^{r+\beta-1}\, e^{-(\alpha x)^{\beta(i+1)}}dx$$

$$I = \int_0^\infty \left[\frac{1}{\alpha}y^{\frac{1}{\beta(i+1)}}\right]^{r+\beta-1} e^{-y}\,\frac{1}{\alpha}\,\frac{1}{[\beta(i+1)]}\,y^{\frac{1}{\beta(i+1)}-1}dy$$

$$I = \left(\frac{1}{\alpha}\right)^{r+\beta-1}\frac{1}{\alpha}\frac{1}{[\beta(i+1)]}\int_0^\infty y^{\frac{r+\beta-1}{\beta(i+1)}}\, e^{-y}\, y^{\frac{1}{\beta(i+1)}-1}dy$$

$$I = \left(\frac{1}{\alpha}\right)^{r+\beta}\frac{1}{[\beta(i+1)]}\int_0^\infty y^{\frac{r+\beta}{\beta(i+1)}-1}\, e^{-y}dy$$

Using the gamma function

$$\Gamma(z) = \int_0^\infty y^{z-1}\, e^{-y}dy$$

$$I = \left(\frac{1}{\alpha}\right)^{r+\beta}\frac{1}{[\beta(i+1)]}\Gamma\left[\frac{r+\beta}{\beta(i+1)}\right]$$

$$I = \left(\frac{1}{\alpha}\right)^{r+\beta}\frac{\Gamma\left[\frac{r+\beta}{\beta(i+1)}\right]}{[\beta(i+1)]}$$

$$E(X^r) = \frac{\theta^{k+1}\beta\alpha^\beta}{\gamma^k k!}\sum_{k=0}^\infty\sum_{j=0}^\infty\sum_{i=0}^\infty c_{i,j,k}\left(\frac{1}{\alpha}\right)^{r+\beta}\frac{\Gamma\left[\frac{r+\beta}{\beta(i+1)}\right]}{[\beta(i+1)]} \qquad (19)$$

4.3 Mean of IGoFre Distribution

By setting $r = 1$, we get the first moment about the origin of IGoFre distribution which represents the mean.

$$Mean = E(X) = \frac{\theta^{k+1}\beta\alpha^{\beta}}{\gamma^{k}k!}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} c_{i,j,k}\left(\frac{1}{\alpha}\right)^{\beta+1}\frac{\Gamma\left[\frac{\beta+1}{\beta(i+1)}\right]}{[\beta(i+1)]} \tag{20}$$

4.4 Variance of IGoFre Distribution

To find the variance, first find the second moment about the origin of IGoFre distribution by setting $r = 2$.

$$E(X^2) = \frac{\theta^{k+1}\beta\alpha^{\beta}}{\gamma^{k}k!}\frac{\theta^{k+1}\beta\alpha^{\beta}}{\gamma^{k}k!}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} c_{i,j,k}\left(\frac{1}{\alpha}\right)^{\beta+2}\frac{\Gamma\left[\frac{\beta+2}{\beta(i+1)}\right]}{[\beta(i+1)]} \tag{21}$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$Var(X) = \left[\frac{\theta^{k+1}\beta\alpha^{\beta}}{\gamma^{k}k!}\frac{\theta^{k+1}\beta\alpha^{\beta}}{\gamma^{k}k!}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} c_{i,j,k}\left(\frac{1}{\alpha}\right)^{\beta+2}\frac{\Gamma\left[\frac{\beta+2}{\beta(i+1)}\right]}{[\beta(i+1)]}\right] -$$

$$\left[\frac{\theta^{k+1}\beta\alpha^{\beta}}{\gamma^{k}k!}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} c_{i,j,k}\left(\frac{1}{\alpha}\right)^{\beta+1}\frac{\Gamma\left[\frac{\beta+1}{\beta(i+1)}\right]}{[\beta(i+1)]}\right]^2 \tag{22}$$

4.5 Renyi Entropy

Renyi entropy is a measure of variation or uncertainty of a random variable. It is a very popular entropy measure in many fields of science such as (engineering, theory of communication, and probability).

It is defined as

$$\eta_R = \frac{1}{1-b} log \int_0^{\infty} f^b\,(x)dx$$

Using the linear representation of PDF of IGoFre distribution

$$\eta_R = \frac{1}{1-b} log \int_0^{\infty}\left(\frac{\theta^{k+1}\beta\alpha^{\beta}}{\gamma^{k}k!}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} c_{i,j,k}\, x^{\beta-1}e^{-(\alpha x)^{\beta(i+1)}}\right)^b dx$$

$$\eta_R = \frac{1}{1-b} log\left(\frac{\theta^{k+1}\beta\alpha^{\beta}}{\gamma^{k}k!}\right)^b\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} c^b{}_{i,j,k}\int_0^{\infty}\left(x^{\beta-1}e^{-(\alpha x)^{\beta(i+1)}}\right)^b (x)dx$$

$$\int_0^{\infty}\left(x^{\beta-1}e^{-(\alpha x)^{\beta(i+1)}}\right)^b (x)dx = \left(\frac{1}{\alpha}\right)^{b(\beta-1)+1}\frac{\Gamma\left[\frac{b(\beta-1)+1}{\beta(i+1)}\right]}{[\beta(i+1)]}$$

Hence,

$$\eta_R = \frac{1}{1-b} log\left(\frac{\theta^{k+1}\beta\alpha^{\beta}}{\gamma^k k!}\right)^b \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} c^b{}_{i,j,k}\left(\frac{1}{\alpha}\right)^{b(\beta-1)+1} \frac{\Gamma\left[\frac{b(\beta-1)+1}{\beta(i+1)}\right]}{[\beta(i+1)]}$$

5. Percentage Points of IGoFre Distribution

The concept of percentage points of distribution is very useful in hypothesis testing. Percentage points simply mean the specified value that a random variable exceeds with a definite probability. If α is the probability that an IGoFre variate, X exceeds the value X_α, then $P(X > X_\alpha) = \alpha$. Tables 1 and 2 below show the percentage points of IGoFre distribution for different values of $(\alpha, \beta, \gamma\ and\ \theta)$

Table 1: Generated Percentage Points of IGoFre distribution for different values of $\alpha, \beta, \gamma\ and\ \theta$.

	IGoFre $(\alpha,\beta,\gamma,\theta)$				
P	q_1 (0.5,0.5,0.5,0.5)	q_2 (0.5,1,1,1)	q_3 (0.5,0.5,1,1)	q_4 (0.5,0.5,0.5,1)	q_5 (0.7,1,1,1)
0.00	0.000000000	0.0000000	0.0000000	0.00000000	0.0000000
0.05	0.008367680	-0.5760763	0.1659320	0.06102552	-0.4114831
0.10	0.018494482	-0.7213462	0.2601701	0.11852580	-0.5152473
0.15	0.032186674	-0.8467604	0.3585016	0.18687392	-0.6048289
0.20	0.050455606	-0.9664997	0.4670608	0.26903841	-0.6903569
0.25	0.074585183	-1.0862149	0.5899314	0.36804858	-0.7758678
0.30	0.106264929	-1.2092696	0.7311665	0.48754650	-0.8637640
0.35	0.147752685	-1.3382646	0.8954760	0.63212143	-0.9559033
0.40	0.202110488	-1.4756260	1.0887360	0.80770084	-1.0540185
0.45	0.273561610	-1.6239359	1.3185839	1.02210202	-1.1599542
0.50	0.368048579	-1.7862039	1.5952622	1.28586415	-1.2758599
0.55	0.494136572	-1.9661760	1.9329239	1.61356066	-1.4044114
0.60	0.664538256	-2.1687665	2.3517741	2.02596395	-1.5491189
0.65	0.898821837	-2.4007456	2.8817896	2.55380668	-1.7148183
0.70	1.228535503	-2.6719345	3.5696170	3.24474923	-1.9085246
0.75	1.707728423	-2.9974862	4.4924617	4.17738682	-2.1410615
0.80	2.437041311	-3.4027284	5.7892804	5.49265441	-2.4305203
0.85	3.628092623	-3.9350999	7.7425055	7.47597691	-2.8107856
0.90	5.822398554	-4.7010777	11.0500659	10.83029736	-3.3579127
0.95	11.061616137	-6.0404327	18.2434138	18.09733153	-4.3145948
0.99	30.764007927	-9.2202988	42.5069549	42.46115280	-6.5859277

Table 2: Generated Percentage Points of IGoFre distribution for different values of $\alpha, \beta, \gamma\ and\ \theta$.

	IGoFre $(\alpha,\beta,\gamma,\theta)$				
P	q_6 (1,1,1,1)	q_7 (1,1,1,0.7)	q_8 (1,1,0.7,0.7)	q_9 (2,0.2,1,5)	q_{10} (2,0.2,2,5)
0.00	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.05	-0.2880382	-0.2099900	-0.1487574	-0.4559387	-0.8901559
0.10	-0.3606731	-0.2654412	-0.2002325	-1.0241599	-1.7067032
0.15	-0.4233802	-0.3140662	-0.2469244	-1.7914688	-2.7134682
0.20	-0.4832498	-0.3611192	-0.2931075	-2.8124639	-3.9773836
0.25	-0.5431074	-0.4087552	-0.3406119	-4.1581909	-5.5745083
0.30	-0.6046348	-0.4583159	-0.3906369	-5.9239405	-7.6030462
0.35	-0.6691323	-0.5108934	-0.4442061	-8.2396129	-10.1949744
0.40	-0.7378130	-0.5675554	-0.5023564	-11.2854013	-13.5321538
0.45	-0.8119679	-0.6294804	-0.5662575	-15.3164406	-17.8711336
0.50	-0.8931020	-0.6980783	-0.6373256	-20.7028907	-23.5832610
0.55	-0.9830880	-0.7751362	-0.7173659	-27.9977206	-31.2223892
0.60	-1.0843833	-0.8630295	-0.8087815	-38.0568049	-41.6447697
0.65	-1.2003728	-0.9650612	-0.9149126	-52.2641421	-56.2338454

0.70	-1.3359672	-1.0860576	-1.0406316	-72.9851569	-77.3526871
0.75	-1.4987431	-1.2335049	-1.1934810	-104.5657286	-109.3411829
0.80	-1.7013642	-1.4199695	-1.3860939	-155.8177779	-160.9973891
0.85	-1.9675499	-1.6690626	-1.6421612	-247.3642661	-252.9114479
0.90	-2.3505389	-2.0339021	-2.0148942	-440.0449870	-445.8358600
0.95	-3.0202164	-2.6842361	-2.6741517	-1018.491778	-1024.094593
0.99	-4.6101494	-4.2577298	-4.25561301	-4623.658275	-4627.386893

Where $q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9$ and q_{10} represent the first, second, third, fourth, fifth, sixth, seventh, eighth, ninth, and tenth quantile density functions respectively.

6. Applications to Real Life Data Sets

To pick the best out of the following models: IGoFre, Gompertz Weibull (GW), Gompertz Burr XII (GBXII), GF, and Gompertz Lomax (GL) distribution, the following criteria are used: negative loglikelihood (-LL), Akaike information criteria (AIC), consistent Akaike information criteria (CAIC), Bayesian information criteria (BIC) and Hannan–Quinn information criterion (HQIC). The goodness of fit tests like Shapiro Wilk statistic and Anderson Darling statistic is also computed, where the distribution with the least value of the above criteria is adjudged the best. Also, the distribution with the highest P-value is considered to be the best.

Data 1: The first data set represents the remission times (in months) of a random sample of 128 bladder cancer patients. It has been used by Oguntunde et al. [34]

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15,2.64, 3.88,5.32,7.39,10.34,14.83,34.26,0.90,2.69,4.18,5.34,7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69

Table 3: Descriptive Statistics for Data 1.

Min	Q1	Median	Mean	Q3	Max.	Standard Deviation	Skewness	Kurtosis
0.080	3.348	6.395	9.366	11.838	79.050	10.508	3.287	18.483

Table 3 reveals that the data is leptokurtic (having kurtosis greater than 3) and heavily positively skewed. The standard deviation of 10.50833 which is greater than the mean shows that the data is dispersed.

Table 4: Performance rating for the fitted models using Data 1.

Distributions	Estimates	-LL	AIC	CAIC	BIC	HQIC
IGoFre	$\hat{\alpha} = 8.2775520$ $\hat{\beta} = 1.1508134$ $\hat{\gamma} = 0.4849761$ $\hat{\theta} = 1.0287830$	409.8628	827.7256	828.0508	839.1337	832.3608
GW	$\hat{\alpha} = 1.29081462$ $\hat{\beta} = 0.21018393$ $\hat{\gamma} = 0.09745496$ $\hat{\theta} = 1.23639879$	410.8044	829.6088	829.934	841.0169	834.2439

GBXII	$\hat{\alpha} = 0.4357691$ $\hat{\beta} = 2.7190203$ $\hat{\gamma} = 0.1971231$ $\hat{\theta} = 1.6656655$	412.0484	832.0968	832.422	843.5049	836.7319
GF	$\hat{\alpha} = 0.01898607$ $\hat{\beta} = 0.95775376$ $\hat{\gamma} = 1.06002108$ $\hat{\theta} = 0.19599511$	413.5953	835.1905	835.5157	846.5986	839.8257
GL	$\hat{\alpha} = 2.2809003$ $\hat{\beta} = 0.6524743$ $\hat{\gamma} = 0.0426511$ $\hat{\theta} = 1.1752488$	414.0363	836.0726	836.3978	847.4807	840.7077

From Table 4, the distribution with the lowest value of each of the criteria above is considered as the best. The results clearly show that IGoFre distribution is the best.

Table 5: Goodness of fit test for the fitted models using Data 1.

Distributions	W	A	P - Values
IGoFre	0.01981018	0.1299716	0.9198
GW	0.04427843	0.266713	0.5676
GBXII	0.07516432	0.4776533	0.4092
GF	0.1215866	0.7449913	0.6089
GL	0.08799927	0.5320612	0.8189

From Table 5 above, the IGoFre distribution has the shortest distance with regards to the Shapiro-Wilk and Anderson-Darling statistics. It also has the highest P - value, and it is thereby regarded as the best out of all the competing models.

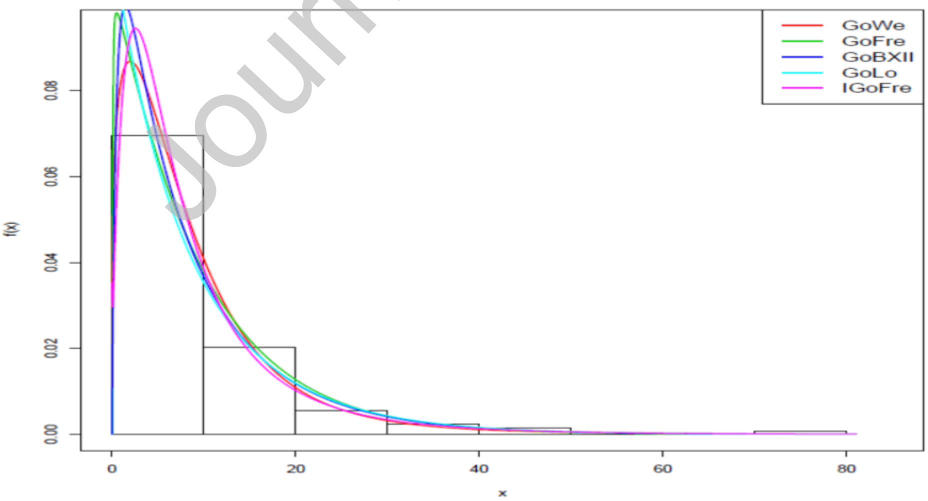

Figure 3: *Histogram and theoretical densities for Data 1*

Figure 3 above compares the distribution of the data on remission times (in months) of a random sample of 128 bladder cancer patients to five theoretical models. By keenly observing the five fitted models, the IGoFre distribution fits the data well.

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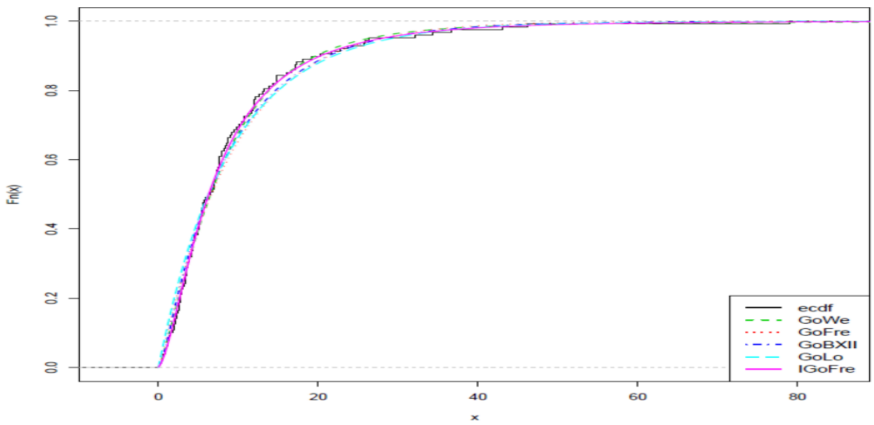

Figure 4: Estimated cdfs of Data 1.

Figure 4 above shows that data 1 follows the IGoFre distribution more than the other four competing models.

Data 2: The second set of data represents the time to failure for 40 suits of turbochargers in diesel engines. The data has been used previously by Alobaidi et al. [35]. The observations are:

1.6, 2.0, 2.6, 3.0, 3.5, 3.9, 4.5, 4.6, 4.8, 5.0, 5.1, 5.3, 5.4, 5.6, 5.8, 6.0, 6.0, 6.1, 6.3, 6.5, 6.5, 6.7, 7.0, 7.1, 7.3, 7.3, 7.3, 7.7, 7.7, 7.8, 7.9, 8.0, 8.1, 8.3, 8.4, 8.4, 8.5, 8.7, 8.8, 9.0

Table 6: Descriptive Statistics for Data 1.

Min	Q1	Median	Mean	Q3	Max.	Standard Deviation	Skewness	Kurtosis
1.600	5.075	6.500	6.253	7.825	9.000	1.956	- 0.663	2.641

Table 7: Performance rating for the fitted models using Data 2.

Distributions	Estimates	-LL	AIC	CAIC	BIC	HQIC
IGoFre	$\hat{\alpha} = 0.1331010844$ $\hat{\beta} = 0.0004361329$ $\hat{\gamma} = 18.9885168262$ $\hat{\theta} = 0.1157310834$	78.04672	164.0934	165.2363	170.849	166.536
GW	$\hat{\alpha} = 0.2344136$ $\hat{\beta} = 1.0298917$ $\hat{\gamma} = 0.1728447$ $\hat{\theta} = 2.2692040$	79.15399	166.308	167.4508	173.0635	168.7506
GBXII	$\hat{\alpha} = 0.004389937$ $\hat{\beta} = 3.230177839$ $\hat{\gamma} = 0.387561058$ $\hat{\theta} = 2.742479866$	82.8526	173.7052	174.8481	180.4607	176.1478
GF	$\hat{\alpha} = 1.4025567$ $\hat{\beta} = 24.0855229$ $\hat{\gamma} = 0.6131221$ $\hat{\theta} = 25.0947912$	80.19111	168.3822	169.5251	175.1377	170.8248

GL	$\hat{\alpha} = 0.04459060$ $\hat{\beta} = 6.56494801$ $\hat{\gamma} = 0.03949853$ $\hat{\theta} = 3.08370062$	80.2226	168.4445	169.5874	175.217	170.8871

From the table above, the distribution with the lowest value of each of the criteria is considered the best. The results clearly show that IGoFre distribution is the best.

Table 8: Goodness of fit test for the fitted models using Data 2.

Distributions	W	A	P - Values
IGoFre	0.010315588	0.09599912	0.9744
GW	0.02342907	0.1715485	0.8453
GBXII	0.08620836	0.6336566	0.9608
GF	0.03658082	0.2864124	0.5246
GL	0.03438217	0.2567791	0.3337

From Table 5 above, the IGoFre distribution has the shortest distance with regard to the Shapiro-Wilk and Anderson-Darling statistics. It also has the highest P-value, it is thereby regarded as the best out of all the competing models.

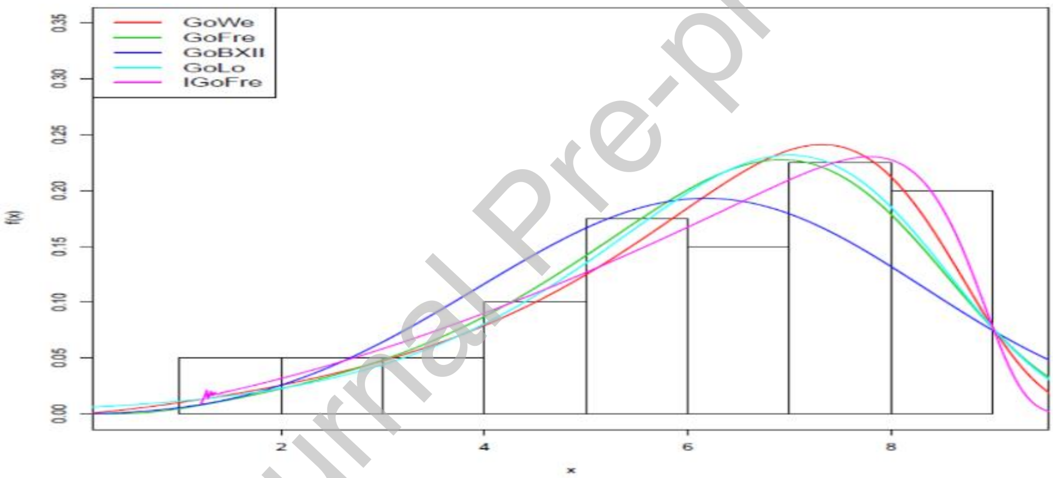

Figure 5: *Histogram and theoretical densities for Data 2*

From the histogram and theoretical densities plot in Figure 5, it shows that the IGoFre distribution fits the data well when compared to GW, GF, GBXII, and GL distributions.

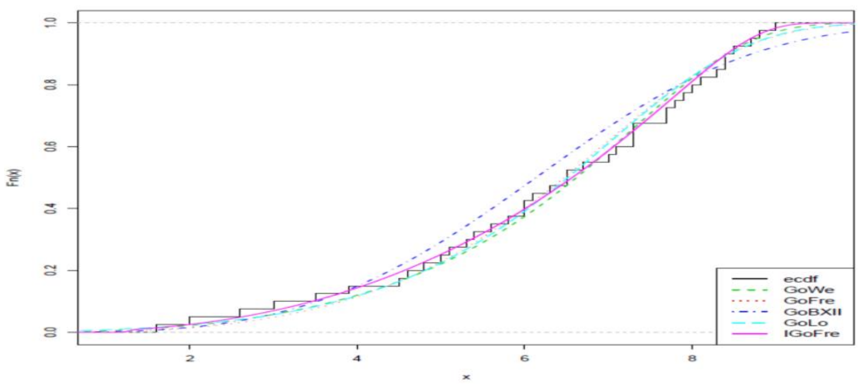

Figure 6: *Estimated cdfs of Data 2.*

Figure 6 above shows that Data 1 fits the IGoFre distribution more than the other four competing models.

7. Discussion

In this work, the distribution of random variables that follow the inverse of the Gompertz-Fréchet distribution, named the IGoFre distribution, was developed and studied. Overall, tables of percentage points provide a convenient way for statisticians to quickly access critical values for various distributions and significance levels, enabling them to make statistical inferences and draw conclusions from the data. The descriptive statistics of the two data sets in Tables 3 and 6 show that Data 1 is negatively skewed while Data 2 is positively skewed. The fact that the IGoFre distribution fits the data well shows that it is a skewed distribution that could have long tails to the left or right depending on the values of the shape parameters. Tables of percentage points generated in Tables 1 and 2 are useful for "tests of hypotheses. The results of applications of the IGoFre distribution to two data sets in tables 4, 5, 7, and 8 reveal that it is the best among the competing models. The histogram and theoretical densities plots in Figures 3 and 5 show that the IGoFre distribution fits the two real-life data sets well. The same applies to the plots of estimated CDFs in Figures 4 and 6.

8. Conclusion

This work proposes a four-parameter continuous distribution known as the Inverted Gompertz Fréchet (IGoFre) distribution, which is an inverse transformation of the Gompertz Frechet distribution proposed by Oguntunde et al. [4]. Some statistical and mathematical properties of the proposed distribution, such as mean, variance, linear representation of the model, and order of statistics of the distribution, were found. From the graph of the histogram and theoretical densities in Figures 3 and 5, it could be observed that the IGoFre distribution is suitable for modeling both positively and negatively skewed data. Also, by using the reliability function, it is evident that the distribution can be used in lifetime studies since the reliability graph tends to decrease as time increases. For hypotheses testing, tables of percentage points of the IGoFre distribution were generated for different values of the parameters. Applications of the proposed distribution to two real data sets reveal that the IGoFre distribution performs better than other competing models.

This study has potential limitations. Being a unimodal distribution, it is not suitable for modeling bimodal and multimodal data. Extending the distribution to model bimodal and multimodal data could be an interesting area for further research.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

References

[1] E. J Thomas, N. C. Lawrence and E. Nosa, Topp-Leone Gompertz Distribution: Properties and Applications, Journal of Data Science, JDS. 18(4) (2020). DOI: 10.6339/JDS.202010_18 (4) 0012

[2] H. Karamikabir, M. Afshari, M. Alizadeh, G. G Hamedani, A new extended generalized Gompertz distribution with statistical properties and simulations, Communications in Statistics – Theory and Methods. 50(2) (2021). https://doi.org/10.1080/03610926.2019.1634209

[3] D. G. Harlow, Applications of the Fréchet distribution function, International Journal of Materials and Product Technology. 17 (2004) 5-6

[4] P. E. Oguntunde, M. A. Khaleel, M. T. Ahmed, H. I. Okagbue, The Gompertz Fréchet Distribution: Properties and Applications, Cogent Mathematics & Statistics, (2019) DOI: 10.1080/25742558.2019.1568662

[5] M. Alizadeh, G. M. Cordeiro, L. G. B. Pinho, & I. Ghosh, The Gompertz-G family of distributions. Journal of Statistical Theory and Practice. 11(1) (2017) 179–207, Doi: 10.1080/15598608.2016.1267668

[6] M.S. Eliwa, M. El – Morshedy, M. Ibrahim, Inverse Gompertz distribution: Properties and different estimation methods with application to complete and censored data, Annals of Data Science, https://doi.org/10.1007/s40745-018-0173-0

[7] E .O. Pelumi, A. K. Mundher, O. Hilary, O. Abiodun, The Gompertz Inverse Exponential distribution, Cogent Mathematics, and Statistics. 5(1), (2019). DOI: 10.1080/25742558.2018.1507122

[8]S. Adeyeye, A. Adewara, E.E. Akarawak, A. Ogunsanya, A. Jamal, Zech distribution: Derivation, properties, and applications to real-life data, RT & A. 2 (68) 17 (2022) online Publication.

[9] M. A. Khaleel, P. E. Oguntunde, M. T. Ahmed, N. A. Ibrahim, Y. F. Loh, The Gompertz Flexible Weibull distribution and its applications, Malaysian Journal of Mathematical Sciences. 14(1) (2020): 169190.

[10] O. Job, A. S. Ogunsanya, Weibull Log Logistic {Exponential} Distribution: Some Properties and Application to Survival Data, International Journal of Statistical Distributions and Applications. (8)1, (2022) 1-13. doi: 10.11648/j.ijsd.20220801.11

[11] Z. Ahmad, Z. Hussain, Flexible Weibull Extended Distribution. MAYFEB Journal of Materials Science, ISSN 2371-8722. 2 (2017), 5-18

[12] A.S. Ogunsanya, W. B. Yahya, T. M. Adegoke, C. Iluno, O. R, Aderele, M.I. Ekum, A New Three-Parameter Weibull Inverse Rayleigh Distribution: Theoretical Development and Applications, Mathematics and Statistics. 9(3), (2021), 249 – 272

[13] A. Z. Afify, H. M. Yousof, G. M. Cordeiro, E. M.M. Ortega and Z. M. Nofal. The Weibull Fréchet distribution and its applications. Journal of Applied Statistics. 17 (2016) 2608-2626, https://doi.org/10.1080/02664763.2016.1142945

[14] M. M. Mansour, G. Aryal, A. Z. Afify, M. Ahmad, The Kumaraswamy Exponentiated Fréchet distribution. Pakistan Journal of Statistics. 34 (3) (2018) 177 – 193

[15] A. Z. Afify, H. M. Yousof, G. M. Cordeiro, Z. M. Nofal, M. Ahmad, The Kumaraswamy Marshall – Olkin Fréchet distribution with applications, Journal of ISOSS, 2(2),(2016) 151 – 168

[16] S. Nadarajah and A. Gupta, The Beta Fréchet distribution, Far East Journal of Theoretical Statistics. 14(1) (2004) 15 -24.

[17] E. Krishna, K. K. Jose, M. M. Ristić, Applications of Marshall–Olkin Fréchet Distribution, Communications in Statistics - Simulation and Computation. 42(1) (2013), 76-89

[18] T. N. Sindu, A. Shafiq, M. Qasem, Al – Mdallal. Exponentiated Transformation of Gumbel Type – II Distribution for Modelling COVID–19 Data. Alexandria Engineering Journal. 60 (2021) 671 – 689.

[19] A. S. Ogunsanya, O. Job, Weibull Half Laplace {Exponential} Distribution. Reliability: Theory & Applications. 18, 1 (72) (2023) online publication

[20] A. Shafiq, S.A. Lone, N.T. Sindhu, Q.M. Al-Mdallal, T. Muhammad, A New Modified Kies Fréchet Distribution: Applications of Mortality Rate of Covid-19, Results in Physics (2021). online publication doi:https://doi.org/10.1016/j.rinp.2021.104638

[21] T.N. Sindhu, S. Anwar, M.K.H Hassan, S.A. Lone, T.A. Abushal, A. Shafiq, An analysis of the New Reliability Model Based on Bathtub-shaped Failure Rate Distribution with Application to Failure Data. Mathematics 11 842 (2023). Online publication https://doi.org/10.3390/ math11040842

[22] A. Shafiq, T.N. Sindhu, S. Dey, S.A. Lone, T.A. Abushal, Statistical Features and Estimation Methods for Half-Logistic Unit-Gompertz Type-1 Model, Mathematics 11(10) (2023) online publication https://doi.org/10.3390/math 11041007

[23] T. N. Sindhu1, Z. Hussain, Predictive Inference and Parameter Estimation from the Half-Normal Distribution for the Left Censored Data, Annals of Data Science, online publication https://doi.org/10.1007/s40745-020-00309-6

[24] T. N. Sindhu, N. Feroze, M. Aslam, Preference of Prior for Bayesian Analysis of the Mixed Burr Type X Distribution under Type 1 Censored Samples. Pak.j.stat.oper.res. 10(1) (2014) 17-39

[25] T. N. Sindhu, Z. Hussain, M. Aslam, Parameter and reliability estimation of inverted Maxwell mixture model, Journal of Statistics and Management Systems. 22(3) (2019) 459-493

[26] A. S. Ogunsanya, E.E. Akarawak, M.I Ekum, On Some Properties of Rayleigh-Cauchy Distribution and its Application to Lifetimes Data. Accepted for Publication in the Journal of Statistics & Management Systems. 24(6). (2021) 1213-1231 https://doi.org/10.1080/09720510.2020.1822499

[27] T. N. Sindhu, A. Atangana. Reliability analysis incorporating exponentiated inverse Weibull distribution and inverse power law. WILEY, DOI: 10.1002/qre.286

[28] J. A., Adewara, J. S., Adeyeye and C. P., Thron. Properties and Applications of the Gompertz Distribution, International Journal of Mathematical Analysis and Optimization: Theory and Applications. 4(1) (2019) 443 - 454

[29] J. A. Adewara, J. S. Adeyeye, M. A. Khaleel, O. L. Aako. Exponentiated Gompertz Exponential (EGoE) distribution: Derivation, Properties and Applications, Journal of the Turkish Statistical Association. 13(1) (2021) 12–28.

[30] A.M. Abd Al-Fattah, A.A. EL-Helbawy, G.R. AL-Dayian, Inverted Kumaraswamy distribution: properties and estimation, Pak.J.Statist. 33(1) (2017) 37 – 61

[31] A.S. Hassan, M. Elgarhy, R. Ragab, Statistical Properties and Estimation of Inverted Topp-Leone Distribution, Journal of Statistics Applications & Probability. 9(2) (2020) 319-331

[32] F. Galton, Enquires into Human Faculty and its Development, Macmillan and Company, London. 1983.

[33] J.J. Moors, A quantile alternative for kurtosis, The Statistician. 37 (1988) 25 – 32.

[34] P. E. Oguntunde, A. O. Adejumo, E. A. Owoloko, Exponential Inverse Exponential (EIE) Distribution with Applications to Lifetime Data, Asian Journal of Scientific Research, ISSN 1992-1454, DOI: 10.3923/ajsr.2017.169.177

[35] M. H. Alobaidi, P. E. Oguntunde, M. A. Khaleel. The transmuted Topp Leone Flexible Weibull (TTLFW) distribution with applications to reliability and lifetime data, Ibn Al-Haitham International Conference for Pure and Applied Sciences (IHICPS), Journal of Physics: Conference Series. 1879 (2021) 022104, IOP Publishing, doi:10.1088/1742-6596/1879/2/022104

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