On a Resonant Fractional Order Multipoint and Riemann-Stieltjes Integral Boundary Value Problems on the Half-line with Two-dimensional Kernel

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***Abstract*—This paper investigates existence of solutions of a resonant fractional order boundary value problem with multipoint and Riemann-Stieltjes integral boundary conditions on the half-line with two-dimensional kernel. We utilised Mawhin's coincidence degree theory to derive our results. The results obtained are validated with examples.**

***Index Terms*—Banach spaces, coincidence degree theory, half-line, resonance, Riemann-Stieltjes integral, two-dimensional kernel.**

I. Introduction

FRACTIONAL differential equation serves as a powerful tool for mathematical modelling of complex phenomena, such as; viscoelastic media, epidemics, electromagnetics, acoustics, control theory, electrochemistry, finance, and materials science found in science and engineering (see [5], [16], [21], [24], [26]. The interest of researchers and scientists have significantly shifted to fractional-order models because, they are more accurate and provide more degrees of freedom than integer-order models.Valuable results have been obtained in the literature on the existence of solutions of fractional order boundary value problems (BVPs) by using different methods.These methods include; coincidence degree theory of Mawhin (see [2], [8], [10], [12], [16], [18], [22], [27], [28], [29], hybrid fixed point theorem [6], Ge and Ren extension of Mawhin coincidence degree theory [9],extension of continuation theorem [25], monotone iterative technique [11] and the references therein. A fractional order BVP is at resonance if the corresponding homogeneous equation has non-trivial solution.

Some scholars have studied resonant fractional order BVPs on finite interval $[0, 1]$ with finite point or integral boundary conditions in which the $\dim\ker L = 1$ and the order $1 < \alpha \leq 2$ (see [1], [7], [12], [15], [23], [31]).

Recently, Zhang and Liu [30] studied the following class of fractional multipoint boundary value problem at resonance with $\dim\ker L = 2$ on an infinite interval, and established that solution exists by using coincidence degree theory

$$D^{\alpha}_{0+}u(t) = f\big(t, u(t),\ D^{\alpha-2}_{0+}u(t), D^{\alpha-1}_{0+}u(t)\big),\ 0 < t < +\infty,$$

subject to;

$$u(0) = 0,\ ,D^{\alpha-2}_{0+}u(0) = \sum_{i=1}^{m}\alpha_i D^{\alpha-2}_{0+}u(\xi_i),$$
$$D^{\alpha-1}_{0+}u(+\infty) = \sum_{j=1}^{n}\beta_j D^{\alpha-1}_{0+}u(\eta_j),$$

$\sum_{i=1}^{m}\alpha_i = 1 = \sum_{j=1}^{n}\beta_j\eta_j$, $\sum_{i=1}^{m}\alpha_i\xi_i = 0 = \sum_{j=1}^{n}\beta_j$ are critical for resonance; where D^{α}_{0+} is the standard Riemann-Liouville fractional derivative of order α, $2 < \alpha \leq 3$, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < +\infty$, and $0 < \eta_1 < \eta_2 < \cdots < \eta_n < +\infty$.

However, the existence of solutions for a resonant fractional order boundary value problems on the half-line with multipoint and Riemann-Stieltjes integral boundary conditions where $\dim\ker L = 2$ and $3 < \alpha \leq 4$ have not been widely reported in the literature. We are motivated by this, to focus on investigating existence of solution for the following resonant fractional order boundary value problem

$$D^{\alpha}_{0+}x(t) = f\Big(t, x(t), D^{\alpha-3}_{0+}x(t), D^{\alpha-2}_{0+}x(t), D^{\alpha-1}_{0+}x(t)\Big),$$
$$x(0) = 0 = D^{\alpha-3}_{0+}x(0),\ D^{\alpha-2}_{0+}x(0) = \sum_{i=1}^{m}\mu_i D^{\alpha-2}_{0+}x(\xi_i),$$
$$D^{\alpha-1}_{0+}x(+\infty) = \int_0^{\eta} D^{\alpha-2}_{0+}x(t)dA(t), \tag{1}$$

where $t \in (0, +\infty)$, $3 < \alpha \leq 4$, $\dim\ker L = 2$, $0 < \xi_1 < \xi_2 < \xi_3 < \cdots < \xi_m < \infty$, $\eta \in (0, +\infty)$ and $A(t)$ is a continuous and bounded variation function on $(0, +\infty)$.

Throughout this investigation, the following assumptions are made:

(H_1) $\sum_{i=1}^{m}\mu_i = 1$, $\sum_{i=1}^{m}\mu_i\xi_i = 0$, $\int_0^{\eta} tdA(t) = 1$, and $\int_0^{\eta} dA(t) = 0$.

(H_2) $\Delta = \Big(1 - \sum_{i=1}^{m}\mu_i e^{-\xi_i}\Big)\Big(\int_0^{\eta}(2+t)e^{-t}dA(t)\Big) + \Big(\int_0^{\eta} e^{-t}dA(t)\Big)\Big(\sum_{j=1}^{m}\mu_i(2+\xi_i)e^{-\xi_i} - 2\Big) \neq 0$

(H_3) There exist nonnegative functions $\rho_1(t), \rho_2(t), \rho_3(t), \rho_4(t), \rho_5(t) \in L^1(0, +\infty)$ such that

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for all $t \in (0, +\infty)$ and $p, q, r, v \in \mathbb{R}$,

$$|f(t,p,q,r,v)| \le \rho_1(t)\frac{|p|}{1+t^\alpha} + \rho_2(t)\frac{|q|}{1+t^2} + \rho_3(t)\frac{|r|}{1+t^{\alpha-1}} + \rho_4(t)\frac{|v|}{1+t^{\alpha-2}} + \rho_5(t),$$

$\Theta := \|\rho_1\|_{L^1} + \|\rho_2\|_{L^1} + \|\rho_3\|_{L^1} + \|\rho_4\|_{L^1}$ and $\|\rho_i\|_{L^1} = \int_0^\infty |\rho_i| dt$, $i = 1, 2, 3, 4$.

(H_4) There exist non-negative constants A_1 and A_2 such that, for all $x \in dom\, L \setminus \ker L$, if one of the following is satisfied:
(i) $|D_{0+}^{\alpha-3}x(t)| > A_1$, for any $t \in (0, A_2]$;
(ii) $|D_{0+}^{\alpha-2}x(t)| > A_1$, for any $t \in (0, A_2]$;
(iii) $|D_{0+}^{\alpha-1}x(t)| > A_1$, for any $t \in (A_2, +\infty)$,
then either $\Pi_1 Nx(t) \neq 0$ or $\Pi_2 Nx(t) \neq 0$.

(H_5) There exists $B > 0$ such that, for any $c_1, c_2 \in \mathbb{R}$ satisfying $|c_1| > B$ or $|c_2| > B$, then either

$$\Pi_1 N(c_1t^{\alpha-1} + c_2t^{\alpha-2}) + \Pi_2 N(c_1t^{\alpha-1} + c_2t^{\alpha-2}) < 0, \tag{2}$$

or

$$\Pi_1 N(c_1t^{\alpha-1} + c_2t^{\alpha-2}) + \Pi_2 N(c_1t^{\alpha-1} + c_2t^{\alpha-2}) > 0. \tag{3}$$

then, the BVP (1) has at least one solution in X provided $H\Theta < 1$ where $H = \left(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)} + \frac{11}{2}\right)$ and $\Theta = \|\rho_1(t)\|_1 + \|\rho_2(t)\|_1 + \|\rho_3(t)\|_1 + \|\rho_4(t)\|_1$.

This paper unlike most of the previous works , focuses on two-dimensional kernel on the half-line with multipoint and Riemann-Stieltjes integral boundary conditions.

The rest of the paper is organized as follows. Section 2 presents some lemmas and definitions which are germane to the study. Section 3 focuses on the main existence results. Section 4 is concerned with examples to validate the results while in section 5, we draw conclusion.

A. Preliminaries

In this section, we recall some basic knowledge of the fractional calculus of Riemann-Liouville type and coincidence degree theory of Mawhin. Some definitions, lemmas and theorems that will be useful in the research study are highlighted.

***Definition** 1:* [10]. The *Riemann-Liouville fractional integral* of order $\alpha > 0$ for a function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0+}^\alpha = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds$$

provided the right - hand side integral is point-wise defined on $(0, +\infty)$.

***Definition** 2:* [10] The *Riemann-Liouville fractional derivative* of order $\alpha > 0$ for a function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0+}^\alpha f(t) = \frac{d^n}{dt^n} I_{0+}^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} f(s) ds$$

where $n = [\alpha] + 1$, provided that the right-hand side integral is point-wise defined on $(0, +\infty)$.

***Lemma** 1:* [18]. If $\alpha > 0$ and f, $D_{0+}^\alpha f \in L^1(0, 1)$, then

$$I_{0+}^\alpha D_{0+}^\alpha f(t) = f(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $n = [\alpha] + 1$, $c_i \in \mathbb{R}$ $(i = 1, 2, \cdots, n)$ are arbitrary constants.

***Lemma** 2:* [7]. Given that $\alpha > \beta > 0$. Suppose that $f(t) \in L^1(0, 1)$, then

$$I_{0+}^\alpha I_{0+}^\beta f(t) = I_{0+}^{\alpha+\beta} f(t),\ D_{0+}^\beta I_{0+}^\alpha f(t) = I_{0+}^{\alpha-\beta} f(t).$$

In particular,

$$D_{0+}^\alpha I_{0+}^\alpha f(t) = f(t).$$

***Lemma** 3:* [4]. Given that $\alpha > 0$, $n \in \mathbb{N}$ and $D = \frac{d}{dx}$. If the fractional derivatives $(D_{0+}^\alpha f)(t)$ and $(D_{0+}^{\alpha+n} f)(t)$ exist, then

$$(D^n D_{0+}^\alpha f)(t) = (D_{0+}^{\alpha+n} f)(t).$$

***Lemma** 4:* [4]. Suppose that $\alpha > 0$, $\lambda > -1$, $t > 0$, then $I_{0+}^\alpha t^\lambda = \dfrac{\Gamma(\lambda+1)}{\Gamma(\lambda+\alpha+1)} t^{\lambda+\alpha}$ and $D_{0+}^\alpha t^\lambda = \dfrac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha}$. In particular, $D_{0+}^\alpha t^{\alpha-m} = 0$, for $m = 1, 2, 3, \cdots, n$ where $n = [\alpha] + 1$.

***Definition** 3:* [19]. Let $n \in \mathbb{R}_+$ and $m = [n]$. The operator

$$D_{0+}^n f = D_{0+}^m I_{0+}^{m-n} f$$

is called Riemann-Liouville fractional differential operator of order n.

If $n = 0$, then $D_{0+}^0 = I$, the identity operator.

***Lemma** 5:* [4] Let $n \in \mathbb{R}_+$ and $m \in \mathbb{N}$ such that $m > n$. Then

$$D_{0+}^n = D_{0+}^m I_{0+}^{m-n}$$

***Definition** 4:* [17]. Let $(X, \|\ .\ \|)$ and $(W, \|\ .\ \|)$ be real Banach spaces.

A linear operator $L : dom\, L \subset X \to W$ is called a Fredholm operator of index zero provided that
(i) $Im\, L$ is a closed subset of W, and
(ii) $\dim \ker L = \text{codim Im} L < +\infty$.

***Definition** 5:* [22]. Let $L : dom\, L \subset X \to W$ be a Fredholm operator, then, there exist continuous projectors $P : X \to X$ and $\Pi : W \to W$ such that $\text{Im}\, P = \ker L$, $\ker \Pi = Im\, L$, $X = \ker L \oplus \ker P$, $W = Im\, L \oplus Im\, \Pi$ and the mapping $L\big|_{dom\, L \cap \ker P} : dom\, L \cap kerP \to Im\, L$ is invertible. We denote the inverse of $L\big|_{dom\, L \cap KerP}$ by $K_p : Im\, L \to dom\, L \cap \ker P$ and the generalized inverse of L is $K_{P,\Pi} : W \to dom\, L \cap \ker P$ where $K_{P,\Pi} = K_P(I - \Pi)$.

Theorem 6: [17]. Let $L : dom\, L \subset X \to W$ be a Fredholm operator of index zero and $N : X \to W$ is

L-compact on $\overline{\Omega}$. Suppose that the following conditions are satisfied.

(i) $Lx \neq \lambda Nx$ for any $x \in (dom\, L \setminus kerL) \cap \partial\Omega, \quad \lambda \in (0, 1)$;

(ii) $Nx \notin ImL$ for any $x \in \ker L \cap \partial\Omega$;

(iii) $\deg\left(\Pi N|_{\ker L}, \ker L \cap \Omega,\ 0\right) \neq 0$, where $\Pi : W \to W$ is a projection such that $ImL = \ker \Pi$. Then, the equation $Lx(t) = Nx(t)$ has at least one solution in $domL \cap \overline{\Omega}$.

II. Main Results

***Lemma** 6:* Suppose (H_1) holds. Then:

(i) $\ker L = \{x(t) \in \mathrm{dom}\, L : x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2}$, for all $t \in (0, +\infty),\ c_1, c_2 \in \mathbb{R}\}$.

(ii) $\mathrm{Im}L = \{w \in W : \Pi_1 w = 0 = \Pi_2 w\}$, where

$$\Pi_1 w = \sum_{i=1}^{m} \mu_i \int_0^{\xi_i} (\xi_i - s)w(s)ds,$$

$$\Pi_2 w = \int_0^{\infty} w(s)ds - \int_0^{\eta}\int_0^{t} (t-s)w(s)dsdA(t).$$

Proof: (i) Consider the homogeneous boundary value problem, $D^{\alpha}_{0+}x(t) = 0$. Since $3 < \alpha \leq 4$, let the solution $x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + c_4 t^{\alpha-4}$.
Using the initial condition $x(0) = 0 \Rightarrow c_4 = 0$.
Then, $x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}$.

$$\begin{aligned} D^{\alpha-3}_{0+}x(t) &= \frac{c_1\Gamma(\alpha)t^2}{\Gamma(3)} + \frac{c_2\Gamma(\alpha-1)t}{\Gamma(2)} + \frac{c_3\Gamma(\alpha-2)}{\Gamma(1)} \\ D^{\alpha-3}_{0+}x(0) &= 0 \Rightarrow c_3 = 0. \\ Hence,\ x(t) &= c_1 t^{\alpha-1} + c_2 t^{\alpha-2}. \end{aligned}$$

Apply the boundary condition to obtain

$$D^{\alpha-2}_{0+}x(t) = c_1\Gamma(\alpha)t + c_2\Gamma(\alpha-1)$$

$$D^{\alpha-2}_{0+}x(0) = c_2\Gamma(\alpha-1) = \sum_{i=1}^{m} \mu_i D^{\alpha-2}_{0+}x(\xi_i)$$

$$c_2\Gamma(\alpha-1) = c_1\Gamma(\alpha)\sum_{i=1}^{m}\mu_i\,\xi_i + c_2\Gamma(\alpha-1)\sum_{i=1}^{m}\mu_i.$$

$$c_2\Gamma(\alpha-1)\left(1 - \sum_{i=1}^{m}\mu_i\right) = c_1\Gamma(\alpha)\sum_{i=1}^{m}\mu_i\,\xi_i = 0.$$

$$\sum_{i=1}^{m}\mu_i = 1, \quad \sum_{i=1}^{m}\mu_i\xi_i = 0. \tag{4}$$

and

$$c_1\Gamma(\alpha)\left(1 - \int_0^{\eta} tdA(t)\right) = c_2\Gamma(\alpha-1)\int_0^{\eta} dA(t) = 0$$

.

$$\int_0^{\eta} tdA(t) = 1, \quad \int_0^{\eta_j} dA(t) = 0. \tag{5}$$

To prove (ii), suppose $w \in \mathrm{Im}\, L$, then there exists $x(t) \in \mathrm{dom}\, L$ such that

$$Lx(t) = w \tag{6}$$

Solving the equation (6)

$$x(t) = I^{\alpha}_{0+}w(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + c_4 t^{\alpha-4}. \tag{7}$$

Applying the initial condition $x(0) = 0 = D^{\alpha-3}_{0+}x(0)$ to equation (7) gives $c_3 = 0, c_4 = 0$, thus

$$x(t) = I^{\alpha}_{0+}w(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}. \tag{8}$$

Apply the boundary condition to equation (8),

$$\begin{aligned} D^{\alpha-2}_{0+}x(t) &= D^{\alpha-2}_{0+}[I^{\alpha}_{0+}w(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}] \\ &= \int_0^{t} (t-s)w(s)ds + c_1\Gamma(\alpha)t + c_2\Gamma(\alpha-1). \end{aligned}$$

From the boundary condition
$D^{\alpha-2}_{0+}x(0) = \sum_{i=1}^{m}\mu_i D^{\alpha-2}_{0+}x(\xi_i)$, we have

$$c_2\Gamma(\alpha-1) = \sum_{i=1}^{m}\mu_i\left(\int_0^{\xi_i}(\xi_i - s)w(s)ds + c_1\Gamma(\alpha)\xi_i + c_2\Gamma(\alpha-1)\right) \tag{9}$$

hence,

$$\sum_{i=1}^{m}\mu_i\int_0^{\xi_i}(\xi_i - s)w(s)ds = 0 = \Pi_1 w \tag{10}$$

To obtain $\Pi_2 w$, apply the boundary condition
$D^{\alpha-1}_{0+}x(+\infty) = \int_0^{\eta} D^{\alpha-2}_{0+}x(t)dA(t)$
to $x(t) = I^{\alpha}_{0+}w(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}$,
then, $D^{\alpha-1}_{0+}x(t) = I^{1}_{0+}w(t) + c_1\Gamma(\alpha) + \dfrac{c_2\Gamma(\alpha-1)}{t}$.

$$\begin{aligned} D^{\alpha-1}_{0+}x(+\infty) &= \lim_{t\to+\infty}\left(\int_0^{t} w(s)ds + c_1\Gamma(\alpha) + \frac{c_2\Gamma(\alpha-1)}{t}\right) \\ &= \int_0^{\eta}\int_0^{t}(t-s)w(s)dsdA(t) \\ &\quad + c_1\Gamma(\alpha)\int_0^{\eta} tdA(t) + c_2\Gamma(\alpha-1)\int_0^{\eta} dA(t). \end{aligned}$$

Hence,

$$\int_0^{\infty} w(s)ds + c_1\Gamma(\alpha) = \int_0^{\eta}\int_0^{t}(t-s)w(s)dsdA(t) + c_1\Gamma(\alpha).$$

and

$$\int_0^{\infty} w(s)ds - \int_0^{\eta}\int_0^{t}(t-s)w(s)dsdA(t) = 0 = \Pi_2 w. \tag{11}$$

Conversely, for any $w \in W$ satisfying (10) and (11), take $x(t) = I^{\alpha}_{0+}w(t)$, then $x(t) \in dom\, L$ and $D^{\alpha}_{0+}x(t) = w \in \mathrm{Im}\, L$. Thus, we have shown that $\mathrm{Im}\, L = \{w \in W : \Pi_1 w = \Pi_2 w = 0\}$. ■

***Definition** 7:* Determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} \Pi_1 e^{-t} & \Pi_2 e^{-t} \\ \Pi_1 te^{-t} & \Pi_2 te^{-t} \end{vmatrix} \tag{12}$$

We apply (10) and (11) to (12) to obtain

$$a_{11}=\sum_{i=1}^{m}\mu_i e^{-\xi_i}-1,\ a_{12}=-\int_0^{\eta}e^{-t}dA(t).$$

$$a_{21}=\sum_{i=1}^{m}\mu_i(2+\xi_i)e^{-\xi_i}-2,\ a_{22}=-\int_0^{\eta}(2+t)e^{-t}dA(t)$$

$\Delta=a_{11}a_{22}-a_{12}a_{21}$

***Definition** 8:* Let $\phi_1,\phi_2 \ :\ W \to W$ such that $\phi_1 w=\frac{1}{\Delta}(a_{22}\Pi_1 w-a_{21}\Pi_2 w)e^{-t}$, $\phi_2 w=\frac{1}{\Delta}(a_{11}\Pi_2 w-a_{12}\Pi_1 w)e^{-t}$. It is easy to show that: $\phi_1(\phi_1 w(t))=\phi_1(t)$, $\phi_1(\phi_2 w(t))=0$, $\phi_2(\phi_1 w(t))=0$ and $\phi_2(\phi_2 w(t))=\phi_2(t)$.

***Lemma** 7:* Suppose (H_1) holds, then $L: dom\,L\subset X\to W$ is a Fredholm operator of index zero.

Proof: We show that $\dim\ker L=\text{codim Im}L$.
Let $P:X\to X$ and $\Pi:W\to W$ be linear projections defined as

$$Px(t)=\frac{1}{\Gamma(\alpha)}D_{0+}^{\alpha-1}x(0)t^{\alpha-1}+\frac{1}{\Gamma(\alpha-1)}D_{0+}^{\alpha-2}x(0)t^{\alpha-2},$$

$\Pi w=\phi_1 w(t)+(\phi_2 w(t))t$, for any $x\in\ker L$.

$$P^2x(t)=P\big(Px(t)\big)=Px(t).$$

P is a continuous linear projection operator such that $\ker L=\text{Im}P$, $x=x-Px+Px$, where $Px\in\ker L$, $x-Px\in\ker P$ then $X=\ker P\oplus\ker L$. From the way the operators ϕ_1 and ϕ_2 are defined, Π is a linear operator. From (8), we can deduce that, given $\Pi w=\phi_1 w+(\phi_2 w)t$

$$\Pi^2 w=\Pi(\phi_1 w+(\phi_2 w)t=\phi_1(\Pi w)+\phi_2((\Pi w))t=\Pi w. \tag{13}$$

Thus, Π is a projection operator. Given $w\in W$ such that $w=\Pi w+(w-\Pi w)$, then $\Pi w\in Im\ \Pi$ and $\Pi(w-\Pi w)=\Pi w-\Pi^2 w=\Pi w-\Pi w=0$. Similarly, $\Pi_1(w-\Pi w)=\Pi_2(w-\Pi w)=0$.
Therefore, $(w-\Pi w)\in ImL=\ker\Pi$. If $w\in ImL\cap Im\,\Pi$, then $w=\Pi w=0$. Consequently, $W=Im\Pi\oplus ImL$ and $\dim\ker L=codim\ ImL=2$. Therefore, L is a Fredholm operator of index zero. ■

Let $(X,\|\cdot\|)$ and $(W,\|\cdot\|)$ be real Banach spaces. Let

$$X=\Big\{x(t): x(t), D_{0+}^{\alpha-3}x(t), D_{0+}^{\alpha-2}x(t),\ D_{0+}^{\alpha-1}x(t)\in C(0,+\infty), \sup_{t>0}\frac{|x(t)|}{1+t^{\alpha}}<+\infty,\ \sup_{t>0}\frac{|D_{0+}^{\alpha-3}x(t)|}{1+t^{2}}<+\infty, \sup_{t>0}\frac{|D_{0+}^{\alpha-2}x(t)|}{1+t^{\alpha-1}}<+\infty,\ \sup_{t>0}\frac{|D_{0+}^{\alpha-1}x(t)|}{1+t^{\alpha-2}}<+\infty\Big\}$$

and $W=L^1(0,+\infty)$ with norms
$\|x(t)\|_X=\max\{\|x(t)\|_0,\ \|D_{0+}^{\alpha-3}x(t)\|_0,\ \|D_{0+}^{\alpha-2}x(t)\|_0,\ \|D_{0+}^{\alpha-1}x(t)\|_0\}$, $\|w\|_W=\|w\|_{L^1}$ where
$\|x(t)\|_0=\sup_{t>0}\frac{|x(t)|}{1+t^{\alpha}}$, $\|D_{0+}^{\alpha-3}x(t)\|_0=\sup_{t>0}\frac{|D_{0+}^{\alpha-3}x(t)|}{1+t^{2}}$,
$\|D_{0+}^{\alpha-2}x(t)\|_0=\sup_{t>0}\frac{|D_{0+}^{\alpha-2}x(t)|}{1+t^{\alpha-1}}$,
$\|D_{0+}^{\alpha-1}x(t)\|_0=\sup_{t>0}\frac{|D_{0+}^{\alpha-1}x(t)|}{1+t^{\alpha-2}}$ and
$\|w\|_{L^1}=\int_0^{+\infty}|w(t)|dt$.

***Lemma** 8:* Let $L_p=L|_{dom\,L\cap\ker P}: dom\,L\cap\ker P\to ImL$ and $K_P: ImL\to domL\cap\ker P$ such that $K_P w=I_{0+}^{\alpha}w=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}w(s)ds$, $w\in ImL$, then, K_P is the inverse of L_P and $\|K_P w\|_X\le\|w\|_{L^1}$.

Proof: To Show that $K_P=L_P^{-1}$. Given any $w\in Im\,L\subset W$, let $K_P w=I_{0+}^{\alpha}w$. Then,
$(L_pK_P)w(t)=D_{0+}^{\alpha}(K_P w(t)=D_{0+}^{\alpha}I_{0+}^{\alpha}w(t)=w(t)$. For $x(t)\in dom\,L\cap\ker P$, we have

$$(K_PL_P)x(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}D_{0+}^{\alpha}x(s)ds =x(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}.$$

It follows that $P(K_PL_Px(t))=0$ since $x(t)=c_1t^{\alpha-1}+c_2t^{\alpha-2}\in\ker L=ImP$.
$Px(t)=x(t)$ and $(K_PL_P)x(t)=x(t)-Px(t)$. Therefore, $x(t)\in dom\,L\cap\ker P$, hence $(K_PL_P)x(t)=x(t)$. Thus, K_P is the inverse of $L|_{dom\,L\cap\ kerP}=L_P$.
Next we show that $\|K_P w\|_X\le\|w\|_{L^1}$.

$$\|K_P w\|_0=\sup_{t>0}\frac{1}{\Gamma(\alpha)}\left|\int_0^t\frac{(t-s)^{\alpha-1}}{1+t^{\alpha}}w(s)ds\right| \le\frac{1}{\Gamma(\alpha)}\|w\|_{L^1}\le\|w\|_{L^1},$$

$$\|D_{0+}^{\alpha-3}K_P w\|_0=\sup_{t>0}\frac{1}{\Gamma(3)}\left|\int_0^t\frac{(t-s)^{2}}{1+t^{2}}w(s)ds\right| \le\|y\|_{L^1},$$

$$\|D_{0+}^{\alpha-2}K_P w\|_0=\sup_{t>0}\frac{1}{\Gamma(2)}\left|\int_0^t\frac{(t-s)}{1+t^{\alpha-1}}w(s)ds\right|\le\|w\|_{L^1},$$

and

$$\|D_{0+}^{\alpha-1}K_P w\|_0=\sup_{t>0}\left|\int_0^t\frac{w(s)}{1+t^{\alpha-2}}ds\right|\le\|w\|_{L^1}.$$

We conclude that $\|K_P w\|_X\le\|w\|_{L^1}$ for any $w\in ImL$. ■

***Lemma** 9:* Assume that (H_3) holds and $\Omega\subset X$ is an open bounded subset such that $dom\,L\cap\overline{\Omega}\neq\emptyset$, then N is L-compact on $\overline{\Omega}$ where $N:\overline{\Omega}\to W$.

Proof: We first show that $\Pi N(\overline{\Omega})$ is bounded. Given that Ω is bounded in X, there exists a constant $M>0$ such that $\|x\|_X\le M$ for any $x\in\overline{\Omega}$. Then by (H_3), we obtain

$$\begin{aligned}|\Pi_1 Nx|&=\Big|\sum_{i=1}^{m}\mu_i\int_0^{\xi_i}(\xi_i-s)f\big(s,x(s),D_{0+}^{\alpha-3}x(s),D_{0+}^{\alpha-2}x(s),D_{0+}^{\alpha-1}x(s)\big)ds\Big|\\&\le\sum_{i=1}^{m}|\mu_i|\int_0^{+\infty}\Big(\Big|\rho_1(s)\frac{|x(s)|}{1+s^{\alpha}}+\rho_4(s)\frac{|D_{0+}^{\alpha-1}x(s)|}{1+s^{\alpha-2}}+\rho_5(s)\Big|\Big)ds\\&\le\sum_{i=1}^{m}|\mu_i|\Big(\Big(\|\rho_1\|_{L^1},\|\rho_2\|_{L^1},\|\rho_3\|_{L^1},\|\rho_4\|_{L^1}\Big)\max\{\|x(t)\|_0,\|D_{0+}^{\alpha-3}x(t)\|_0,\|D_{0+}^{\alpha-2}x(t)\|_0,\|D_{0+}^{\alpha-1}x(t)\|_0\}+\rho_5\|_{L^1}\Big)\\&\le\Theta|x\|_X+\|\rho_5\|_{L^1}:=M_1,\end{aligned}$$

$$\begin{aligned}\big|\Pi_2 Nx\big| &= \Big|\int_0^\infty f\big(s,x(s),D_{0+}^{\alpha-3}x(s),D_{0+}^{\alpha-2}x(s),D_{0+}^{\alpha-1}x(s)\\&\quad\big)ds-\int_0^\eta\int_0^t (t-s)f\Big(s,x(s),D_{0+}^{\alpha-3}x(s),D_{0+}^{\alpha-2}x(s),\\&\quad D_{0+}^{\alpha-1}x(s)\Big)dsdA(t)\Big|\\&\le\Big|\int_0^\infty f\big(s,x(s),D_{0+}^{\alpha-3}x(s),D_{0+}^{\alpha-2}x(s),D_{0+}^{\alpha-1}x(s)\\&\quad\big)ds\Big|+\Big|\int_0^\eta\int_0^t(t-s)f\Big(s,x(s),D_{0+}^{\alpha-3}x(s),\\&\quad D_{0+}^{\alpha-2}x(s),D_{0+}^{\alpha-1}x(s)\Big)dsdA(t)\Big|\\&\le\int_0^\infty\Big|f\Big(s,x(s),D_{0+}^{\alpha-3}x(s),D_{0+}^{\alpha-2}x(s),D_{0+}^{\alpha-1}x(s)\\&\quad\Big)\Big|ds+\int_0^\eta\int_0^t\Big|f\Big(s,x(s),D_{0+}^{\alpha-2}x(s),D_{0+}^{\alpha-2}x(s),\\&\quad D_{0+}^{\alpha-1}x(s)\Big)\Big|dsdA(t)\\&\le\Theta\|x\|_X+\|\rho_5\|_{L^1}\\&\quad+\int_0^\eta\int_0^t\Big|f\Big(s,x(s),D_{0+}^{\alpha-3}x(s),D_{0+}^{\alpha-2}x(s),\\&\quad D_{0+}^{\alpha-1}x(s)\Big)\Big|dsdA(t)\\&\le\Theta\|x\|_X+\|\rho_5\|_{L^1}\\&\quad+\int_0^\eta\int_0^\infty\Big|f\Big(s,x(s),D_{0+}^{\alpha-3}x(s),D_{0+}^{\alpha-2}x(s),\\&\quad+D_{0+}^{\alpha-1}x(s)\Big)\Big|dsdA(t)\\&\le M_1+\int_0^\eta M_1 dA(t)\\&=M_1\text{ since }\int_0^\eta dA(t)=0\end{aligned}$$

Thus,

$$\begin{aligned}\|\Pi Nx\|_{L^1}&=\int_0^{+\infty}|\Pi Nx(s)|ds\\&\le\int_0^{+\infty}|\Pi_1 Nx(s)|ds+\int_0^{+\infty}|\Pi_2 Nx(s)s|ds\\&\le\frac{1}{|\Delta|}(|a_{22}|M_1+|a_{21}|M_1)\\&\quad+\frac{1}{|\Delta|}(|a_{12}|M_1+|a_{11}|M_1)\\&=\frac{1}{|\Delta|}(|a_{11}|+|a_{12}|+|a_{21}|+|a_{22}|)M_1:=M.\end{aligned}$$

Thus, $\Pi N(\overline{\Omega})$ is bounded.
Next we prove that $K_{P,\Pi}N(\overline{\Omega})$ is compact on $(0,+\infty)$. It is sufficient to show that $K_{P,\Pi}N(\overline{\Omega})$ is:

(i) bounded;
(ii) equicontinuous on any subcompact interval of $(0,+\infty)$;
(iii) equiconvergent at infinity.

(i). Given any $x\in\overline{\Omega}$,

$$Nx(t)=f\left(s,x(s),D_{0+}^{\alpha-3}x(s),D_{0+}^{\alpha-2}x(s),D_{0+}^{\alpha-1}x(s)\right),$$

$$\begin{aligned}\|Nx(t)\|_{L^1}&=\int_0^{+\infty}|f(s,x(s),D_{0+}^{\alpha-3}x(s),D_{0+}^{\alpha-2}x(s),\\&\quad D_{0+}^{\alpha-1}x(s))|ds\\&\le\Theta\|x\|_X+\|\rho_5\|_{L^1}:=M_1.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{|K_{P,\Pi}Nx(t)}{1+t^\alpha}|&=\Big|\frac{1}{\Gamma(\alpha)}\int_0^t\frac{(t-s)^{\alpha-1}}{1+t^\alpha}(I-\Pi)Nx(s))ds\Big|\\&\le\frac{1}{\Gamma(\alpha)}\int_0^{+\infty}(|Nx(s)|+|\Pi Nx(s)|)ds\\&\le\frac{1}{\Gamma(\alpha)}(M_1+M),\end{aligned}$$

$$\begin{aligned}\frac{|D_{0+}^{\alpha-3}K_{P,\Pi}Nx(t)|}{1+t^2}&=\Big|\frac{1}{\Gamma(3)}\int_0^t\frac{(t-s)^{3-1}}{1+t^2}(I-\Pi)Nx(s)ds\Big|\\&\le\int_0^{+\infty}(|Nx(s)|+|\Pi Nx(s)|)ds\\&=(\|Nx(t)\|_{L^1}+\|\Pi Nx(t)\|_{L^1})\\&\le M_1+M.\end{aligned}$$

$$\begin{aligned}\frac{|D_{0+}^{\alpha-2}K_{P,\Pi}Nx(t)|}{1+t^{\alpha-1}}&=\Big|\frac{1}{\Gamma(2)}\int_0^t\frac{(t-s)^{2-1}}{1+t^{\alpha-1}}(I-\Pi)Nx(s)ds\Big|\\&\le\int_0^{+\infty}(|Nx(s)|+|\Pi Nx(s)|)ds\\&\le M_1+M\end{aligned}$$

and

$$\begin{aligned}\frac{|D_{0+}^{\alpha-1}K_{P,\Pi}Nx(t)|}{1+t^{\alpha-2}}&=\Big|\int_0^t\frac{1}{1+t^{\alpha-2}}(I-\Pi)Nx(s)ds\Big|\\&\le\int_0^t|(I-\Pi)Nx(s)|ds\\&=(\|Nx(t)\|_{L^1}+\|\Pi Nx(t)\|_{L^1})\\&\le M_1+M.\end{aligned}$$

Therefore, $K_{P,\Pi}N(\overline{\Omega})$ is bounded.
(ii) Next, we show that $K_{P,\Pi}N$ is equicontinous on any subcompact interval of $(0,+\infty)$. Let $x\in\overline{\Omega}$, by hypothesis (H_3),

$$\begin{aligned}|Nx(s)|=|f(s,p,q,r,v)|&\le\rho_1(s)\frac{|p|}{1+s^\alpha}+\rho_2(s)\frac{|q|}{1+s^2}\\&\quad+\rho_3(s)\frac{|r|}{1+s^{\alpha-1}}+\rho_4(s)\frac{|v|}{1+s^{\alpha-2}}+\rho_5(s)\\&=\rho_1(s)\frac{|x(s)|}{1+s^\alpha}+\rho_2(s)\frac{|D_{0+}^{\alpha-3}x(s)|}{1+s^2}\\&\quad+\rho_3(s)\frac{|D_{0+}^{\alpha-2}x(s)|}{1+s^{\alpha-1}}+\rho_4(s)\frac{|D_{0+}^{\alpha-1}x(s)|}{1+s^{\alpha-2}}+\rho_5(s).\end{aligned}$$

Suppose $\omega>0$ is any real number in $(0,+\infty)$.
Let $t_1,\ t_2\in[0,\omega]$ such that $t_1<t_2$, then

$$\begin{aligned}&\Big|\frac{K_{P,\Pi}Nx(t_2)}{1+t_2^\alpha}-\frac{K_{P,\Pi}Nx(t_1)}{1+t_1^\alpha}\Big|\\&=\Big|\frac{1}{\Gamma(\alpha)}\Big(\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^\alpha}(I-\Pi)Nx(s)ds\\&\quad-\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{1+t_1^\alpha}(I-\Pi)Nx(s)ds\Big)\Big|\\&\le\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\Big|\frac{(t_2-s)^{\alpha-1}}{1+t_2^\alpha}\Big||(I-\Pi)Nx(s)|ds\\&\quad+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\Big|\frac{(t_2-s)^{\alpha-1}}{1+t_2^\alpha}\\&\quad-\frac{(t_1-s)^{\alpha-1}}{1+t_2^\alpha}\Big||(I-\Pi)Nx(s)|ds\to0\\&\quad\text{as }t_1\to t_2.\end{aligned}$$

$$\begin{aligned}&\left|\frac{D_{0+}^{\alpha-3}K_{P,\Pi}Nx(t_2)}{1+t_2^2}-\frac{D_{0+}^{\alpha-3}K_{P,\Pi}Nx(t_1)}{1+t_1^2}\right|\\&=\frac{1}{\Gamma(3)}\Big(\Big|\int_0^{t_2}\frac{(t_2-s)^2}{1+t_2^2}(I-\Pi)Nx(s)ds\\&-\int_0^{t_1}\frac{(t_1-s)^2}{1+t_1^2}(I-\Pi)Nx(s)ds\Big|\Big)\\&\le\int_{t_1}^{t_2}\left|\frac{(t_2-s)^2}{1+t_2^2}\right||(I-\Pi)Nx(s)|ds\\&+\int_0^{t_1}\left|\frac{(t_2-s)^2}{1+t_2^2}-\frac{(t_1-s)^2}{1+t_1^2}\right|\\&|I-\Pi)Nx(s)|ds\quad\to 0 \text{ as } t_1\to t_2\end{aligned}$$

$$\begin{aligned}&\left|\frac{D_{0+}^{\alpha-2}K_{P,\Pi}Nx(t_2)}{1+t_2^{\alpha-1}}-\frac{D_{0+}^{\alpha-2}K_{P,\Pi}Nx(t_1)}{1+t_1^{\alpha-1}}\right|\\&=\Big|\int_0^{t_2}\frac{t_2-s}{1+t_2^{\alpha-1}}(I-\Pi)Nx(s)ds\\&-\int_0^{t_1}\frac{t_1-s}{1+t_1^{\alpha-1}}(I-\Pi)Nx(s)ds\Big|\\&\le\int_{t_1}^{t_2}\left|\frac{t_2-s}{1+t_2^{\alpha-1}}\right||(I-\Pi)Nx(s)|ds\\&+\int_0^{t_1}\left|\frac{t_2-s}{1+t_2^{\alpha-1}}-\frac{t_1-s}{1+t_1^{\alpha-1}}\right|\\&|I-\Pi)Nx(s)|ds\quad\to 0 \text{ as } t_1\to t_2\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{D_{0+}^{\alpha-1}K_{P,\Pi}Nx(t_2)}{1+t_2^{\alpha-2}}-\frac{D_{0+}^{\alpha-1}K_{P,\Pi}Nx(t_1)}{1+t_1^{\alpha-2}}\right|\\&=\Big|\int_0^{t_2}\frac{(I-\Pi)Nx(s)}{1+t_2^{\alpha-2}}ds\\&-\int_0^{t_1}\frac{(I-\Pi)Nx(s)}{1+t_1^{\alpha-2}}ds\Big|\\&\le\int_{t_1}^{t_2}\left|\frac{(I-\Pi)Nx(s)}{1+t_2^{\alpha-2}}\right|ds\\&+\int_0^{t_1}\left|\frac{1}{1+t_2^{\alpha-2}}-\frac{1}{1+t_1^{\alpha-2}}\right|\\&|(I-\Pi)Nx(s)|ds\quad\to 0 \text{ as } t_1\to t_2.\end{aligned}$$

Thus, $K_{P,\Pi}N(\overline{\Omega})$ is equicontinuous.
(iii) To show that $K_{P,\Pi}N(\overline{\Omega})$ is equiconvergent at infinity, consider

$$\begin{aligned}\int_0^{+\infty}|(I-\Pi)Nx(t)|dt&\le\int_0^{+\infty}|Nx(t)|dt\\&+\int_0^{+\infty}|\Pi Nx(t)|dt\\&\le\|Nx(t)\|_{L^1}+\|\Pi Nx(t)\|_{L^1}\\&=M_1+M.\end{aligned}$$

Thus, given $\epsilon>0$ there exists a positive real number K such that

$$\int_K^{+\infty}|(I-\Pi)Nx(t)|dt<\epsilon.$$

With the given $\epsilon>0$ there exists a constant $K_1>K>0$ such that for any $t_1,\ t_2\ge K_1$ and $0\le s\le K$,

$$\lim_{t\to\infty}\frac{(t-K)^{\alpha-1}}{1+t^\alpha}=0,\ \lim_{t\to\infty}\frac{(t-K)^2}{1+t^2}=1,$$

$$\lim_{t\to\infty}\frac{(t-K)}{1+t^{\alpha-1}}=0,\ \text{and}\ \lim_{t\to\infty}\frac{1}{1+t^{\alpha-2}}=0.$$

$$\left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^\alpha}-\frac{(t_2-s)^{\alpha-1}}{1+t_2^\alpha}\right|<\epsilon,\ \left|\frac{(t_1-K)^2}{1+t_1^2}-\frac{(t_2-K)^2}{1+t_2^2}\right|<\epsilon,$$

$$\left|\frac{(t_1-K)}{1+t_1^{\alpha-1}}-\frac{(t_2-K)}{1+t_2^{\alpha-1}}\right|<\epsilon\ \text{and}\left|\frac{1}{1+t_1^{\alpha-2}}-\frac{1}{1+t_2^{\alpha-2}}\right|<\epsilon.$$

Therefore, for any $t_1,t_2\ge K_1>K>0$,

$$\begin{aligned}&\left|\frac{K_{P,\Pi}Nx(t_1)}{1+t_1^\alpha}-\frac{K_{P,\Pi}Nx(t_2)}{1+t_2^\alpha}\right|\\&\le\frac{1}{\Gamma(\alpha)}\Big|\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{1+t_1^\alpha}(I-\Pi)Nx(s)ds\\&-\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^\alpha}(I-\Pi)Nx(s)ds\Big|\\&\le\frac{\epsilon}{\Gamma(\alpha)}\int_0^K|(I-\Pi)Nx(s)|ds\\&+\frac{2\epsilon}{\Gamma(\alpha)}\int_K^{+\infty}|(I-\Pi)Nx(s)|ds\\&<\frac{\epsilon}{\Gamma(\alpha)}(M_1+M)+\frac{2\epsilon}{\Gamma(\alpha)}=\frac{(M_1+M+2)\epsilon}{\Gamma(\alpha)}.\end{aligned}$$

Similarly, we establish that

$$\left|\frac{D_{0+}^{\alpha-2}K_{P,\Pi}Nx(t_1)}{1+t_1^{\alpha-1}}-\frac{D_{0+}^{\alpha-2}K_{P,\Pi}Nx(t_2)}{1+t_2^{\alpha-1}}\right|<(M_1+M+2)\epsilon$$

$$\left|\frac{D_{0+}^{\alpha-3}K_{P,\Pi}Nx(t_1)}{1+t_1^2}-\frac{D_{0+}^{\alpha-3}K_{P,\Pi}Nx(t_2)}{1+t_2^2}\right|<\frac{(M_1+M+2)\epsilon}{2}$$

and

$$\begin{aligned}&\left|\frac{D_{0+}^{\alpha-1}K_{P,\Pi}Nx(t_1)}{1+t_1^{\alpha-2}}-\frac{D_{0+}^{\alpha-1}K_{P,\Pi}Nx(t_2)}{1+t_2^{\alpha-2}}\right|\\&\le\int_K^{+\infty}\left|\frac{1}{1+t_1^{\alpha-2}}-\frac{1}{1+t_2^{\alpha-2}}\right||(I-\Pi)Nx(s)|ds\\&<(M_1+M)\epsilon.\end{aligned}$$

We conclude that $K_{P,\Pi}N(\overline{\Omega})$ is equiconvergent at infinity. Hence, it follows that $K_{P,\Pi}N(\overline{\Omega})$ is relatively compact. Hence, N is L-compact on $\overline{\Omega}$. ■

Lemma 10: Suppose that (H_3) and (H_4) hold, then the set $\Omega_1=\{x\in dom\,L|\ker L:Lx(t)=\lambda Nx(t),\ \lambda\in(0,1)\}$ is bounded in X provided $H\Theta<1$.

Proof: Let $x\in\Omega_1$ and $Nx\in ImL$

$$\Pi_1Nx(t)=\Pi_2Nx(t)=0.$$

Hence, from assumption (H_4), there exist $t_0\in(0,A_2]$ and $t_1\in(A_2,+\infty)$ such that $|D_{0+}^{\alpha-3}x(t_0)|\le A_1$, $|D_{0+}^{\alpha-2}x(t_0)|\le A_1$ and $|D_{0+}^{\alpha-1}x(t_1)|\le A_1$. Combining this with the additive rule of fractional derivative,

$$\begin{aligned}|D_{0+}^{\alpha-1}x(t)|&=\left|D_{0+}^{\alpha-1}x(t_1)+\int_{t_1}^tD_{0+}^\alpha x(s)ds\right|\\&\le A_1+\int_{t_1}^t|Nx(s)|ds\\&=A_1+\|Nx\|_{L^1}\end{aligned}\quad. \tag{14}$$

$$\begin{aligned}|D_{0+}^{\alpha-2}x(0)| &= |D_{0+}^{\alpha-2}x(t_0) - D^{-1}D^{\alpha-1}x(t)| \\ &\leq |D_{0+}^{\alpha-2}x(t_0)| + \left|\int_0^{t_0} D_{x+}^{\alpha-1}x(s)ds\right| \\ &= 2A_1 + \|Nx\|_{L^1}\end{aligned} \tag{15}$$

$$\begin{aligned}|D_{0+}^{\alpha-3}x(0)| &= |D_{0+}^{\alpha-3}x(t_0) - D^{-1}D^{\alpha-2}x(t)| \\ &\leq |D_{0+}^{\alpha-3}x(t_0)| + \left|\int_0^{t_0} D_{x+}^{\alpha-2}x(s)ds\right| \\ &= 3A_1 + \|Nx\|_{L^1}\end{aligned} \tag{16}$$

From the definition of P, we get

$$\begin{aligned}\|Px(t)\|_0 &= \sup_{t>0}\frac{1}{1+t^\alpha}\left|\frac{1}{\Gamma(\alpha)}D_{0+}^{\alpha-1}x(0)t^{\alpha-1}\right. \\ &\quad \left.+\frac{1}{\Gamma(\alpha-1)}D_{0+}^{\alpha-2}x(0)t^{\alpha-2}\right| \\ &\leq \frac{1}{\Gamma(\alpha)}(A_1+\|Nx\|_{L^1}) \\ &\quad + \frac{1}{\Gamma(\alpha-1)}(2A_1+\|Nx\|_{L^1})\end{aligned} \tag{17}$$

$$\begin{aligned}\|D_{0+}^{\alpha-1}Px(t)\|_0 &= \sup_{t>0}\frac{|D_{0+}^{\alpha-1}x(t)|}{1+t^{\alpha-2}} \\ &\leq A_1 + \|Nx\|_{L^1}\end{aligned} \tag{18}$$

$$\begin{aligned}\|D_{0+}^{\alpha-2}Px(t)\|_0 &= \sup_{t>0}\frac{|D_{0+}^{\alpha-1}x(0)t + D_{0+}^{\alpha-2}x(0)|}{1+t^{\alpha-1}} \\ &\leq A_1 + \|Nx\|_{L^1} + 2A_1 + \|Nx\|_{L^1}\end{aligned} \tag{19}$$

and

$$\begin{aligned}\|D_{0+}^{\alpha-3}Px(t)\|_0 &= \sup_{t>0}\frac{\frac{1}{2}|D_{0+}^{\alpha-1}x(0)t^2 + D_{0+}^{\alpha-2}x(0)t|}{1+t^2} \\ &\leq \frac{1}{2}(A_1+\|Nx\|_{L^1}) + 2A_1 + \|Nx\|_{L^1}\end{aligned} \tag{20}$$

$$\begin{aligned}\|Px(t)\|_X &= \max\{\|Px(t)\|_0, \|D_{0+}^{\alpha-3}Px(t)\|_0, \\ &\qquad \|D_{0+}^{\alpha-2}Px(t)\|_0, \|D_{0+}^{\alpha-1}Px(t)\|_0\} \\ &\leq \|Px(t)\|_0 + \|D_{0+}^{\alpha-3}Px(t)\|_0 \\ &\quad + \|D_{0+}^{\alpha-2}Px(t)\|_0 + \|D_{0+}^{\alpha-1}Px(t)\|_0 \\ &= \left(\frac{5}{2}+\frac{1}{\Gamma(\alpha)}\right)(A_1+\|Nx\|_{L^1}) \\ &\quad + \left(2+\frac{1}{\Gamma(\alpha-1)}\right)(2A_1+\|Nx\|_{L^1}).\end{aligned} \tag{21}$$

Observe that $(I-P)x \in dom\,L \cap \ker P$ and $LPx = 0$. By definition, the operator $K_P : ImL \to dom\,L \cap \ker P$ is such that for any $w \in ImL$, $K_P w = I_{0+}^{\alpha} w$. Thus

$$\begin{aligned}\|(I-P)x\|_X &= \|K_P L(I-P)x\|_X \\ &\leq \|L(I-P)x\|_{L^1} \\ &= \|Lx\|_{L^1} \\ &\leq \|Nx\|_{L^1}.\end{aligned} \tag{22}$$

Combining (21) and (22) we get

$$\begin{aligned}\|x\|_X &= \|Px\|_X + \|(I-P)x\|_X \\ &\leq \left(\frac{5}{2}+\frac{1}{\Gamma(\alpha)}\right)(A_1+\|Nx\|_{L^1}) \\ &\quad + \left(2+\frac{1}{\Gamma(\alpha-1)}\right)(2A_1+\|Nx\|_{L^1}) + \|Nx\|_{L^1} \\ &= \left(\frac{13}{2}+\frac{1}{\Gamma(\alpha)}+\frac{2}{\Gamma(\alpha-1)}\right)A_1 \\ &\quad + \left(\frac{11}{2}+\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha-1)}\right)\|Nx\|_{L^1} \\ &= GA_1 + H\|Nx\|_{L^1} \leq GA_1 + H(\Theta\|x\|_X + \|\rho_5\|_{L^1})\end{aligned}$$

Hence,

$$\begin{aligned}(1-H\Theta)\|x\|_X &\leq GA_1 + H\|\rho_5\|_{L^1} \\ \|x\|_X &\leq \frac{GA_1 + H\|\rho_4\|_{L^1}}{1-H\Theta}\end{aligned}$$

where

$$G = \left(\frac{13}{2}+\frac{1}{\Gamma(\alpha)}+\frac{2}{\Gamma(\alpha-1)}\right)$$

and

$$H = \left(\frac{11}{2}+\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha-1)}\right)$$

. Hence, Ω_1 is bounded provided $H\Theta < 1$ ∎

Lemma 11: Suppose that (H_5) holds, then the set $\Omega_2 = \{x \in \ker L : Nx \in ImL\}$ is bounded in X.

Proof: Let $x \in \Omega_2$, where $x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2}$, $c_1,\ c_2 \in \mathbb{R}$ and $\Pi_1 Nx(t) = \Pi_2 Nx(t) = 0$. Since $Nx \in ImL = \ker \Pi$. By (H_5), it follows that $|c_1| \leq B$ and $|c_2| \leq B$.

$$\|D_{0+}^{\alpha-1}x(t)\|_0 = \sup_{t>0}\frac{|D_{0+}^{\alpha-1}x(t)|}{1+t^{\alpha-2}} < |c_1\Gamma(\alpha)| \leq B\Gamma(\alpha)$$

$$\|x\|_0 = \sup_{t>0}\frac{|x(t)|}{1+t^\alpha} < |c_1| + |c_2| \leq 2B.$$

$$\begin{aligned}\|D_{0+}^{\alpha-2}x(t)\|_0 &= \sup_{t>0}\frac{|D_{0+}^{\alpha-2}x(t)|}{1+t^{\alpha-1}} \\ &\leq |c_1|\Gamma(\alpha) + |c_2|\Gamma(\alpha-1) \\ &= (\Gamma(\alpha)+\Gamma(\alpha-1))B.\end{aligned}$$

$$\begin{aligned}\|D_{0+}^{\alpha-3}x(t)\|_0 &= \sup_{t>0}\frac{|D_{0+}^{\alpha-3}x(t)|}{1+t^2} \\ &\leq \frac{B}{2}\Gamma(\alpha) + B\Gamma(\alpha-1) \\ &= \left(\frac{\Gamma(\alpha)}{2}+\Gamma(\alpha-1)\right)B.\end{aligned}$$

$$\begin{aligned}\|x\|_X &= \max\{\|x\|_0, \|D_{0+}^{\alpha-3}x\|_0, \|D_{0+}^{\alpha-2}x\|_0, \\ &\qquad \|D_{0+}^{\alpha-1}x\|_0\} \\ &\leq 2B + \left(\frac{\Gamma(\alpha)}{2}+\Gamma(\alpha-1)\right)B \\ &\quad + (\Gamma(\alpha)+\Gamma(\alpha-1))B + B\Gamma(\alpha) \\ &= [2+\frac{5}{2}\Gamma(\alpha)+2\Gamma(\alpha-1)]B.\end{aligned}$$

We conclude that Ω_2 is bounded in X. ∎

***Lemma** 12:* Suppose that the assumption (H_5) holds, then the set

$$\Omega_3 = \{x \in \ker L : \nu\lambda Jx(t) + (1-\lambda)\Pi Nx(t) = 0,\ \lambda \in [0,1]\}.$$

is bounded in X, where $\nu = \begin{cases} -1, & \text{if } 2 \text{ holds} \\ +1, & \text{if } 3 \text{ holds} \end{cases}$,
and $J : \ker L \to Im\Pi$ is a linear isomorphism defined by

$$J(c_1 t^{\alpha-1} + c_2 t^{\alpha-2}) = \frac{1}{\Delta}(a_{22}|c_1| - a_{21}|c_2|)e^{-t} + \frac{1}{\Delta}(-a_{12}|c_1| + a_{11}|c_2|)te^{-t},$$
$$c_1, c_2 \in \mathbb{R}.$$

Proof: If (H_5) holds, $x \in \Omega_3$ can be written as $x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2}$, with $c_1, c_2 \in \mathbb{R}$.
If $\nu = -1, \lambda Jx(t) = (1-\lambda)\Pi Nx(t),\ \lambda \in [0,1]$. By using similar argument as in the proof of lemma 11, it is required only to show that $|c_1| \le B$ and $|c_2| \le B$.
For instance, in $\lambda Jx(t) = (1-\lambda)\Pi Nx(t) = 0,\ \lambda \in [0,1]$; if $\lambda = 0$, then $\Pi Nx(t) = 0$, then

$$\frac{1}{\Delta}\big(a_{22}\Pi_1 Nx(t) - a_{21}\Pi_2 Nx(t)\big)e^{-t} + \frac{1}{\Delta}\big(-a_{12}\Pi_1 Nx(t) + a_{11}\Pi_2 Nx(t)\big)te^{-t} = 0.$$

Thus,

$$\begin{cases} a_{22}\Pi_1 Nx(t) - a_{21}\Pi_2 Nx(t) = 0, \\ -a_{12}\Pi_1 Nx(t) + a_{11}\Pi_2 Nx(t) = 0. \end{cases}$$

Since $\Delta \ne 0$, then $\Pi_1 Nx(t) = 0 = \Pi_2 Nx(t)$.
By assumption (H_5), we have $|c_1| \le B$, $|c_2| \le B$.
Suppose $\lambda = 1$ then $Jx(t) = 0$, thus

$$\frac{1}{\Delta}(a_{22}|c_1| - a_{21}|c_2|)e^{-t} + \frac{1}{\Delta}(-a_{12}|c_1| + a_{11}|c_2|)te^{-t} = 0.$$

Since $\Delta \ne 0$, it follows that

$$\begin{cases} a_{22}|c_1| - a_{21}|c_2| = 0, \\ -a_{12}|c_1| + a_{11}|c_2| = 0 \end{cases}$$

and we get $c_1 = c_2 = 0$.
With $\lambda \in (0,1)$, by the equation $\lambda Jx(t) = (1 - \lambda\Pi Nx(t))$, we obtain

$$\lambda\left[\frac{1}{\Delta}(a_{22}|c_1| - a_{21}|c_2|)e^{-t} + \frac{1}{\Delta}(-a_{12}|c_1| + a_{11}|c_2|)te^{-t}\right] = (1-\lambda)\left[\frac{1}{\Delta}(a_{22}\Pi_1 Nx(t) - a_{21}\Pi_2 Nx(t)e^{-t} + \frac{1}{\Delta}(-a_{12}\Pi_1 Nx(t) + a_{11}\Pi_2 Nx(t)te^{-t}\right]$$

from which we obtain

$$\begin{cases} \lambda a_{22}|c_1| - \lambda a_{21}|c_2| & = (1-\lambda)a_{22}\Pi_1 Nx(t) \\ & \quad -(1-\lambda)a_{21}\Pi_2 Nx(t), \\ -\lambda a_{12}|c_1| + \lambda a_{11}|c_2| & = -(1-\lambda)a_{12}\Pi_1 Nx(t) \\ & \quad +(1-\lambda)a_{11}\Pi_2 Nx(t). \end{cases}$$

Since $\Delta \ne 0$, then

$$\begin{cases} \lambda|c_1| = (1-\lambda)\Pi_1 Nx(t), \\ \lambda|c_2| = (1-\lambda)\Pi_2 Nx(t) \end{cases}$$

Then, if $|c_1| > B$ and $|c_2| > B$, then by (2)

$$0 < \lambda(|c_1| + |c_2|) = (1-\lambda)(\Pi_1 Nx(t) + \Pi_2 Nx(t)) < 0$$

a contradiction.
If $\nu = +1$, then $\lambda Jx(t) = -(1-\lambda)\Pi Nx(t)), \lambda \in [0,1]$. We claim that $|c_1| < B$ and $|c_2| < B$.
If this claim would not hold, then by (3),

$$0 < \lambda(|c_1| + |c_2|) = -(1-\lambda)(\Pi_1 Nx(t) + \Pi_2 Nx(t)) < 0$$

is also a contradiction. Hence, Ω_3 is bounded in X. ■

Theorem 9: Assume $(H_1) - (H_5)$ hold, then, the BVP (1) has at least one solution in X provided $H\Theta < 1$ where $H = \left(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)} + \frac{11}{2}\right)$ and $\Theta = \|\rho_1(t)\|_1 + \|\rho_2(t)\|_1 + \|\rho_3(t)\|_1 + \|\rho_4(t)\|_1$

Proof: . Assume $\Omega \subset X$ is a bounded open set such that $\cup_{i=1}^{3}\Omega_i \subset \Omega$, $i = 1,2,3$, then by lemma 9 we have shown that N is L-compact on $\overline{\Omega}$. By applying lemma 10 and lemma 11, we get

(i) $Lx(t) \ne \lambda Nx(t)$ for any $x \in (dom\, L|_{\ker L}) \cap \partial\Omega$, $\lambda \in (0,1)$;
(ii) $Nx(t) \notin ImL$ for any $x \in \ker L \cap \partial\Omega$ where $\partial\Omega$ is the boundary of Ω

Lastly, we show that $\deg\{\Pi N|_{\ker L}, \Omega \cap \ker L, 0\} \ne 0$. To verify this, we define

$$H(x,\lambda) = \lambda Jx(t) + (1-\lambda)\Pi Nx(t).$$

From lemma 11, we assert that $H(x,\lambda) \ne 0$ i.e. if $\lambda = 0$, then $H(x,0) = \Pi Nx(t)$.
If $\lambda = 1$, we obtain $H(x,1) = Jx(t)$.
For any $x \in \ker L \cap \partial\Omega,\ \lambda \in [0,1]$, by the homotopy property of the Brouwer degree, we obtain

$$\begin{aligned} \deg\{\Pi N|_{\ker L}, \Omega \cap \ker L, 0\} &= \deg\{H(\cdot,0), \Omega \cap \ker L, 0\} \\ &= \deg\{H(\cdot,1), \Omega \cap \ker L, 0\} \\ &= \deg\{\nu J, \Omega \cap \ker L, 0\} \\ &\ne 0. \end{aligned}$$

By theorem 6, it follows that $Lx(t) = Nx(t)$ has at least one solution in $dom\ L \cap \overline{\Omega}$ in X. ■

III. Example

Example 1. Consider the boundary value problem:

$$\begin{aligned} &D_{0+}^{\frac{7}{2}}x(t) = f\Big(t, x(t), D_{0+}^{\frac{1}{2}}x(t), D_{0+}^{\frac{3}{2}}x(t), D_{0+}^{\frac{5}{2}}x(t)\Big), \\ &x(0) = 0 = D_{0+}^{\frac{1}{2}}x(0),\ D_{0+}^{\frac{3}{2}}x(0) = 4D_{0+}^{\frac{3}{2}}x(\frac{1}{2}) - 3D_{0+}^{\frac{3}{2}}x(\frac{2}{3}) \\ &D_{0+}^{\frac{5}{2}}x(+\infty) = \int_0^1 D_{0+}^{\frac{3}{2}}x(t)dA(t),\ t \in (0,+\infty), \end{aligned} \tag{23}$$

$A(t) = 6(t^2 - t)$, and

$$\begin{aligned} f\Big(t, x(t), D_{0+}^{\frac{1}{2}}x(t), D_{0+}^{\frac{3}{2}}x(t), D_{0+}^{\frac{5}{2}}x(t)\Big) &= \frac{1}{50}e^{-4t}\frac{\sin x(t)}{1+t^{\frac{7}{2}}} \\ &+ \frac{1}{20}e^{-4t}\frac{\sin\Big(D_{0+}^{\frac{1}{2}}x(t)\Big)}{1+t^2} \\ &+ \frac{1}{10}e^{-5t}\frac{D_{0+}^{\frac{3}{2}}x(t)}{1+t^{\frac{5}{2}}},\ t \in (0,1] \\ &= \frac{1}{30}e^{-3t}\frac{\cos(D_{0+}^{\frac{5}{2}}x(t))}{1+t^{\frac{3}{2}}},\ t \in (1,+\infty). \end{aligned}$$

Corresponding to the BVP (1), here

$$\alpha=\frac{7}{2},\mu_1=4,\mu_2=-3,\xi_1=\frac{1}{2},\xi_2=\frac{2}{3},m=2,\eta=1$$

, We check assumptions H_1-H_5 for the existence of at least a solution for the BVP.
H_1: Resonance assumption.
We see that
$\sum_{i=1}^{2}\mu_i = 1$, $\sum_{i=1}^{2}\mu_i\xi_i = 0$, $\int_0^1 tdA(t) = 1$, and $\int_0^1 dA(t) = 0$. So ,assumption H_1 holds, the BVP is a resonant problem.
Next, we show that $\Delta=a_{11}a_{22}-a_{12}a_{21}\neq 0$

$$\begin{aligned}\Delta&=(-0.1091)(31.2437)-(0.6218)(-0.0407)\\&=-3.3834\neq 0\end{aligned}$$

The assumption H_2 holds.
Assumption H_3: We show that $H\Theta<1$. Let,
$\rho_1(t)=\frac{1}{50}e^{-4t},\rho_2(t)=\frac{1}{20}e^{-4t},\rho_3(t)=\frac{1}{10}e^{-5t}$,
$\rho_4(t)=\frac{1}{30}e^{-3t},\rho_5(t)=0$

$$\begin{aligned}&\Theta=0.00491+0.01227+0.01987+0.00055=0.0376.\\&H=6.5532.\end{aligned}$$

$H\Theta=0.2464<1$
Hence, the assumption H_3 is satisfied.
Next, we verify assumption H_4.

$$\begin{aligned}&\Pi_1Nx(t)\\&=\mu_1\int_0^{\xi_1}(\xi_1-s)Nx(s)ds+\mu_2\int_0^{\xi_2}(\xi_2-s)Nx(s)ds\\&=4\int_0^{\frac{1}{2}}(\frac{1}{2}-s)f(s,x(s),D_{0+}^{\frac{1}{2}}x(s),D_{0+}^{\frac{3}{2}}x(s),D_{0+}^{\frac{5}{2}}x(s))ds\\&-3\int_0^{\frac{2}{3}}(\frac{2}{3}-s)f(s,x(s),D_{0+}^{\frac{1}{2}}x(s),D_{0+}^{\frac{3}{2}}x(s),D_{0+}^{\frac{5}{2}}x(s))ds.\end{aligned}$$

If $D_{0+}^{\frac{3}{2}}x(s)>A_1$, then
$f(s,x(s),D_{0+}^{\frac{1}{2}}x(s),D_{0+}^{\frac{3}{2}}x(s),D_{0+}^{\frac{5}{2}}x(s))$
$>\frac{1}{10}e^{-5s}A_1-\frac{7}{100}e^{-4s}$.
If $D_{0+}^{\frac{3}{2}}x(s)<-A_1$, then
$f(s,x(s),D_{0+}^{\frac{1}{2}}x(s),D_{0+}^{\frac{3}{2}}x(s),D_{0+}^{\frac{5}{2}}x(s))$
$<\frac{7}{100}e^{-4s}-\frac{1}{10}e^{-5s}A_1$.

$$\begin{aligned}\Pi_1Nx(t)>4&\int_0^{\frac{1}{2}}(\frac{1}{2}-s)(\frac{1}{10}e^{-5s}A_1-\frac{7}{100}e^{-4s})ds\\&-3\int_0^{\frac{2}{3}}(\frac{2}{3}-s)(\frac{7}{100}e^{-4s}-\frac{1}{10}e^{-5s}A_1)ds\end{aligned}$$

Setting $A_1=3$ then $\Pi_1Nx(t)\neq 0$.
Next, we show that $\Pi_2Nx(t)\neq 0$

$$\begin{aligned}\Pi_2Nx(t)&=\int_0^\infty w(s)ds-\int_0^1\int_0^t(t-s)w(s)dsdA(t)\\&>\int_0^1(\frac{1}{10}e^{-5s}A_1-\frac{7}{100}e^{-4s})ds\\&+\int_1^\infty-\frac{1}{30}e^{-3s}ds\\&-\int_0^1\int_1^\infty(t-s)(\frac{1}{30}e^{-3s})dsdA(t)\\&\text{where } A(t)=6(t^2-t),\\&=0.01987A_1-0.01718\neq 0.\end{aligned}$$

So, the assumption H_4 holds.
Lastly, we verify assumption H_5.

$$\begin{aligned}\Pi_1Nx(t)=4&\int_0^{\frac{1}{2}}(\frac{1}{2}-s)\Big(\frac{7}{100}e^{-4s}+\frac{1}{10}e^{-5s}(c_1\Gamma(\frac{7}{2})s\\&+c_2\Gamma(\frac{5}{2}))\Big)ds-3\int_0^{\frac{2}{3}}(\frac{2}{3}-s)\Big(\frac{7}{100}e^{-4s}\\&+\frac{1}{10}e^{-5s}(c_1\Gamma(\frac{7}{2})s+c_2\Gamma(\frac{5}{2}))\Big)ds\\&=-0.0029-0.0030c_1+0.6267c_2.\end{aligned}$$

$$\begin{aligned}\Pi_2Nx(t)=&\int_0^1\Big(\frac{7}{100}e^{-4s}+\frac{1}{10}e^{-5s}(c_1\Gamma(\frac{7}{2})s\\&+c_2\Gamma(\frac{5}{2}))\Big)ds+\frac{1}{30}\int_1^\infty e^{-3s}ds\\&-\frac{1}{30}\int_0^1\int_1^\infty(t-s)e^{-3s}dsdA(t)\\&=0.01276c_1+0.02641c_2+0.00662.\end{aligned}$$

$\Pi_1Nx(t)+\Pi_2Nx(t)=0.00372+0.00976c_1+0.65311c_2$.

Let $B=10$. Then if $|c_1|>10$ or $|c_2|>10$ then,
$\Pi_1Nx(t)+\Pi_2Nx(t)>0$
Hence, the assumption H_5 holds.
Since conditions (H_1-H_5) of theorem (9) hold, the boundary value problem (23) has at least one solution in X.

Example 2. Consider the boundary value problem:
$D_{0+}^{\frac{7}{2}}x(t)=f\Big(t,x(t),D_{0+}^{\frac{1}{2}}x(t),D_{0+}^{\frac{3}{2}}x(t),D_{0+}^{\frac{5}{2}}x(t)\Big)$,
subject to:

$$\begin{aligned}&x(0)=0=D_{0+}^{\frac{1}{2}}x(0),\ D_{0+}^{\frac{3}{2}}x(0)=\frac{5}{3}D_{0+}^{\frac{3}{2}}x(\frac{2}{5})-\frac{2}{3}D_{0+}^{\frac{3}{2}}x(1)\\&D_{0+}^{\frac{5}{2}}x(+\infty)=\int_0^1 D_{0+}^{\frac{3}{2}}x(t)dA(t),t\in(0,+\infty)\end{aligned}\tag{24}$$

where $A(t)=4(t^3-t)$,

$$\begin{aligned}f\Big(t,x(t),D_{0+}^{\frac{1}{2}}x(t),D_{0+}^{\frac{3}{2}}x(t),D_{0+}^{\frac{5}{2}}x(t)\Big)&=\frac{1}{20}e^{-3t}cos(\frac{x(t)}{1+t^{\frac{7}{2}}})\\&+\frac{1}{10}h_1(t)e^{-2t}D_{0+}^{\frac{3}{2}}x(t)+\frac{1}{15}h_2(t)e^{-3t}sin(\frac{D_{0+}^{\frac{5}{2}}x(t)}{1+t^{\frac{3}{2}}}),\end{aligned}$$

$$h_1(t)=\begin{cases}1, & t\in[0,1]\\0, & t\in(1,+\infty)\end{cases};$$

$$h_2(t)=\begin{cases}0, & t\in[0,1]\\1, & t\in(1,+\infty)\end{cases}$$

Corresponding to the BVP (1),

$$\alpha = \frac{7}{2}, \mu_1 = \frac{5}{3}, \mu_2 = -\frac{2}{3}, \xi_1 = \frac{2}{5}, \xi_2 = 1, m = 2, \eta = 1$$

We can easily verify that $(H_1) - (H_3)$ are satisfied .
$\Theta = 0.06842, H = 6.55309$ and $H\Theta = 0.4484 < 1$.
Also,

$$\begin{aligned}\Delta &= (-0.1293)(0.8923) - (28.74)(-0.0546) \\ &= 1.4538 \neq 0\end{aligned}$$

Next, we verify the assumption (H_4). Take $A_1 = 10$. Then, if $|D_{0+}^{\frac{3}{2}}x(t)| > A_1$ holds, for any t $\in [0,1]$, then we have $f(s, x(s), D_{0+}^{\frac{1}{2}}x(s), D_{0+}^{\frac{3}{2}}x(s), D_{0+}^{\frac{5}{2}}x(s))$
$> \frac{1}{10}e^{-2s}A_1 - \frac{1}{20}e^{-3s}$.
If $D_{0+}^{\frac{3}{2}}x(s) < -A_1$ holds for any t $\in [0,1]$, then $f(s, x(s), D_{0+}^{\frac{1}{2}}x(s), D_{0+}^{\frac{3}{2}}x(s), D_{0+}^{\frac{5}{2}}x(s))$
$< \frac{1}{20}e^{-3s} - \frac{1}{10}e^{-2s}A_1$. So

$$\begin{aligned}&\Pi_1 Nx(t) \\ &= \mu_1 \int_0^{\xi_1} (\xi_1 - s)Nx(s)ds + \mu_2 \int_0^{\xi_2} (\xi_2 - s)Nx(s)ds \\ &> \frac{5}{3} \int_0^{\frac{2}{5}} (\frac{2}{5} - s)(\frac{1}{10}e^{-2s}A_1 - \frac{1}{20}e^{-3s})ds \\ &- \frac{2}{3} \int_0^1 (1-s)(\frac{1}{20}e^{-3s} - \frac{1}{10}e^{-2s}A_1)ds. \\ &= 0.02922A_1 - 0.01219 > 0.\end{aligned}$$

Therefore, $\Pi_1 Nx(t) \neq 0$.

$$\begin{aligned}\Pi_2 Nx(t) &> \int_0^1 (\frac{1}{10}e^{-2s}A_1 - \frac{1}{20}e^{-3s})ds \\ &+ \int_1^\infty \frac{7}{60}e^{-3s}ds \\ &- \int_0^1 \int_1^t (t-s)(\frac{7}{60}e^{-3s} - \frac{1}{10}A_1)dsdA(t) \\ &= 0.07429A_1 - 0.04299 \neq 0.\end{aligned}$$

So, the assumption H_4 holds.
Choosing $B = 25$, if $|c_1| > B, |c_2| > B$, then (H_5) also hold. Since conditions $(H_1 - H_5)$ of theorem (9) are satisfied, the boundary value problem (24) has at least one solution in X.

IV. Conclusion

The study has established existence of solution for a resonant fractional order multipoint and Riemann-Stieltjes integral boundary value problem on half-line with the $\dim kerL = 2$ using coincidence degree theory. The result was illustrated with examples. The outcome of the research will further enrich the existing literature in the field.

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