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Mixed fractional order p-Laplacian boundary value problem with a two-dimensional Kernel at resonance on an unbounded domain

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ABSTRACT

In this paper, by using the Ge and Ren extension of coincidence degree theory, we established the existence of a solution for a resonant mixed fractional order p-Laplacian boundary value problem (BVP) on the half-line. In the process, we solved the corresponding homogeneous fractional order BVP for conditions critical for resonance and showed that the operator $A(x,\lambda)(t)$ constructed from the abstract equation $Mx(t) = Nx(t)$ is relatively compact. The results are demonstrated with an example.

Introduction

The purpose of this paper is to establish the existence of at least one solution for the mixed fractional order p-Laplacian boundary value problem:

$$^{c}D^{\beta}_{0+}\phi_p\left(D^{\alpha}_{0+}x(t)\right) = h\Big(t, x(t), D^{\alpha-1}_{0+}x(t), D^{\alpha}_{0+}x(t)\Big), \quad t \in (0,+\infty) \tag{1}$$

subject to boundary conditions

$$\begin{aligned} &x(0) = I^{2-\alpha}_{0+}x(0) = 0, \quad D^{\alpha-1}_{0+}x(0) = \sum_{i=1}^{m} l_i D^{\alpha-1}_{0+}x(\xi_i), \\ &D^{\alpha}_{0+}x(+\infty) = \sum_{j=1}^{n} w_j D^{\alpha}_{0+}x(\ell_j) \end{aligned} \tag{2}$$

where $\beta \in (0,1]$, $\alpha \in (2,3]$, $l_i \in \mathbb{R}$, $i = 1,2,3,\dots,m$, $w_j \in \mathbb{R}$, $j = 1,2,3,\dots,n$.

$^{c}D^{\beta}_{0+}$ is the Caputo fractional derivative, D^{α}_{0+} is the Riemann–Liouville fractional derivative, $2 < \alpha+\beta \le 4$, $0 < \xi_1 < \xi_2 < \xi_3 < \cdots < \xi_m < +\infty$, $0 < \ell_1 < \ell_2 < \ell_3 < \cdots < \ell_n < +\infty$, $h : [0,+\infty) \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a continuous function and the p-Laplacian operator $\phi_p(s) = |s|^{p-2}s$, $p \neq 2$, $\phi_p^{-1} = \phi_q$ and $\frac{1}{p} + \frac{1}{q} = 1$.

The fractional order p-Laplacian BVP (1)–(2) is nonlinear, hence we apply the Ge and Ren extension of coincidence degree for the existence results.

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Fractional order equations are tools for modeling complex phenomena such as found in electromagnetics, acoustics, control theory, electrochemistry, finance, and material science see [1–9] and the references therein.

Researchers have focused attention on fractional order boundary value problems with or without a p-Laplacian operator, due to its ability to explain the dynamical behavior of some physical systems. Some authors have used Caputo fractional order derivatives to develop mathematical models for some deadly infectious diseases such as Lassa fever [10], and Covid-19 [11,12]. In these studies, the authors applied the Laplace–Adomian decomposition method and numerical simulations to the fractional-order Caputo derivatives to produce the approximate solutions of the models analytically.

Some studies have focused on the existence of solutions of the fractional order p-Laplacian boundary value problems. The motive has been to determine whether such problems are solvable or not. The author in [13] established that a solution exists for the resonant p-Laplacian boundary value problem:

$$ {}_{0}^{c}D_{t}^{\beta}\left(\phi_{p}\left({}_{0}^{c}D_{t}^{\alpha}x\right)\right)=f\left(t,x,{}_{0}^{c}D_{t}^{\alpha}x\right),\quad t\in[0,1] $$

$$ x(0)=0,\quad {}_{0}^{c}D_{t}^{\alpha}x(0)={}_{0}^{c}D_{t}^{\alpha}x(1), $$

where $0<\alpha,\beta\leq 1$, ${}_{0}^{c}D_{t}^{\alpha}$ and ${}_{0}^{c}D_{t}^{\beta}$ are Caputo fractional derivatives, $f:[0,1]\times\mathbb{R}^{2}\to\mathbb{R}$ is a continuous function, $\phi_{p}(s)=|s|^{p-2}s\quad(s\neq 0)$, $\phi_{p}(0)=0$, $p>1$, by using the continuation theorem of coincidence degree theory, when the dimension of the kernel equals one.

In another study [14], the authors derived existence results for a class of p-Laplacian fractional differential equations with integral boundary conditions by using Schaefer's fixed point theorem and Banach contraction mapping principle

$$ {}^{c}D_{0+}^{\beta}\phi_{p}\left({}^{c}D_{0+}^{\alpha}x(t)\right)=f(t,x(t)),\quad t\in(0,1), $$

$$ x(0)=\int_{0}^{1}g(s)x(s)ds,\quad x(1)=0, $$

$$ \phi_{p}\left({}^{c}D_{0+}^{\alpha}x(0)\right)=\phi_{p}\left({}^{c}D_{0+}^{\alpha}x(1)\right)=\int_{o}^{1}h(s)\phi_{p}\left({}^{c}D_{0+}^{\alpha}x(s)\right)ds, $$

where $1<\alpha,\ \beta\leq 2,\ 3<\alpha+\beta\leq 4$, ${}^{c}D_{0+}^{\alpha}$ and ${}^{c}D_{0+}^{\beta}$ are the Caputo fractional derivatives, $f\in C([0,1]\times\mathbb{R},\mathbb{R})$, $g,h\in C([0,1],\mathbb{R})$, and $\phi_{p}(s)=|s|^{p-2}s,\ p>1$.

Researchers in [15] studied a resonant fractional order boundary value problem with one-dimensional kernel on the half-line;

$$ {}^{c}D_{+0}^{a}\phi_{p}\left(D_{0+}^{b}u(t)\right)=e^{-t}w\left(t,u(t),D_{0+}^{b}u(t)\right),\quad t\in(0,+\infty), $$

$$ I_{0+}^{1-b}u(0)=0,\quad \phi_{p}\left(D_{0+}^{b}u(\infty)\right)=\phi_{p}\left(D_{0+}^{b}u(0)\right), $$

where ${}^{c}D_{0+}^{a}$ is the left Caputo fractional derivative on the half-line and D_{0+}^{b} is the right Riemann–Liouville fractional derivative on the half-line, $0<a,b\leq 1$,$0<a+b<2$, $\phi_{p}(r)=|r|^{p-2}r$, $p\neq 2$, with $\phi_{p}^{-1}=\phi_{q}$ and $\frac{1}{p}+\frac{1}{q}=1$, $w:[0,+\infty)\times\mathbb{R}^{2}\to\mathbb{R}$ is a continuous function. The authors applied the Ge and Ren coincidence degree theorem to the problem with nonlinear differential operator to obtain the existence results .

The growing interest in the study of fractional-order models among researchers and scientists is motivated by its better description of systems associated with its higher degrees of freedom. On the other hand, fractional calculus has become an excellent set of tools for describing the memory and hereditary properties of various materials and processes (see [16].

Previous studies on mixed fractional order problems have focused on one-dimensional kernel and the ones with two-dimensional kernels are on the bounded domains. This paper intends to fill this gap by investigating the existence of the solution of a mixed fractional order p-Laplacian boundary value problem at resonance when the dimension of the kernel equals two in an unbounded domain. In this study, the differential operator is nonlinear, hence, the adoption of the extension of coincidence degree theorem.

The paper is structured as follows. Section "Materials and methods" presents some relevant lemmas and definitions which will be used in the proof of the main existence results. Section "Results and discussion" focuses on the main existence results. In Section "Example", we demonstrate the results with an example.

Materials and methods

This section presents some relevant definitions, lemmas, and theorems from fractional calculus and coincidence degree theory.

Definition 1 (*[16]*)**.** The Riemann–Liouville and Caputo fractional integral of order $\alpha>0$ for a function $h:(0,+\infty)\to\mathbb{R}$ is given by

$$ I_{0+}^{\alpha}h(t)=\frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-s)^{\alpha-1}h(s)\,ds, $$

provided the right-hand side integral is pointwise defined on $(0,+\infty)$.

Definition 2 ([16,17]). The Riemann–Liouville fractional derivative of order $\alpha > 0$ for a function $h : (0, +\infty) \to \mathbb{R}$ is given by

$$\begin{cases} D^{\alpha}_{0+} h(t) = \frac{d^n}{dt^n} I^{n-\alpha}_{0+} h(t) \\ \qquad\qquad = \dfrac{1}{\Gamma(n-\alpha)} \dfrac{d^n}{dt^n} \displaystyle\int_0^t (t-s)^{n-\alpha-1} h(s) ds \end{cases} \tag{3}$$

where $n = [\alpha] + 1$, and $[\alpha]$ is the greatest integer $\le \alpha$ provided that the right-hand side integral is pointwise defined on $(0, +\infty)$.

Definition 3 ([16]). The Caputo fractional derivative of order $\beta > 0$ for a function $h : (0, +\infty) \to \mathbb{R}$ is given by

$$^{c}D^{\beta}_{0+} h(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} h^n(s) ds \tag{4}$$

Lemma 1 ([18]). *Let $\alpha > 0$, then the general solution of the Riemann–Liouville fractional order differential equation $D^{\alpha}_{0+} x(t) = 0$ is*

$$x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n} \tag{5}$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$ and $n = [\alpha] + 1$ is the smallest integer greater than or equal to α.

Lemma 2 ([18]). *If $\alpha > 0$ and h, $D^{\alpha}_{0+} h \in L^1(0, 1)$, then*

$$I^{\alpha}_{0+} D^{\alpha}_{0+} h(t) = h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \tag{6}$$

where $n = [\alpha] + 1$, $\quad c_i \in \mathbb{R}, \pi = 1, 2, \ldots, n$ are arbitrary constants.

Lemma 3 ([18]). *Let $\alpha > 0$, then the general solution of the Caputo fractional order differential equation $^{c}D^{\alpha}_{0+} x(t) = 0$ is*

$$x(t) = d_0 + d_1 t^1 + d_2 t^2 + \cdots + d_n t^n \tag{7}$$

where $d_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$ and $n = [\alpha] + 1$ is the smallest integer greater than or equal to α.

Lemma 4 ([18]). *If $\alpha > 0$ and h, $D^{\alpha}_{0+} h \in L^1(0, 1)$, then*

$$(I^{\alpha}_{0+})^{c} D^{\alpha}_{0+} h(t) = h(t) + d_0 + d_1 t^1 + d_2 t^2 + \cdots + d_n t^n, \tag{8}$$

where $n = [\alpha] + 1$, $\quad d_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$ are arbitrary constants.

Lemma 5 ([19]). *Given that $\alpha > \beta > 0$. If $h(t) \in L^1(0, 1)$ then,*

$$I^{\alpha}_{0+} I^{\beta}_{0+} h(t) = I^{\alpha+\beta}_{0+} h(t), \quad D^{\beta}_{0+} I^{\alpha}_{0+} h(t) = I^{-\beta}_{0+} I^{\alpha}_{0+} h(t) = I^{\alpha-\beta}_{0+} h(t).$$

In particular,

$$D^{\alpha}_{0+} I^{\alpha}_{0+} h(t) = I^{-\alpha}_{0+} I^{\alpha}_{0+} h(t) = h(t).$$

Lemma 6 ([20]). *Suppose that $\alpha > 0$, $\omega > -1$, $t > 0$. Then,*

$$I^{\alpha}_{0+} t^{\omega} = \frac{\Gamma(\omega+1)}{\Gamma(\omega+\alpha+1)} t^{\omega+\alpha}, \tag{9}$$

and

$$D^{\alpha}_{0+} t^{\omega} = \frac{\Gamma(\omega+1)}{\Gamma(\omega-\alpha+1)} t^{\omega-\alpha}. \tag{10}$$

Definition 4 ([18]). Gamma function for $\alpha \in \mathbb{R}$ is given by

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt \tag{11}$$

where $\Gamma(\alpha) = (\alpha - 1)!$, $\alpha \in R^+$.

Let $(X, \|\cdot\|_X)$ and $(Z, \|\cdot\|_Z)$ be any two Banach spaces. Let M be a continuous operator such that

$$M : \text{dom}\, M \subset X \longrightarrow Z, \text{ and } Mx(t) = {}^{c}D^{\beta}_{0+} \phi_p \left(D^{\alpha}_{0+} x(t)\right),$$

for any $x \in dom\, M$.

Define the operator $N_{\lambda} x : \overline{\Omega} \longrightarrow Z$ such that

$$N_{\lambda} x = \lambda h\Big(t, x(t), D^{\alpha}_{0+} x(t), D^{\alpha-1}_{0+} x(t)\Big), \quad \lambda \in [0, 1],\ t \in (0, +\infty), \Omega \subset X$$

is an open bounded set. Then the fractional differential equation (1) can be written in abstract form as

$$Mx(t) = N_{\lambda}x(t). \tag{12}$$

Definition 5 (*[21]*). An operator $M : X \cap dom\, M \longrightarrow Z$ is said to be quasilinear if

(i) $Im\, M = M(X \cap dom\, M)$ is a closed subset of Z.
(ii) $\ker M = \{x \in X \cap dom\, M : Mx = 0\}$ is linearly homeomorphic to $\mathbb{R}^n$.

Definition 6 (*[21]*). An operator $T : X \longrightarrow Z$ is said to be bounded if $T(\Omega) \subset Z$ is bounded for any bounded subset $\Omega \subset X$. Let $X_1 = \ker M$ and X_2 be the complement of the space of X_1 in X such that $X = X_1 \bigoplus X_2$. Also, let $Z_1 \subset Z$ be a subspace of Z and Z_2 be the complement space of Z_1 in Z, so that $Z = Z_1 \bigoplus Z_2$. Let $P : X \to X_1$ is a projector and $Q : Z \longrightarrow Z_1$ is a semi-projector and $\Omega \subset X$ an open bounded set with $\theta \in \Omega$ as the origin in a linear space. Define $\Theta_{\lambda} = \{x \in \bar{\Omega} : Mx(t) = N_{\lambda}x(t),\ \lambda \in [0,1]\}$ and let $N_{\lambda} : \bar{\Omega} \to Z$, be a continuous operator.

Definition 7 (*[14]*). The mapping $Q : Z \to Z_1$ is a semi-projector if

$$Q^2 z = Qz \ and\ Q(kz) = kQz,\ z \in Z,\ k \in \mathbb{R}.$$

Definition 8 (*[22]*). An operator $\phi_p : \mathbb{R} \to \mathbb{R}$ is p-Laplacian if it satisfies the following properties:

(i) ϕ_p is continuous, monotonically increasing, and invertible such that $\phi_p^{-1} = \phi_q$ and for

$$q > 1,\ \frac{1}{p} + \frac{1}{q} = 1$$

(ii) $\phi_p(x+y) \le \phi_p(x) + \phi_p(y),\ if\ \ 1 < p < 2.$
(iii) $\phi_p(x+y) \le 2^{p-2}\left(\phi_p(x) + \phi_p(y)\right),\ if\ \ p > 2.$

Definition 9 (*[21]*). The operator N_{λ} is said to be M-compact in $\bar{\Omega}$ if there exists a vector subspace $Z_1 \subset Z$ such that $\dim Z_1 = \dim X_1$, $(X_1 = \ker M,\ \Omega \subset X)$ and a compact and continuous operator, $A : \bar{\Omega} \times [0,1] \to X_2$ such that for $\lambda \in [0,1]$, the following conditions hold:

(i) $(I-Q)N_{\lambda}\left(\bar{\Omega}\right) \subset Im\, M \subset (I-Q)Z$;
(ii) $QN_{\lambda}x = 0 \Longleftrightarrow QNx = 0,\ \lambda \in (0,1)$;
(iii) $A(\cdot, 0)$ is the zero operator;
(iv) $A(\cdot, \lambda) = (I-P)$;
(v) $M[P + A(\cdot,\lambda)] = (I-Q)N_{\lambda}$.

Theorem 1 (*[21]*). *Let $\left(X, \|\cdot\|_X\right)$ and $\left(Z, \|\cdot\|_Z\right)$ be any two Banach spaces and $\Omega \subset X$ an open bounded set. If the following properties hold,*

(i) The operator $M : X \cap dom\, M \longrightarrow Z$ is quasilinear;
(ii) The operator $N_{\lambda} : \bar{\Omega} \longrightarrow Z$, $\lambda \in [0,1]$ is M-compact;
(iii) $QNx \neq 0$, $\forall x \in \ker M \cap \partial\Omega$;
(iv) $Mx \neq N_{\lambda}x$, $\lambda \in [0,\ 1]$, $x \in \partial\Omega$;
(v) $deg\{JQN,\ \Omega \cap \ker M, 0\} \neq 0$, where the operator $J : W_1 \to X_1$ is a homeomorphism such that $J(\theta) = \theta$ and the degree is the Brouwer degree,

then the abstract equation $Mx(t) = Nx(t)$ has at least one solution in $\bar{\Omega} \cap dom\, M$.

$$\text{Let } X = \Big\{ x(t) = C^2[0,+\infty) : \left(x(t), D_{0+}^{\alpha-1}x(t), D_{0+}^{\alpha}x(t)\right) \in L^1(0,+\infty),\ \sup_{t\in[0,+\infty)} e^{-nt}|x(t)|,$$
$$\sup_{t\in[0,+\infty)} e^{-nt}\left|D_{0+}^{\alpha-1}x(t)\right| \text{ and } \sup_{t\in[0,+\infty)} \left|D_{0+}^{\alpha}x(t)\right| \text{ exist, for } n \in \mathbb{R}^+ \Big\}$$

with norms

$$\|x\|_X = \max\left\{ \|x(t)\|_0,\ \|D_{0+}^{\alpha-1}x(t)\|_{\infty},\ \|D_{0+}^{\alpha}x(t)\|_{\infty}, \right\}$$

where

$$\|x(t)\|_0 = \sup_{t\in[0,+\infty)} e^{-nt}\left|x(t)\right|,\ \|D_{0+}^{\alpha-1}x(t)\|_{\infty} = \sup_{t\in[0,+\infty)} e^{-nt}\left|D_{0+}^{\alpha-1}x(t)\right|,$$
$$\|D_{0+}^{\alpha}x(t)\|_{\infty} = \sup_{t\in[0,+\infty)} \left|D_{0+}^{\alpha}x(t)\right| \text{ and} \tag{13}$$
$$\|z\|_Z = \|z\|_{L^1} \text{ where } \|z\|_{L^1} = \int_0^{+\infty} |z(t)|dt.$$

Lemma 7. *If* $\sum_{i=1}^{m} l_i\xi_i = 0$, $\sum_{i=1}^{m} l_i = 1$, $\sum_{j=1}^{n} w_j = 1$, *and* $\sum_{j=1}^{n} w_j \ell_j^{-1} = 0$, *then* $\ker M = \left\{x \in \text{dom } M : x(t) = d_1 t^{\alpha} + d_2 t^{\alpha-1}\right\}$, $d_1, d_2 \in \mathbb{R}$, $t \in [0, +\infty)$ *and* $\dim \ker M = 2$.

Proof. If $x \in \ker M$, then

$$ {}^cD_{0+}^{\beta}\phi_p\left(D_{0+}^{\alpha}x(t)\right) = 0 \tag{14}$$

$\phi_p\left(D_{0+}^{\alpha}x(t)\right) = I_{0+}^{p}(0) = c, \quad c \in \mathbb{R}.$

So, $D_{0+}^{\alpha}x(t) = \phi_q(c)$ and by Lemma 2,

$$\begin{aligned} I_{0+}^{\alpha}D_{0+}^{\alpha}x(t) &= I_{0+}^{\alpha}\phi_q(c) + d_2t^{\alpha-1} + d_3t^{\alpha-2} + d_4t^{\alpha-3} \\ x(t) &= \frac{\phi_q(c)}{\Gamma(\alpha+1)}t^{\alpha} + d_2t^{\alpha-1} + d_3t^{\alpha-2} + d_4t^{\alpha-3} \\ &= d_1t^{\alpha} + d_2t^{\alpha-1} + d_3t^{\alpha-2} + d_4t^{\alpha-3}. \end{aligned}$$

Applying $x(0) = 0 = I_{0+}^{2-\alpha}x(0)$, we obtain

$$x(t) = d_1t^{\alpha} + d_2t^{\alpha-1}. \tag{15}$$

By applying the boundary conditions (2) to (15), we obtain

$$d_2\left(1 - \sum_{i=i}^{m} l_i\right) = d_1\sum_{i=i}^{m} l_i\xi_i = 0,$$

$$\implies \sum_{i=i}^{m} l_i = 1, \quad \sum_{i=i}^{m} l_i\xi_i = 0. \tag{16}$$

Also,

$$d_1\left(1 - \sum_{j=i}^{n} w_j\right) = d_2\sum_{j=1}^{n} w_i\ell_j^{-1},$$

$$\implies \sum_{j=1}^{n} w_j = 1, \quad \sum_{j=1}^{n} w_j\ell_j^{-1} = 0. \tag{17}$$

$\ker M = \left\{x(t)\middle| x(t) = d_1t^{\alpha} + d_2t^{\alpha-1}, \quad d_1, d_2 \in \mathbb{R}, \quad t \in (0, +\infty)\right\}$. Since $\ker M$ depends on two coefficients, then $\dim \ker M = 2$. □

$$(H_1): \sum_{i=i}^{m} l_i = 1, \sum_{i=i}^{m} l_i\xi_i = 0, \sum_{j=1}^{n} w_j = 1 \text{ and } \sum_{j=1}^{n} w_j\ell_j^{-1} = 0.$$

Lemma 8. *The following statement holds:*

$$\begin{aligned} Im\, M &= \left\{z \in Z : Q_1z = Q_2z = 0\right\} \text{ where} \\ Q_1z &= \sum_{i-1}^{m} l_i\int_0^{\xi_i}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\xi_i}(\xi_i - r)^{\beta-1}z(r)dr\right)ds \\ Q_2z &= \phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\infty}(t-s)^{\beta-1}z(s)ds\right) - \sum_{j=1}^{n} w_j\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\ell_j}(\ell_j - s)^{\beta-1}z(s)ds\right) \end{aligned}$$

and the operator $M : \text{dom } M \subset X \to Z$ *is quasi-linear.*

Proof. Consider ${}^cD_{0+}^{\beta}\phi_p\left(D_{0+}^{\alpha}x(t)\right) = z(t)$, for $z(t) \in \text{Im } M$ and $x \in \text{dom } M$, then

$$x(t) = I_{0+}^{\alpha}\phi_q\left(I_{0+}^{\beta}z(t)\right) + d_1t^{\alpha} + d_2t^{\alpha-1} + d_3t^{\alpha-2} + d_4t^{\alpha-3}.$$

From the initial conditions $x(0) = I_{0+}^{2-\alpha}x(0) = 0$, $d_3 = 0$, $d_4 = 0$.

Hence,

$$x(t) = I_{0+}^{\alpha}\phi_q\left(I_{0+}^{\beta}z(t)\right) + d_1t^{\alpha} + d_2t^{\alpha-1}. \tag{18}$$

Applying the boundary conditions (2) to (18), we have

$$d_2 = \sum_{i=1}^{m} l_i\left(\int_0^{\xi_i}\phi_q\left(I_{0+}^{\beta}z(r)dr\right)ds + d_1\xi_i + d_2\right)$$

$$= \sum_{i=1}^{m} l_i \int_0^{\xi_i} \phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^{\xi_i} \left(\xi_i - r\right)^{\beta-1} z(r)dr \right) + d_1 \sum_{i-1}^{m} l_i \xi_i + d_2 \sum_{i=1}^{m} l_i$$

By (16), we obtain

$$d_2 = \sum_{i=1}^{m} l_i \int_0^{\xi_i} \phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^{\xi_i} \left(\xi_i - r\right)^{\beta-1} z(r)dr \right) ds + d_2.$$

Thus,

$$\sum_{i=1}^{m} l_i \int_0^{\xi_i} \phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^{\xi_i} \left(\xi_i - r\right)^{\beta-1} z(r)dr \right) ds = 0 := Q_1 z \tag{19}$$

Also,

$$\begin{aligned} &\phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^{\infty} (t-s)^{\beta-1} z(s)ds \right) + d_1 \\ &= \sum_{j=1}^{n} w_j D_{0+}^{\alpha} x(\ell_j) \\ &= \sum_{j=1}^{n} w_j \left(\phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^{\ell_j} (\ell_j - s)^{\beta-1} z(s)ds \right) + d_1 + d_2 \ell_j^{-1} \right) \\ &= \sum_{j=1}^{n} w_j \left(\phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^{\ell_j} (\ell_j - s)^{\beta-1} z(s)ds \right) \right) + d_1 \sum_{j=1}^{n} w_j + d_2 \sum_{j=1}^{n} w_j \ell_j^{-1}. \end{aligned}$$

By using (16), we have

$$\phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^{+\infty} (t-s)^{\beta-1} z(s)ds \right) + d_1 = \sum_{j=1}^{n} w_j \phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^{\ell_j} (\ell_j - s)^{\beta-1} z(s)ds \right) + d_1.$$

Hence,

$$\phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^{+\infty} (t-s)^{\beta-1} z(s)ds \right) - \sum_{j=1}^{n} w_j \phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^{\ell_j} (\ell_j - s)^{\beta-1} z(s)ds \right) = 0 := Q_2 z. \quad \square \tag{20}$$

We have established that $\operatorname{Im} M = \left\{ z \in Z : Q_1 z = Q_2 z = 0 \right\}$.

For $x \in \operatorname{dom} M$, it is observed that $\dim \ker M = 2$ and $\operatorname{Im} M$ is a closed subset of Z. Thus, M is a quasi-linear operator.

Let

$$D = \begin{vmatrix} Q_1 t^{\alpha-1} e^{-t} & Q_2 t^{\alpha-1} e^{-t} \\ Q_1 t^{\alpha} e^{-t} & Q_2 t^{\alpha} e^{-t} \end{vmatrix} := \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},$$

where

$$\begin{aligned} Q_1 z &= \sum_{i=1}^{m} l_i \int_0^{\xi_i} \phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^{\xi_i} \left(\xi_i - r\right)^{\beta-1} z(r)dr \right) ds, \quad \text{and} \\ Q_2 z &= \phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^{+\infty} (t-s)^{\beta-1} z(s)ds \right) - \sum_{j=1}^{n} w_j \phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^{\ell_j} (\ell j - s)^{\beta-1} z(s)ds \right) \end{aligned}$$

(H_2): Assume that $D = a_{11}a_{22} - a_{12}a_{21} \neq 0$,

We define the projector $P : X \longrightarrow X_1$ as

$$Px(t) = \frac{D_{0+}^{\alpha} x(+\infty) t^{\alpha}}{\Gamma(\alpha+1)} + \frac{D_{0+}^{\alpha-1} x(0) t^{\alpha-1}}{\Gamma(\alpha)}, \quad x \in X.$$

Next, we define operators $L_1, L_2 : Z \longrightarrow Z_1$ as

$$L_1 z = \frac{1}{D} \left(a_{22} Q_1 z - a_{21} Q_2 z \right) e^{-t}, \;\; L_2 z = \frac{1}{D} \left(-a_{12} Q_1 z + a_{11} Q_2 z \right) e^{-t},$$

and the operator $Q : Z \longrightarrow Z_1$ as

$$Qz = \left(L_1 z \cdot t^{\alpha-1} \right) + \left(L_2 z \cdot t^{\alpha} \right)$$

where Z_1 is the complement space of the $\operatorname{Im} M$ in Z.

Lemma 9. *The operator $Q : Z \longrightarrow Z_1$ is a semi-projector.*

Proof. We make the following computations.

$$\begin{aligned} L_1 \left(L_1 z \right) t^{\alpha-1} &= \frac{1}{D} \left(L_1 \left(a_{22} Q_1 z t^{\alpha-1} - a_{21} Q_2 z t^{\alpha-1} \right) e^{-t} \right) \\ &= \frac{1}{D} \left(a_{22} a_{11} - a_{21} a_{12} \right) \left(L_1 z \right) = L_1 z. \end{aligned} \tag{21}$$

$$\begin{aligned} L_1\left(L_2 z\right) t^\alpha &= \frac{1}{D}\left(a_{22} Q_1(L_2 z) t^\alpha e^{-t} - a_{21} Q_2(L_2 z) t^\alpha e^{-t}\right) \\ &= \frac{1}{D}\cdot 0(L_2 z) \ = 0 \end{aligned} \tag{22}$$

$$\begin{aligned} L_2\left(L_1 z\right) t^{\alpha-1} &= \frac{1}{D}\left(-a_{12} Q_1(L_1 z) t^{\alpha-1} + a_{11} Q_2(L_1 z) t^{\alpha-1}\right) e^{-t} \\ &= \frac{1}{D}\left(-a_{12}a_{11} + a_{11}a_{12}\right)(L_1 z) \ = 0 \end{aligned} \tag{23}$$

$$\begin{aligned} L_2\left(L_2 z\right) t^\alpha &= \frac{1}{D}\left(-a_{12} Q_1(L_2 z) t^\alpha + a_{11} Q_2(L_2 z) t^\alpha\right) e^{-t} \\ &= \frac{1}{D}\left(-a_{12}a_{21} + a_{11}a_{22}\right)(L_2 z) \ = L_2 z. \end{aligned} \tag{24}$$

From (21)–(24), we obtain

$$\begin{aligned} Q^2 z &= Q\left((L_1 z)\cdot t^{\alpha-1} + (L_2 z)\cdot t^\alpha\right) \\ &= (L_1 z)\cdot t^{\alpha-1} + (L_2 z)\cdot t^\alpha \ = Qz. \end{aligned} \tag{25}$$

For any $\mu \in \mathbb{R}$,

$$\begin{aligned} L_1 \mu z &= \frac{1}{D}\left(a_{22} Q_1 \mu z - a_{21} Q_2 \mu z\right) e^{-t} \\ &= \mu \cdot \frac{1}{D}\left(a_{22} Q_1 z - a_{21} Q_2 z\right) e^{-t} \ = \mu L_1 z \end{aligned} \tag{26}$$

$$\begin{aligned} L_2 \mu z &= \frac{1}{D}\left(-a_{12} Q_1 \mu z - a_{11} Q_2 \mu z\right) e^{-t} \\ &= \mu \cdot \frac{1}{D}\left(-a_{12} Q_1 \mu z + a_{11} Q_2 z\right) e^{-t} \ = \mu L_2 z. \end{aligned} \tag{27}$$

$$Q\mu z = (L_1 \mu z)\cdot t^{\alpha-1} + (L_2 \mu z)\cdot t^\alpha.$$

By using (26) and (27), we obtain

$$\begin{aligned} Q\mu z &= \mu\left((L_1 z)\cdot t^{\alpha-1} + (L_2 z)\cdot t^\alpha\right) \\ &= \mu Q z. \end{aligned} \tag{28}$$

Therefore, by Definition 7, $Q : Z \to Z_1$ is a semi-projector. □

Lemma 10. *If h is an L^1- Carathéodory function, then $N_\lambda : \overline{\Omega} \to Z$ is M-compact on the closure of an open and bounded $\Omega \subset X$.*

Proof. To prove condition 9(i), consider $(I - Q)\, N_\lambda x(t)$,

$$\begin{aligned} Q\,(I - Q)\, N_\lambda x(t) &= Q N_\lambda x(t) - Q^2 N_\lambda x(t) \\ &= Q N_\lambda x(t) - Q N_\lambda x(t) = 0. \end{aligned}$$

Thus, $Q\,(I - Q)\, N_\lambda x(t) \subset Im\, M \ = \ker Q$. Also for $z \in \operatorname{Im} M$, $Qz = 0$. Thus, $z \in \ker Q$ i.e $z \in (I - Q)z$.

Therefore, $Q\,(I - Q)\, N_\lambda x(t) \subset Im\, M \subset (I - Q)z$.

Next, to prove 9(ii), let $Q N_\lambda x(t) = 0$ for $\lambda \in [0, 1]$. Then,

$$\begin{aligned} Q N_\lambda x(t) \ &= \ 0 \ = Q\left(\lambda h\left(t, x(t), D^{\alpha-1}_{0+} x(t), D^\alpha_{0+} x(t)\right)\right) \\ &= \ \lambda Q N x(t) \\ &= \ 0. \end{aligned}$$

Conversely, if $Q N x(t) = 0$, then

$$\begin{aligned} Q N x(t) \ &= \ 0 = L_1\left(Q N_\lambda x(t)\right)\cdot t^{\alpha-1} + L_2\left(Q N_\lambda x(t)\right)\cdot t^\alpha \\ &= \ \frac{1}{D}\Big[a_{22} Q_1 (Q N_\lambda x(t)) t^{\alpha-1} e^{-t} - a_{21} Q_2 (Q N_\lambda x(t)) t^{\alpha-1} e^{-t} \\ &\quad - a_{12} Q_1 (Q N_\lambda x(t)) t^\alpha e^{-t} + a_{11} Q_2 (Q N_\lambda x(t)) t^\alpha e^{-t}\Big] \\ &= \ \frac{1}{D}\left[a_{22} Q_1 t^{\alpha-1} e^{-t} - a_{21} Q_2 t^{\alpha-1} e^{-t} - a_{12} Q_1 t^\alpha e^{-t} + a_{11} Q_2 t^\alpha e^{-t}\right]\cdot Q N_\lambda x. \\ &= \ \frac{1}{D}(D + D)\, Q N_\lambda x(t) = 2 Q N_\lambda x(t) \ = 0. \end{aligned}$$

Then, $Q N_\lambda x(t) = 0$.

To prove 9(iii), 9(iv), and 9(v), we define an operator

$A : X \times [0, 1] \longrightarrow X_2$ as

$$A(x, \lambda)(t) = I^\alpha_{0+}\left[\phi_q\left(I^\beta_{0+}(I - Q) N_\lambda x(t)\right) - D^\alpha_{0+} x(+\infty)\right] - I^{\alpha-1}\left(D^{\alpha-1}_{0+} x(0)\right), \lambda \in [0, 1]. \tag{29}$$

From 9(ii), $QN_\lambda x(t) = 0$. If $N_\lambda = \left\{ x \in \overline{\Omega} : Mx(t) = N_\lambda x(t), \quad \lambda \in [0,1] \right\}$, then ${}^cD^{\beta}_{0+}\phi_p\left(D^{\alpha}_{0+}x(t)\right) = N_\lambda x(t) \in \text{Im M} = \ker Q$.

$A(x, 0) = 0$. Hence a zero operator and 9(iii) is satisfied.

$$\begin{aligned}
A(x,\lambda)(t) &= I^{\alpha}_{0+}\left[\phi_q\left(I^{\beta}_{0+}(I-Q)N_\lambda x(t)\right) - D^{\alpha}_{0+}x(+\infty)\right] - I^{\alpha-1}\left(D^{\alpha-1}_{0+}x(0)\right)\\
&= I^{\alpha}_{0+}\left[\phi_q\left(I^{\beta}_{0+}N_\lambda x(t)\right) - D^{\alpha}_{0+}x(+\infty)\right] - I^{\alpha-1}\left(D^{\alpha-1}_{0+}x(0)\right)\\
&= x(t) - I^{\alpha}_{0+}\left(D^{\alpha}_{0+}x(+\infty)\right) - I^{\alpha-1}\left(D^{\alpha-1}_{0+}x(0)\right)\\
&= x(t) - \left[\frac{D^{\alpha}_{0+}x(+\infty)}{\Gamma(\alpha+1)}t^{\alpha} + \frac{1}{\Gamma(\alpha)}D^{\alpha-1}x(0)t^{\alpha-1}\right]\\
&= x(t) - Px(t).
\end{aligned}$$

Thus,

$$A(x,\lambda)(t) = (1-P)\,x(t). \tag{30}$$

Next, for any $x \in \overline{\Omega}$,

$$\begin{aligned}
&M\left[Px + A(x,\lambda)\right](t)\\
&= M\left[\frac{D^{\alpha}_{0+}x(+\infty)}{\Gamma(\alpha+1)}t^{\alpha} + \frac{1}{\Gamma(\alpha)}D^{\alpha-1}_{0+}x(0)t^{\alpha-1} + I^{\alpha}_{0+}\left[\phi_q\left(I^{\beta}_{0+}(I-Q)N_\lambda x(t)\right) - D^{\alpha}_{0+}x(+\infty)\right]\right.\\
&\left. - I^{\alpha-1}\left(D^{\alpha-1}_{0+}x(0)\right)\right]\\
&= {}^cD^{\beta}_{0+}\phi_p\left[D^{\alpha}_{0+}\left(I^{\alpha}_{0+}\left(\phi_q\left(I^{\beta}_{0+}\left(I-Q\right)N_\lambda x(t)\right)\right)\right)\right]\\
&= {}^cD^{\beta}_{0+}\phi_p\left(\phi_q\left(I^{\beta}_{0+}\left(I-Q\right)N_\lambda x(t)\right)\right)\\
&= {}^cD^{\beta}_{0+}\left(I^{\beta}_{0+}\left(I-Q\right)N_\lambda x(t)\right)\\
&= \left(I-Q\right)N_\lambda x(t)
\end{aligned}$$

Hence,

$$M\left[Px + A(x,\lambda)\right](t) = \left(I-Q\right)N_\lambda x(t). \tag{31}$$

Thus, 9(iv) and 9(v) of the Definition 9 are satisfied.

Next, we prove that A is relatively compact for $\lambda \in [0,1]$ in the following steps;

Step 1: $A(x,\lambda)(t)$ is uniformly bounded in X.

Step 2: $A(x,\lambda)(t)$ is equicontinuous on every compact subset of $(0,+\infty)$.

Step 3: $A(x,\lambda)(t)$ is equiconvergent at infinity.

Step 1: Let $\Omega \subset X$ be an open bounded set and $B > 0$ such that $\|x\|_X < B$ for any $x \in \overline{\Omega}$. Since h is a L^1- Carathéodory function, then for a.e, $t \in [0,+\infty)$ and $\lambda \in [0,1]$, there exists $F_B : [0,+\infty) \longrightarrow [0,+\infty)$ such that $\int_0^{+\infty} F_B(t)dt < +\infty$, $\left|h\left(t, x(t), D^{\alpha-1}_{0+}x(t), D^{\alpha}_{0+}x(t)\right)\right| \le F_B(t)$ hence,

$$\int_0^{+\infty}\left|(I-Q)\,N_\lambda x(s)\right|ds \le \|F_B(s)\|_Z + \|QNx(s)\|_Z. \tag{32}$$

Hence, for any $x \in \overline{\Omega}$,

$$\begin{aligned}
\|A(x,\lambda)\|_0 &= \sup_{t\in[0,+\infty)} e^{-nt}\left|A(x,\lambda)(t)\right|\\
&= \sup_{t\in[0,+\infty)} e^{-nt}\left|I^{\alpha}\left(\phi_q\left(I^{\beta}_{0+}(I-Q)N_\lambda x(r)\right) - D^{\alpha}_{0+}x(+\infty)\right) - I^{\alpha-1}\left(D^{\alpha-1}_{0+}x(0)\right)\right|\\
&\le \sup_{t\in[0,+\infty)} e^{-nt}\left[\phi_q\left(\|F_B(s)\|_Z + \|QNx(s)\|_Z\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{D^{\alpha}_{0+}x(+\infty)}{\Gamma(\alpha+1)}t^{\alpha}\right]\\
&\quad + \sup_{t\in[0,+\infty)} e^{-nt}\left[\frac{D^{\alpha-1}_{0+}x(0)}{\Gamma(\alpha)}t^{\alpha-1}\right].
\end{aligned}$$

Let $D^{\alpha}_{0+}x(+\infty) = m_1, \quad D^{\alpha-1}_{0+}x(0) = m_2$, then

$$\|A(x,\lambda)\|_0 \le \phi_q\left(\|F_B(s)\|_Z + \|QNx\|_Z\right) + m_1 + \alpha m_2. \tag{33}$$

$$
\begin{aligned}
\| D_{0+}^{\alpha-1} A(x,\lambda) \|_\infty &= \sup_{t\in[0,+\infty)} e^{-nt} \left| D_{0+}^{\alpha-1} A(x,\lambda)(t) \right| \\
&= \sup_{t\in[0,+\infty)} e^{-nt} \Big| D_{0+}^{\alpha-1} \Big(I^{\alpha} \left(\phi_q \left(I_{0+}^{\beta} (I-Q) N_\lambda x(r) \right) - D_{0+}^{\alpha} x(+\infty) \right) \Big) \\
&\quad - I^{\alpha-1} \left(D_{0+}^{\alpha-1} x(0) \right) \Big| \\
&\le \sup_{t\in[0,+\infty)} t e^{-nt} \left[\phi_q \left(\| F_B(s) \|_Z + \| QNx(r) \|_Z \right) + m_1 \right] + \sup_{t\in[0,+\infty)} e^{-nt} m_2,
\end{aligned}
$$

hence

$$
\| D_{0+}^{\alpha-1} A(x,\lambda) \|_\infty \le \phi_q \left(\| F_B(s) \|_Z + \| QNx \|_Z \right) + m_1 + m_2 \tag{34}
$$

Next,

$$
\begin{aligned}
\| D_{0+}^{\alpha} A(x,\lambda) \|_\infty &= \sup_{t\in[0,+\infty)} \left| D_{0+}^{\alpha} A(x,\lambda)(t) \right| \\
&= \sup_{t\in[0,+\infty)} \Big| D_{0+}^{\alpha} \Big(I^{\alpha} \left(\phi_q \left(I_{0+}^{\beta} (I-Q) N_\lambda x(r) \right) - D_{0+}^{\alpha} x(+\infty) \right) \\
&\quad - I^{\alpha-1} \left(D_{0+}^{\alpha-1} x(0) \right) \Big) \Big| \\
&\le \sup_{t\in[0,+\infty)} \left(\phi_q \left(I_{0+}^{\beta} (I-Q) N_\lambda x(r) \right) + m_1 \right) \\
&\le \phi_q \left(\| F_B(s) \|_Z + \| QNx \|_Z \right) + m_1,
\end{aligned}
$$

hence

$$
\| D_{0+}^{\alpha} A(x,\lambda) \|_\infty \le \phi_q \left(\| F_B(s) \|_Z + \| QNx \|_Z \right) + m_1. \tag{35}
$$

From (33), (34), and (35), we have

$$
\left\| A(x,\lambda) \right\|_X \le \max \left(\| A(x,\lambda) \|_0 , \left\| D_{0+}^{\alpha-1} A(x,\lambda) \right\|_\infty , \left\| D_{0+}^{\alpha} A(x,\lambda) \right\|_\infty \right) = C
$$

where C is an arbitrary constant. Hence, $A(x,\lambda)$ is uniformly bounded in Z.

Step 2: Next, we show that $A(x,\lambda)\overline{\Omega}$ is equicontinuous in a compact subinterval of $[0,+\infty)$. Let $M>0$, $t_1,t_2 \in [0,M]$ with $t_1 < t_2$, $x \in \overline{\Omega}$ and $\lambda \in [0,1]$, we get

$$
\begin{aligned}
& \left| e^{-nt_2} A(x,\lambda)(t_2) - e^{-nt_1} A(x,\lambda)(t_1) \right| \\
&= \Bigg| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{e^{nt_2}} \Big(\phi_q \Big(I_{0+}^{\beta} (I-Q) N_\lambda x(r) \Big) - D_{0+}^{\alpha} x(+\infty) \Big) ds \\
&\quad - \frac{1}{\Gamma(\alpha-1)} \int_0^{t_2} \frac{(t_2-s)^{\alpha-2}}{e^{nt_2}} \left(D_{0+}^{\alpha-1} x(0) \right) ds \\
&\quad - \Bigg(\frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{e^{nt_1}} \Big(\phi_q \Big(I_{0+}^{\beta} (I-Q) N_\lambda x(r) \Big) - |D_{0+}^{\alpha} x(+\infty) \Big) ds \\
&\quad - \frac{1}{\Gamma(\alpha-1)} \int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{e^{nt_1}} \left(D_{0+}^{\alpha-1} x(0) \right) ds \Bigg) \Bigg| \\
&\le \frac{1}{\Gamma(\alpha+1)} \phi_q \Big(\| F_B(s) \|_Z + \| QNx \|_Z \Big) \left| \frac{t_2^{\alpha}}{e^{nt_2}} - \frac{t_1^{\alpha}}{e^{nt_1}} \right| + \frac{m_1}{\Gamma(\alpha+1)} \left| \frac{t_1^{\alpha}}{e^{nt_1}} - \frac{t_2^{\alpha}}{e^{nt_2}} \right| \\
&\quad + \frac{m_2}{\Gamma(\alpha)} \left| \frac{t_1^{\alpha-1}}{e^{nt_1}} - \frac{t_2^{\alpha-1}}{e^{nt_2}} \right| \longrightarrow 0 \text{ as } t_1 \to t_2.
\end{aligned}
$$

$$
\begin{aligned}
& \left| e^{-nt_2} D_{0+}^{\alpha-1} A(x,\lambda)(t_2) - e^{-nt_1} D_{0+}^{\alpha-1} A(x,\lambda)(t_1) \right| \\
&= \left| \frac{D_{0+}^{\alpha-1} \Big(I_{0+}^{\alpha} \Big(\phi_q \Big[I_{0+}^{\beta} (I-Q) N_\lambda x(r) \Big] - D_{0+}^{\alpha} x(+\infty) \Big) - I_{0+}^{\alpha-1} \left(D_{0+}^{\alpha-1} x(0) \right) \Big)}{e^{nt_2}} \right. \\
&\quad \left. - \frac{D_{0+}^{\alpha-1} \Big[I_{0+}^{\alpha} \Big(\phi_q \Big(I_{0+}^{\beta} (I-Q) N_\lambda x(r) \Big) - D_{0+}^{\alpha} x(+\infty) \Big) - I_{0+}^{\alpha-1} \left(D_{0+}^{\alpha-1} x(0) \right) \Big]}{e^{nt_1}} \right| \\
&\le \phi_q \Big(\| F_B(s) \|_Z + \| QNx \|_Z \Big) \left| \frac{t_2}{e^{nt_2}} - \frac{t_1}{e^{nt_1}} \right| + m_1 \left| \frac{t_1}{e^{nt_1}} - \frac{t_2}{e^{nt_2}} \right| \longrightarrow 0 \text{ as } t_1 \to t_2
\end{aligned}
$$

and

$$\Big|D^{\alpha}_{0+}A(x,\lambda)(t_2) - D^{\alpha}_{0+}A(x,\lambda)(t_1)\Big|$$
$$= \Bigg|D^{\alpha}_{0+}\Big[I^{\alpha}_{0+}\Big(\phi_q\Big(\frac{1}{\Gamma(\beta)}\int_0^{t_2}(t_2-s)^{\beta-1}(I-Q)N_\lambda x(r)ds\Big) - D^{\alpha}_{0+}x(+\infty)\Big) - I^{\alpha-1}_{0+}\left(D^{\alpha-1}_{0+}x(0)\right)\Big]$$
$$-\ D^{\alpha}_{0+}\Big[I^{\alpha}_{0+}\Big(\phi_q\Big(\frac{1}{\Gamma(\beta)}\int_0^{t_1}(t_1-s)^{\beta-1}(I-Q)N_\lambda x(r)ds\Big) - D^{\alpha}_{0+}x(+\infty)\Big) - I^{\alpha-1}_{0+}\left(D^{\alpha-1}_{0+}x(0)\right)\Big]\Bigg|$$
$$\leq \phi_q\Big(\frac{1}{\Gamma(\beta)}\int_0^{t_2}\Big|(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}\Big|\,\big|(I-Q)N_\lambda x(r)\big|\,ds\Big)$$
$$+\ \ \phi_q\Big(\frac{1}{\Gamma(\beta)}\int_{t_1}^{t_2}\big|t_1-s\big|^{\beta-1}\big|(I-Q)N_\lambda x(r)\big|\,ds\Big] \to\ 0 \text{ as } t_1 \to\ t_2.$$

Thus, $A(x,\lambda)\overline{\Omega}$ is equicontinuous on $[0,M]$.

Step 3:

Next, we show that $A(x,\lambda)\overline{\Omega}$ is equiconvergent at $+\infty$.

For any $x \in \overline{\Omega}$, we have

$$\int_0^{+\infty}\big|(I-Q)N_\lambda x(s)\big|\,ds| \leq \|F_B(s)\|_Z + \|QNx(s)\|_Z,$$

Hence, given $\epsilon > 0$, there exists $K > 0$ such that for $r \geq K$, then

$$\Bigg|\phi_q\Big(\frac{1}{\Gamma(\beta)}\int_K^{+\infty}(t-r)^{\beta-1}(I-Q)N_\lambda x(r)dr\Big)\Bigg| < \epsilon, \quad \forall\ x \in \overline{\Omega}$$

In addition, since $\lim_{t\to\infty}\frac{(t-K)^{\alpha}}{e^{nt}} = 0, \quad \lim_{t\to\infty}\frac{(t-K)^{\alpha-1}}{e^{nt}} = 0$ and $\lim_{t\to\infty}\frac{t-K}{e^{nt}} = 0$, then for the given ϵ there exists $k_1 > K > 0$ such that for any $t_1, t_2 \geq K$ and $t \in [0,K]$, we have

$$\left|\frac{t_2^{\alpha}}{e^{nt_2}} - \frac{t_1^{\alpha}}{e^{nt_1}}\right| < \epsilon,\ \left|\frac{t_2^{\alpha-1}}{e^{nt_2}} - \frac{t_1^{\alpha-1}}{e^{nt_1}}\right| < \epsilon,\ \left|t_2^{\beta} - t_1^{\beta}\right| < \epsilon \text{ for } \beta \in (0,1] \quad \text{and}$$
$$\left|\frac{t_2}{e^{nt_2}} - \frac{t_1}{e^{nt_1}}\right| < \epsilon.$$

Thus, for $t_1, t_2 \geq k_1$, we have

$$\left|e^{-nt_2}A(x,\lambda)(t_2) - e^{-nt_1}A(x,\lambda)(t_1)\right|$$
$$= \Bigg|\frac{e^{-nt_2}}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}\left[\phi_q\left(I^{\beta}_{0+}(I-Q)N_\lambda x(r)\right) - D^{\alpha}_{0+}x(+\infty)\right]ds - e^{-nt_2}I^{\alpha-1}_{0+}\left(D^{\alpha-1}_{0+}x(0)\right)$$
$$-\Big(\frac{e^{-nt_1}}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\left[\phi_q\left(I^{\beta}_{0+}(I-Q)N_\lambda x(r)\right) - D^{\alpha}_{0+}x(+\infty)\right]ds - e^{-nt_1}I^{\alpha-1}_{0+}\left(D^{\alpha-1}_{0+}x(0)\right)\Big)\Bigg|$$
$$\leq \frac{1}{\Gamma(\alpha+1)}\phi_q\Big(\big\|F_B(s)\big\|_Z + \|QNx\|_Z\Big)\left|\frac{t_2^{\alpha}}{e^{nt_2}} - \frac{t_1^{\alpha}}{e^{nt_1}}\right|$$
$$+\ \frac{m_1}{\Gamma(\alpha+1)}\left|\frac{t_1^{\alpha}}{e^{nt_1}} - \frac{t_2^{\alpha}}{e^{nt_2}}\right| + \frac{m_2}{\Gamma(\alpha)}\left|\frac{t_1^{\alpha-1}}{e^{nt_1}} - \frac{t_2^{\alpha-1}}{e^{nt_2}}\right|$$
$$< \left[\frac{1}{\Gamma(\alpha+1)}\Big(\phi_q\big(\big\|F_B(s)\big\|_Z + \|QNx\|_Z + m_1\big) + \frac{m_2}{\Gamma(\alpha)}\Big)\right]\epsilon.$$

$$\left|e^{-nt_2}D^{\alpha-1}_{0+}A(x,\lambda)(t_2) - e^{-nt_1}D^{\alpha-1}_{0+}A(x,\lambda)(t_1)\right|$$
$$= \Bigg|\frac{D^{\alpha-1}_{0+}\Big[I^{\alpha}_{0+}\left(\phi_q\left(I^{\beta}_{0+}(I-Q)N_\lambda x(r)\right) - D^{\alpha}_{0+}x(+\infty)\right) - I^{\alpha-1}_{0+}\left(D^{\alpha-1}_{0+}x(0)\right)\Big]}{e^{nt_2}}$$
$$-\ \frac{D^{\alpha-1}_{0+}\Big[I^{\alpha}_{0+}\Big(\phi_q\left(I^{\beta}_{0+}(I-Q)N_\lambda x(r)\right) - D^{\alpha}_{0+}x(+\infty)\Big) - I^{\alpha-1}_{0+}\left(D^{\alpha-1}_{0+}x(0)\right)\Big]}{e^{nt_1}}\Bigg|$$
$$\leq \phi_q\Big(\big\|F_B(s)\big\|_Z + \|QNx\|_Z\Big)\left|\frac{t_2}{e^{nt_2}} - \frac{t_1}{e^{nt_1}}\right| + m_1\left|\frac{t_1}{e^{nt_2}} - \frac{t_2}{e^{nt_1}}\right| + m_2\epsilon$$
$$< \Big(\phi_q\left(\big\|F_B(s)\big\|_Z + \|QNx\|_Z\right) + m_1 + m_2\Big)\epsilon$$

and

$$\Big|D^{\alpha}_{0+}A(x,\lambda)(t_2) - D^{\alpha}_{0+}A(x,\lambda)(t_1)\Big|$$

$$= \left| D^{\alpha}_{0+}\left[I^{\alpha}_{0+}\left(\phi_q\left(I^{\beta}_{0+}(I-Q)N_\lambda x(r)\right) - D^{\alpha}_{0+}x(+\infty)\right) - I^{\alpha-1}_{0+}\left(D^{\alpha-1}_{0+}x(0)\right)\right]\right.$$
$$\left. - D^{\alpha}_{0+}\left[I^{\alpha}_{0+}\left(\phi_q\left(I^{\beta}_{0+}(I-Q)N_\lambda x(r)\right) - D^{\alpha}_{0+}x(+\infty)\right) - I^{\alpha-1}_{0+}\left(D^{\alpha-1}_{0+}x(0)\right)\right]\right|$$
$$\leq \phi_q\left(\left|(I-Q)N_\lambda x(r)\right| \frac{|t_2^{\beta} - t_1^{\beta}|}{\Gamma(\beta+1)}\right)$$
$$< \phi_q\left(\frac{\left|(I-Q)N_\lambda x(r)\right|}{\Gamma(\beta+1)}\epsilon\right)$$

Thus, $A(x,\lambda)\overline{\Omega}$ is equiconvergent at $+\infty$. Since $A(x,\lambda)\overline{\Omega}$ is bounded, equicontinuous, and equiconvergent at infinity, consequently, $A : X\times[0,1] \longrightarrow X_2$ is relatively compact. □

Results and discussion

In this section, we state and prove the existence results for the fractional order p-Laplacian boundary value problem (1)–(2).

Theorem 2. *Let h be a Carathéodory function. If $(H_1), (H_2)$, and the following conditions hold:*

(H_3) There exist nonnegative functions $h_i(t) \in Z,\ i = 1,2,3,4,$ such that for all $t \in [0,+\infty)$,

$$\left|f(t,v_1,v_2,v_3)\right| \leq \frac{h_1(t)|v_1|^{p-1}}{(e^{nt})^{p-1}} + \frac{h_2(t)|v_2|^{p-1}}{(e^{nt})^{p-1}} + \frac{h_3(t)|v_3|^{p-1}}{(e^{nt})^{p-1}} + h_4(t),\ (v_1,v_2,v_3) \in \mathbb{R}^3,$$

(H_4) There exists a constant $B > 0$ such that for $x \in dom\ M$, $t \in [0,+\infty)$ if $\left|D^{\alpha}_{0+}x(t)\right| > B$ or $\left|D^{\alpha-1}_{0+}x(t)\right| > B$, then either

$$Q_1Nx(t) \neq 0 \quad or \quad Q_2Nx(t) \neq 0;$$

(H_5) There exists $C > 0$ such that for $|d_1| > C$ or $|d_2| > C$ with $d_1, d_2 \in \mathbb{R}$, and $x \in X$ either

$$d_1Q_1Nx(t) + d_2Q_2Nx(t) < 0 \tag{36}$$

or

$$d_1Q_1Nx(t) + d_2Q_2Nx(t) > 0 \tag{37}$$

then, the BVP (1)–(2) has at least one solution in X provided;

$$2^{2q-4}\frac{1}{\Gamma(\beta+1)}\left[\frac{\|h_1\|_1^{q-1}}{\Gamma(\alpha+1)} + \|h_2\|_1^{q-1} + \|h_3\|_1^{q-1}\right] < 1,\ if\ 1<p<2,$$

or

$$\frac{1}{\Gamma(\beta+1)}\left[\frac{\|h_1\|_1^{q-1}}{\Gamma(\alpha+1)} + \|h_2\|_1^{q-1} + \|h_3\|_1^{q-1}\right] < 1,\ if\ p>2.$$

Before proving Theorem 2, we state and prove the following lemmas;

Lemma 11. *If assumptions (H_2) and (H_3) of Theorem 2 hold, then set*

$$\Omega_1 = \left\{ x \in dom\ M \setminus \ker\ M, \quad Mx = N_\lambda x, \quad \lambda \in [0,1]\right\}$$

is bounded.

Proof. Suppose $x \in \Omega_1$, then $Mx = N_\lambda x$ and $QN_\lambda x = 0$, since $N_\lambda x \in Im\ M = \ker Q$. By (18) in Lemma 8

$$x(t) = d_1t^{\alpha} + d_2t^{\alpha-1} + I^{\alpha}_{0+}\phi_q\left(I^{\beta}_{0+}z(t)\right),$$
$$\text{From } {}^cD^{\beta}_{0+}\phi_p\left(D^{\alpha}_{0+}x(t)\right) = z(t),$$
$$x(t) = d_1t^{\alpha} + d_2t^{\alpha-1} + I^{\alpha}_{0+}D^{\alpha}_{0+}x(t),$$

$$|x(t)| = \left| d_1t^{\alpha} + d_2t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1}D^{\alpha}_{0+}x(r)dr\right|, \quad d_1, d_2 \in \mathbb{R}.$$

By definition,

$$\|x\|_0 = \sup_{t\in[0,+\infty)} e^{-nt}\left| d_1t^{\alpha} + d_2t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1}D^{\alpha}_{0+}x(r)dr\right|$$

$$
\begin{aligned}
&\le \sup_{t\in[0,+\infty)} e^{-nt}t^{\alpha}\left|d_1\right| + \sup_{t\in[0,+\infty)} e^{-nt}t^{\alpha-1}\left|d_2\right| \\
&\quad + \sup_{t\in[0,+\infty)} e^{-nt}t^{\alpha}\frac{1}{\Gamma(\alpha+1)}\int_0^{+\infty}\left|D_{0+}^{\alpha}x(r)\right|dr
\end{aligned}
$$

$$
\|x\|_0 \le 2C + \frac{1}{\Gamma(\alpha+1)}\|D_{0+}^{\alpha}x\|_{\infty} := M_1. \tag{38}
$$

$$
\begin{aligned}
\|D_{0+}^{\alpha-1}x(t)\|_{\infty} &= \sup_{t\in[0,+\infty)} e^{-nt}\left|D_{0+}^{\alpha-1}\left(d_1t^{\alpha}+d_2t^{\alpha-1}+I_{0+}^{\alpha}D_{0+}^{\alpha}x(t)\right)\right| \\
&\le \Gamma(\alpha+1)\left|d_1\right| + \Gamma(\alpha)\left|d_2\right| + \sup_{t\in[0,+\infty)} te^{-nt}\int_0^{+\infty}|D_{0+}^{\alpha}x|dr \\
&\le \Gamma(\alpha+1)\left|d_1\right| + \Gamma(\alpha)\left|d_2\right| + \|D_{0+}^{\alpha}x\|_{\infty}
\end{aligned}
$$

$$
\|D_{0+}^{\alpha-1}x(t)\|_{\infty} \le C(\alpha+1)\Gamma(\alpha) + \|D_{0+}^{\alpha}x\|_{\infty} := M_2. \tag{39}
$$

$$
\|D_{0+}^{\alpha}x(t)\|_{\infty} = \sup_{t\in[0,+\infty)}\left|D_{0+}^{\alpha}x(t)\right| := M_3. \tag{40}
$$

Hence, from the assumption (H_4) of Theorem 2, there exists a constant $B>0$ such that for any $t_1 \in [0,+\infty)$, $\left|D_{0+}^{\alpha}x(t_1)\right| < B$. By the condition (H_2) of Theorem 2 and from the BVP (1)–(2)

$$
{}^{c}D_{0+}^{\beta}\phi_p\left(D_{0+}^{\alpha}x(t)\right) = \lambda h\left(t, x(t), D_{0+}^{\alpha-1}x(t), D_{0+}^{\alpha}x(t)\right).
$$

We obtain

$$
\begin{aligned}
&\left|\phi_p\left(D_{0+}^{\alpha}x(t)\right)\right| \\
&= \left|\phi_p\left(D_{0+}^{\alpha}x(t_1)\right) - \frac{\lambda}{\Gamma(\beta)}\int_{t_1}^{t}(r-t_1)^{\beta-1}f\left(r, x(r), D_{0+}^{\alpha-1}x(r), D_{0+}^{\alpha}x(r)\right)dr\right| \\
&\le \left|\phi_p\left(D_{0+}^{\alpha}x(t_1)\right)\right| + \frac{\lambda}{\Gamma(\beta)}\int_{t_1}^{t}(r-t_1)^{\beta-1}\left|\frac{h_1(r)\left|v_1\right|^{p-1}}{(e^{nr})^{p-1}} + \frac{h_2(r)\left|v_2\right|^{p-1}}{(e^{nr})^{p-1}} + h_3(r)\left|v_3\right|^{p-1} + h_4(r)\right|dr \\
&\le \phi_p(B) + \frac{1}{\Gamma(\beta+1)}\Big[\|h_1\|_1\,\phi_p\left(\|x\|_0\right) + \|h_2\|_1\,\phi_p\left(\|D_{0+}^{\alpha-1}x\|_{\infty}\right) \\
&+\ \|h_3\|_1\,\phi_p\left(\|D_{0+}^{\alpha}x\|_{\infty}\right) + \|h_4\|_1\Big], \\
&\le \phi_p(B) + \frac{1}{\Gamma(\beta+1)}\Big[\|h_1\|_1\,\phi_p\left(2C + \frac{1}{\Gamma(\alpha+1)}\|D_{0+}^{\alpha}x\|_{\infty}\right) \\
&+\ \|h_2\|_1\,\phi_p\left(C(\alpha+1)\Gamma(\alpha) + \|D_{0+}^{\alpha}x\|_{\infty}\right) + \|h_3\|_1\,\phi_p\left(\|D_{0+}^{\alpha}x\|_{\infty}\right) + \|h_4\|_1\Big],
\end{aligned}
$$

by (38)–(40). Thus,

$$
\begin{aligned}
\phi_p\left(\|D_{0+}^{\alpha}x(t)\|_{\infty}\right) \le \phi_p(B) + \frac{1}{\Gamma(\beta+1)}\Big[&\|h_1\|_1\,\phi_p\left(2C + \frac{1}{\Gamma(\alpha+1)}\|D_{0+}^{\alpha}x\|_{\infty}\right) \\
&+\ \|h_2\|_1\,\phi_p\left(C(\alpha+1)\Gamma(\alpha) + \|D_{0+}^{\alpha}x\|_{\infty}\right) + \|h_3\|_1\,\phi_p\left(\|D_{0+}^{\alpha}x\|_{\infty}\right) + \|h_4\|_1\Big]
\end{aligned}
$$

If $1 < p < 2$, then

$$
\begin{aligned}
\|D_{0+}^{\alpha}x(t)\|_{\infty} \le 2^{2q-4}\Big\{&B + \frac{1}{\Gamma(\beta+1)}\left[2C\|h_1\|_1^{q-1} + C(\alpha+1)\Gamma(\alpha)\|h_2\|_1^{q-1} + \|h_4\|_1^{q-1}\right] \\
&+ \frac{1}{\Gamma(\beta+1)}\left[\frac{\|h_1\|_1^{q-1}}{\Gamma(\alpha+1)} + \|h_2\|_1^{q-1} + \|h_3\|_1^{q-1}\right]\|D_{0+}^{\alpha}x(t)\|_{\infty}\Big\}.
\end{aligned}
$$

Hence, by (13)

$$
\|D_{0+}^{\alpha}x(t)\|_{\infty} \le \frac{2^{2q-4}\left\{B + \frac{1}{\Gamma(\beta+1)}\left[2C\|h_1\|_1^{q-1} + C(\alpha+1)\Gamma(\alpha)\|h_2\|_1^{q-1} + \|h_4\|_1^{q-1}\right]\right\}}{1 - 2^{2q-4}\frac{1}{\Gamma(\beta+1)}\left[\frac{\|h_1\|_1^{q-1}}{\Gamma(\alpha+1)} + \|h_2\|_1^{q-1} + \|h_3\|_1^{q-1}\right]}
$$

Similarly, if $p > 2$, then

$$\|D_{0+}^{\alpha}x(t)\|_{\infty} \le \frac{\left\{B + \frac{1}{\Gamma(\beta+1)}\left[2C\|h_1\|_1^{q-1} + C(\alpha+1)\Gamma(\alpha)\|h_2\|_1^{q-1} + \|h_4\|_1^{q-1}\right]\right\}}{1 - \frac{1}{\Gamma(\beta+1)}\left[\frac{\|h_1\|_1^{q-1}}{\Gamma(\alpha+1)} + \|h_2\|_1^{q-1} + \|h_3\|_1^{q-1}\right]}$$

Hence,

$$\begin{aligned}\|x\|_X &= \max\left\{\|x\|_0,\ \|D_{0+}^{\alpha-1}x\|_{\infty},\ \|D_{0+}^{\alpha}x\|_{\infty}\right\}\\ &= \max\left\{2C + \frac{1}{\Gamma(\alpha+1)}\|D_{0+}^{\alpha}x\|_{\infty}\|, C(\alpha+1)\Gamma\alpha + \|D_{0+}^{\alpha}x\|_{\infty}, D_{0+}^{\alpha}x\|_{\infty}\right\}\\ &\le \max\left\{M_1, M_2, M_3\right\} := M.\end{aligned}$$

Therefore, Ω_1 is bounded in X. □

Lemma 12. *If (H_5) of Theorem 2 holds, then $\Omega_2 = \{x \in \ker M : Nx \in Im\, M\}$ is bounded.*

Proof. Suppose $x \in \Omega_2$, with $x = d_1 t^{\alpha} + d_2 t^{\alpha-1}$, $d_1, d_2 \in \mathbb{R}$.

Since $Nx \in \operatorname{Im} M = \ker Q$, then $Q_1 Nx = 0 = Q_2 Nx$. From the condition (H_5) of Theorem 2, it follows that there exists a nonnegative constant C, such that $|d_1| < C$ and $|d_2 < C|$, then

$$\begin{aligned}\|x\|_X &= \max\left\{\|x\|_0,\ \|D_{0+}^{\alpha-1}x\|_{\infty},\ \|D_{0+}^{\alpha}x\|_{\infty}\right\}\\ &= \max\left\{\sup_{t\in[0,+\infty)} e^{-nt}t^{\alpha}|d_1| + \sup_{t\in[0,+\infty)} e^{-nt}t^{\alpha-1}|d_2|,\ \sup_{t\in[0,+\infty)} e^{-nt}\left|D_{0+}^{\alpha-1}\left(d_1t^{\alpha} + d_2t^{\alpha-1}\right)\right|,\right.\\ &\qquad \left.\sup_{t\in[0,+\infty)}\left|D_{0+}^{\alpha}\left(d_1t^{\alpha} + d_2t^{\alpha-1}\right)\right|\right\}\\ &= \max\left\{|d_1| + |d_2|,\ \Gamma(\alpha+1)|d_1| + \Gamma(\alpha)|d_2|,\ \Gamma(\alpha+1)|d_1|\right\}\\ &\le |d_1| + |d_2| + 2\Gamma(\alpha+1)|d_1| + \Gamma(\alpha)|d_2|.\end{aligned}$$

Therefore, Ω_2 is bounded. □

Proof of Theorem 2. We have earlier proved that M is *quasi-linear* and N_{λ} is M-compact on $\overline{\Omega}$. Also, from Lemmas 11 and 12, we have shown that conditions (iii) and (iv) of Theorem 1 (Ge & Ren) hold.

Lastly, we prove that the condition (v) of Theorem 1 also holds.

Define a homeomorphism $J : \operatorname{Im} Q \longrightarrow \ker M$ by

$$J(d_1t^{\alpha} + d_2t^{\alpha-1}) = \frac{1}{D}\left(\left(a_{11}|d_1| + a_{12}|d_2|\right)t^{\alpha-1} + \left(a_{21}|d_1| + a_{22}|d_2|\right)t^{\alpha}\right)e^{-t}. \tag{41}$$

If (36) holds, for any $x \in \operatorname{dom} M \cap \partial\Omega$, $x(t) = d_1t^{\alpha} + d_2t^{\alpha-1} \neq 0$, define a homotopy mapping

$$H(x,\lambda) = -\lambda x + (1-\lambda)JQNx, \quad \lambda \in [0,1]. \tag{42}$$

Then, $H(x,0) = JQNx \neq 0$ and $H(x,1) = -x \neq 0$. For $\lambda \in (0,1)$, we suppose that $H(x,\lambda) = 0$, then from (41) and (42), we obtain

$$\begin{aligned}\lambda[a_{11}|d_1| + a_{12}|d_2|] &= (1-\lambda)\left[a_{11}Q_1N\left(d_1t^{\alpha} + d_2t^{\alpha-1}\right) + d_{12}Q_2N\left(d_1t^{\alpha} + d_2t^{\alpha-1}\right)\right]\\ \lambda[a_{21}|d_1| + a_{22}|d_2|] &= (1-\lambda)\left[a_{21}Q_1N\left(d_1t^{\alpha} + d_2t^{\alpha-1}\right) + d_{22}Q_2N\left(d_1t^{\alpha} + d_2t^{\alpha-1}\right)\right]\end{aligned}$$

or

$$\begin{aligned}a_{11}[\lambda|d_1| - (1-\lambda)Q_1N\left(d_1t^{\alpha} + d_2t^{\alpha-1}\right)] + a_{12}[\lambda|d_2| - (1-\lambda)Q_2N\left(d_1t^{\alpha} + d_2t^{\alpha-1}\right)] &= 0\\ a_{21}[\lambda|d_1| - (1-\lambda)Q_1N\left(d_1t^{\alpha} + d_2t^{\alpha-1}\right)] + a_{22}[\lambda|d_2| - (1-\lambda)Q_2N\left(d_1t^{\alpha} + d_2t^{\alpha-1}\right)] &= 0.\end{aligned}$$

Assume (H_2), we have

$$\begin{aligned}\lambda|d_1| &= (1-\lambda)Q_1N\left(d_1t^{\alpha} + d_2t^{\alpha-1}\right)\\ \lambda|d_2| &= (1-\lambda)Q_2N\left(d_1t^{\alpha} + d_2t^{\alpha-1}\right).\end{aligned}$$

Hence,

$$|d_1| + |d_2| = \frac{1-\lambda}{\lambda}\left(Q_1N\left(d_1t^{\alpha} + d_2t^{\alpha-1}\right) + Q_2N\left(d_1t^{\alpha} + d_2t^{\alpha-1}\right)\right) < 0$$

which contradicts $|d_1| + |d_2| \ge 0$.

If (37) holds, then we define

$$H(x,\lambda) = -\lambda x - (1-\lambda)JQNx, \quad \lambda \in [0,1].$$

Then, by a similar argument,

$$\left|d_1\right|+\left|d_2\right| = -\frac{1-\lambda}{\lambda}\Big(Q_1 N\left(d_1 t^{\alpha}+d_2 t^{\alpha-1}\right)+Q_2 N\left(d_1 t^{\alpha}+d_2 t^{\alpha-1}\right)\Big) < 0$$

which also contradicts $|d_1|+|d_2| \geq 0$.

Thus, by the homotopy property of Brouwer degree, we have

$$\begin{aligned}
\deg(JQN,\ \Omega\cap M,0) &= \deg(H(\cdot,0),\ \Omega\cap\ker M,\ 0)\\
&= \deg(H(\cdot,1),\ \Omega\cap\ker M,\ 0)\\
&= \deg(\pm J,\ \Omega\cap\ker M,\ 0)\\
&= \mathrm{Sgn}\left(\frac{1}{D}\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}\right)\\
&= \pm 1 \neq 0.
\end{aligned}$$

Therefore, at least one solution exists in X for the BVP (1)–(2)

Example

Consider the fractional order BVP

$$^{c}D_{0+}^{1/2}\phi_3\left(D_{0+}^{5/2}x(t)\right)=h\left(t,x(t),D_{0+}^{3/2}x(t),D_{0+}^{5/2}x(t)\right), t\in[0,+\infty)$$

subject to the boundary conditions

$$x(0)=I_{0+}^{-\frac{1}{2}}x(0)=0,\qquad D_{0+}^{3/2}x(0)=3D_{0+}^{3/2}x(2)-2D_{0+}^{3/2}x(3),$$
$$D_{0+}^{5/2}x(+\infty)=-4D_{0+}^{5/2}x(4)+5D_{0+}^{5/2}x(5), \text{and}$$
$$h\left(x,x(t),D_{0+}^{3/2}x(t),D_{0+}^{5/2}x(t)\right)=\frac{e^{-7t}\sin x}{5}+\frac{e^{-10t}\cos D_{0+}^{3/2}x(t)}{2}+\frac{e^{-5t}\sin D_{0+}^{5/2}x(t)}{8}.$$

Here, $\beta=1/2$, $\alpha=5/2$, $p=3$, $l_1=3$, $l_2=-2$, $\xi_1=2$, $\xi_2=3$, $\omega_1=-4$, $\omega_2=5$, $\ell_1=4$, $\ell_2=5$, $m=n=2$

$$\sum_{i=1}^{2} l_i=1,\ \sum_{i=1}^{2} l_i\xi_i=0,\ \sum_{j=1}\omega_j=1 \text{ and } \sum_{i=1}^{2}\omega_j\ell_j^{-1}=(-4)^{-1}+5\left(5^{-1}\right)=0.$$

Condition (H_1) holds.

Next, by computing $a_{11}=4.0241$, $a_{12}=-0.5115$, $a_{21}=-1.2529$, and $a_{22}=-1.07309$, we obtain $D=a_{11}a_{22}-a_{12}a_{21}=-4.9591 \neq 0$,

Hence, (H_2) holds.

$\|h_1(t)\|_{L^1}=\frac{1}{35}$, $\|h_2(t)\|_{L^1}=\frac{1}{20}$ and $\|h_3(t)\|_{L^1}=\frac{1}{40}$.

$$\text{Since } p\geq 2,\ \frac{1}{\Gamma(\beta+1)}\left[\frac{\|h_1\|_1^{q-1}}{\Gamma(\alpha+1)}+\|h_2\|_1^{q-1}+\|h_3\|_1^{q-1}\right]$$
$$=\frac{1}{\Gamma(3/2)}\left[\frac{\left(\frac{1}{35}\right)^{1/2}}{\Gamma(7/2)}+\left(\frac{1}{20}\right)^{1/2}+\left(\frac{1}{40}\right)^{1/2}\right]$$
$$=0.4881<1$$

Hence, (H_3) is satisfied.

Let $B=60$ for any $x\in dom\,M$. Assume $|D_{0+}^{3/2}x(t)|>B$ holds for $t\in[0,+\infty)$. Then from the continuity of $D_{0+}^{3/2}x(t)$, either $D_{0+}^{3/2}x(t)>B$ or $D_{0+}^{3/2}x(t)<-B$.

If $D_{0+}^{3/2}x(t)>B$ holds for any

$$\begin{aligned}
Q_1 Nx(t) &= \sum_{i=1}^{2} l_i\int_0^{\xi_i}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\xi_i}(\xi_i-r)^{\beta-1}Nx(r)\,dr\right)ds\\
&> 3\int_0^2\phi_{3/2}\left(\frac{1}{\Gamma(1/2)}\int_0^2(2-r)^{-1/2}\left(\frac{Be^{-10r}}{2}-\frac{e^{-7r}}{5}-\frac{e^{-5r}}{8}\right)dr\right)ds\\
&\quad -2\int_0^3\phi_{3/2}\left(\frac{1}{\Gamma(1/2)}\int_0^3(3-r)^{-1/2}\left(\frac{Be^{-10r}}{2}-\frac{e^{-7r}}{5}-\frac{e^{-5r}}{8}\right)dr\right)ds=0.6628>0
\end{aligned}$$

Similarly, if $|D_{0^+}^{5/2} x(t)| > B$, then either $D_{0^+}^{5/2} x(t) > B$ or $D_{0^+}^{5/2} x(t) < -B$, for $t \in [0, +\infty)$. If $D_{0^+}^{5/2} x(t) > B$, for $t \in [0, +\infty)$, then

$$\begin{aligned} Q_2 N x(t) &= \phi_{3/2}\left(\frac{1}{\Gamma(1/2)}\int_0^{+\infty}(t-r)^{-1/2} N x(r)\, dr\right) \\ &\quad - \sum_{j=1}^{2}\omega_j \phi_{3/2}\left(\frac{1}{\Gamma(1/2)}\int_0^{\ell_j}\left(\ell_j - r\right)^{-1/2} N x(r)\, dr\right) \\ &= 4\phi_{3/2}(0.412143) - 5\phi_{3/2}(0.366491) = -0.45899 < 0 \end{aligned}$$

Since $Q_1 N x(t) \neq 0$ or $Q_2 N x(t) \neq 0$, then the condition (H_4) is satisfied.

Lastly, If $C = 120$, for any $d_1, d_2 \in \mathbb{R}$, if $|d_1|$ or $|d_2| > C$. Given $\alpha = 5/2$,

$$\begin{aligned} &Q_1 N\left(d_1 t^{\alpha} + d_2 t^{\alpha-1}\right) + Q_2 N\left(d_1 t^{\alpha} + d_2 t^{\alpha-1}\right), \\ &\quad = 3\int_0^2 \phi_{3/2}(4.197)\, ds - 2\int_0^3 \phi_{3/2}(3343)\, ds + 0 + 4\phi_{3/2}(2.86655) \\ &- 5\phi_{3/2}(2.54925) = 0.0961 > 0 \end{aligned}$$

Thus, $d_1 Q_1 N\left(d_1 t^{\alpha} + d_2 t^{\alpha-1}\right) + d_2 Q_2 N\left(d_1 t^{\alpha} + d_2 t^{\alpha-1}\right) \neq 0$, hence, condition (H_5) holds. Therefore, from Theorem 2, the fractional BVP ("Example") has at least one solution in $dom\ M \cap \partial\Omega$.

Conclusions

The study has investigated a mixed fractional-order p-Laplacian BVP with two-dimensional kernel on an unbounded domain. By using the Ge and Ren extension of coincidence degree theory and some assumptions, we have shown that the mixed fractional order BVP (1)–(2) has at least a solution. The outcome of this investigation is supported by the findings of the studies on mixed fractional order p-Laplacian BVPs with lower dimensions of kernel on bounded or unbounded domain [13–15]. The result is new and demonstrated with an example. In contrast to this study, some researchers have used Laplace Adomian decomposition method and numerical simulation approach to obtain an approximate analytical solutions to fractional order derivative models [10–12].

CRediT authorship contribution statement

Ezekiel K. Ojo: Conceptualized the problem, Worked on the methodology, Manuscript preparation. **Samuel A. Iyase:** Supervised the work. **Timothy A. Anake:** Supervised the work.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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