Iyase and Imaga Boundary Value Problems (2022) 2022:47
https://doi.org/10.1186/s13661-022-01629-7

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Higher-order p-Laplacian boundary value problems with resonance of dimension two on the half-line

S.A. Iyase[1] and O.F. Imaga[1*]

*Correspondence: imaga.ogbu@covenantuniversity.edu.ng
[1]Department of Mathematics, Covenant University, Ota, Nigeria

Abstract

We apply the extension of coincidence degree to obtain sufficient conditions for the existence of at least one solution for a class of higher-order p-Laplacian boundary value problems with two-dimensional kernel on the half-line. The result obtained improves and generalizes some of the known results on p-Laplacian boundary value problems in the literature. We also validate our result with an example.

Keywords: Coincidence degree; Half-line; Higher order; p-Laplacian; Resonance; Two dimension

1 Introduction

This paper is concerned with the existence of solution for the following higher-order p-Laplacian boundary value problem:

$$\big(\phi_p\big(y^{(n-1)}(t)\big)\big)' = h\big(t, y(t), y'(t), \dots, y^{(n-1)}(t)\big), \quad t \in (0,\infty), n \geq 3, \tag{1.1}$$

$$\begin{aligned} &y^{(n-2)}(\infty) = \sum_{i=1}^{m} \alpha_i y^{(n-2)}(\xi_i), \qquad y^{(n-3)}(0) + y^{(n-2)}(0) = \sum_{j=1}^{m} \beta_j y^{(n-3)}(\eta_j), \\ &y^{(n-1)}(\infty) = 0, \qquad y^{(i)}(0) = 0, \quad i = 0, 1, 2, \dots, n-4, \end{aligned} \tag{1.2}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $1/p + 1/q = 1$, $\phi_q = \phi_p^{-1}$, $h : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is a Caratheodory's function, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < \infty$, $0 < \eta_1 < \eta_2 < \cdots < \eta_m < \infty, \alpha_i, \beta_j \in \mathbb{R}$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, m$, $\sum_{i=1}^{m} \alpha_i = \sum_{j=1}^{m} \beta_j = \sum_{j=1}^{m} \beta_j \eta_j = 1$.

Our result will be based on the extension of Mawhin's continuation theorem by Ge and Ren [6]. Higher-order resonant boundary value problems have in recent years become of great interest to various researchers, see for example [1, 3–5, 7, 8, 12, 13] and the references therein. Some of the results utilized Mawhin's coincidence degree theory [14] which has continued to play a significant role in the study of boundary value problems when the differential operator is linear. However, when the differential operator is nonlinear, Mawhin's continuation theorem can no longer be applied directly as was the case in the above ref-

© The Author(s) 2022. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

erences. For some results on the application of the extension of coincidence degree by Ge and Ren, see [10, 11, 13] and the references therein.

p-Laplacian boundary value problems have found applications in diverse areas such as in nonlinear elasticity, blood flow models, non-Newtonian mechanics, glaciology, etc. Although there have been some results on p-Laplacian boundary value problems at resonance with a two-dimensional kernel, see for example [9], to the best of our knowledge this is the first paper on higher-order p-Laplacian boundary value problems with a resonance of dimension two on the half-line. (1.1)–(1.2) is a problem at resonance if $Ly = (\phi_p(y^{(n-1)}(t))) = 0$ has nontrivial solutions under the given boundary conditions. Generally, resonance problems can be cast in the abstract form $Ly = Ny$, where L is not an invertible operator.

The organization of this paper is as follows. In Sect. 2, we recall some technical results such as definitions, theorems, and lemmas. In Sect. 3, we state and prove the main existence result, and in Sect. 4, we provide an example to demonstrate our results.

2 Some technical results

We recall some notations, definitions, lemmas, and theorems.

Definition 2.1 Let Y and Z be two Banach spaces with $\|\cdot\|_Y$ and $\|\cdot\|_Z$ respectively. The operator $L : Y \to Z$ is quasi-linear if

(i) $\operatorname{Im} L = L(Y \cap \operatorname{dom} L)$ is a closed subset of Z,

(ii) $\ker L = \{y \in Y \cap \operatorname{dom} M : Ly = 0\}$ is linearly homeomorphic to $\mathbb{R}^n$.

Let $P : Y \to Y_1$ and $Q : Z \to Z$ be projections such that $\operatorname{Im} P = \ker L$, $\ker Q = \operatorname{Im} L$. Let $Y_1 = \ker L$, $Z_2 = \operatorname{Im} L$ and Z_1, Y_2 be the complement spaces of Z_2 in Z, Y_1 in Y. Then

$$Y = Y_1 \oplus Y_2, \qquad Z = Z_1 \oplus Z_2.$$

Definition 2.2 Let Y be a Banach space with $Y_1 \subset Y$. The mapping $Q : Y \to Y_1$ is a semi-projector if $Q^2 y = Qy$ and $Q(\sigma y) = \sigma Qy$, $y \in Y$, $\sigma \in \mathbb{R}$.

Definition 2.3 Let $L : Y \cap \operatorname{dom} L \to Z$ be a quasi-linear operator. Let $Y_1 = \ker L$ and $W \subset Y$ be an open and bounded set with $0 \in W$. Then $L_\sigma : \overline{W} \to Z$, $\sigma \in [0,1]$ is said to be L-compact in $\overline{W}$ if $L_\sigma : \overline{W} \to Z$ is a continuous operator, and there exists an operator $R : \overline{W} \times [0,1] \to Y_2$ which is continuous and compact such that, for $\sigma \in [0,1]$,

$$\text{(i)} \quad (I - Q)N_\sigma(\overline{W}) \subset \operatorname{Im} L \subset (I - Q)Z, \tag{2.1}$$

$$\text{(ii)} \quad QN_\sigma y = 0, \quad \sigma \in (0,1) \quad \text{iff} \quad QNy = 0, \tag{2.2}$$

$$\text{(iii)} \quad R(\cdot, 0) \text{ is the zero operator}, \tag{2.3}$$

$$\text{(iv)} \quad R(\cdot,\sigma)|_{\Omega_\sigma} = (I - P)|_{\Omega_\sigma}, \quad \text{where } \Omega_\sigma = \{y \in \overline{W} : Ly = N_\sigma y\}, \tag{2.4}$$

$$\text{(v)} \quad L\big[P + R(\cdot,\sigma)\big] = (I - Q)N_\sigma, \tag{2.5}$$

where Q is a semi-projector.

Definition 2.4 ([15]) Let $\phi_p : \mathbb{R} \to \mathbb{R}$, then ϕ_p satisfies the following conditions:

$$\text{(i)} \quad \phi_p(u+v) \le \left(\phi_p(u) + \phi_p(v)\right), \quad 1 < p \le 2, \tag{2.6}$$

$$\text{(ii)} \quad \phi_p(u+v) \le 2^{p-2}\left(\phi_p(u) + \phi_p(v)\right), \quad p > 2. \tag{2.7}$$

In what follows, we shall need the following space:

$$\begin{aligned} Y = \Big\{ & y : [0,\infty) \to \mathbb{R} : y, \left(\phi_p\left(y^{(n-1)}\right)\right) \in AC[0,\infty), \lim_{t\to\infty} e^{-t}\left|y^{(i)}(t)\right| \text{ exists}, \\ & 0 \le i \le n-1, \left(\phi_p\left(y^{(n-1)}\right)\right)' \in L^1[0,\infty) \Big\} \end{aligned} \tag{2.8}$$

with the norm

$$\|y\| = \max_{0\le i\le n-1} \sup_{t\in(0,\infty)} \left|y^{(i)}(t)\right| e^{-t}. \tag{2.9}$$

Then Y is a Banach space.

Definition 2.5 ([14]) $h : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is $L^1[0,\infty)$ Caratheodory if it satisfies the following conditions:

(i) For each $y \in \mathbb{R}^n$, the mapping $t \to h(t,y)$ is Lebesgue measurable,
(ii) For a.e. $t \in [0,\infty)$, the mapping $y \to h(t,y)$ is continuous on $\mathbb{R}^n$,
(iii) For each $r > 0$, there exists $\alpha_r \in L^1[0,\infty)$ such that for a.e. $t \in [0,\infty)$ and every y such that $\|y\| \le r$ we have $|h(t,y)| < \alpha_r$.

Theorem 2.1 ([2]) *Let X be the space of all continuous and bounded vector-valued functions on $[0,\infty)$ and $X_1 \subset X$. Then X_1 is relatively compact if*

(i) *X_1 is bounded in X,*
(ii) *all functions from X_1 are equicontinuous on any compact subinterval of $[0,\infty)$,*
(iii) *all functions from X_1 are equiconvergent at infinity.*

Let $L : \operatorname{dom} L \subset Y \to Z$ where

$$\begin{aligned} \operatorname{dom} L = \Bigg\{ & y \in Y : \left(\phi_p\left(y^{(n-1)}\right)\right)' \in L^1[0,\infty), y^{(n-2)}(\infty) = \sum_{i=1}^{m} \alpha_i y^{(n-2)}(\xi_i) \\ & y^{(n-3)}(0) + y^{(n-2)}(0) = \sum_{j=1}^{m} \beta_j y^{(n-3)}(\eta_j), y^{(n-1)}(\infty) = 0, \\ & y^{(i)}(0) = 0, i = 0,1,2,\dots,(n-4) \Bigg\} \end{aligned} \tag{2.10}$$

and $N_\sigma : Y \to Z$ is defined by $N_\sigma y = \sigma h(t, y(t), \dots, y^{(n-1)}(t))$. Thus (1.1)–(1.2) is of the form

$$Lu = N_\sigma y \quad \text{when } \sigma = 1. \tag{2.11}$$

Theorem 2.2 ([6]) *Let $W \subset Y$ be an open and bounded set with $0 \in W$. Let $L : Y \cap \operatorname{dom} L \to Z$ be a quasi-linear operator and $N_\sigma : \overline{W} \to Z$, $\sigma \in [0,1]$ be L-compact. In addition, if the following hold*:

(i) $Ly \neq N_\sigma y$, $y \in \partial W \cap \operatorname{dom} L$, $\sigma \in (0,1)$,
(ii) $\deg(JQN, W \cap \ker L, 0) \neq 0$, *where* $N = N_1$ *and* $J : \operatorname{Im} Q \to \ker L$ *is the homeomorphism with* $J(0) = 0$,

then the abstract equation $Ly = Ny$ *has at least one solution in* $\operatorname{dom} L \cap \overline{W}$.

In what follows we assume the following conditions:

$$(A_1) \quad \sum_{i=1}^m \alpha_i = \sum_{j=1}^m \beta_j = 1, \qquad \sum_{j=1}^m \beta_j \eta_j = 1, \tag{2.12}$$

$$(A_2) \quad \Delta = \begin{vmatrix} Q_1 t^{n-3} e^{-t} & Q_2 t^{n-3} e^{-t} \\ Q_1 t^{n-2} e^{-t} & Q_2 t^{n-2} e^{-t} \end{vmatrix} = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} = c_{11}c_{22} - c_{12}c_{21} \neq 0, \tag{2.13}$$

where

$$Q_1 z = \sum_{j=1}^m \beta_j \int_0^{\eta_j} \int_0^s \phi_q \left(\int_v^\infty z(\tau)\, d\tau \right) dv\, ds, \tag{2.14}$$

$$Q_2 z = \sum_{i=1}^m \alpha_i \int_{\xi_i}^\infty \phi_q \left(\int_s^\infty z(\tau)\, d\tau \right) ds. \tag{2.15}$$

Lemma 2.1 *Suppose that* (A_1) *and* (A_2) *hold. Then*
(i) $\ker L = \{ y \in \operatorname{dom} L : y(t) = at^{n-3} + bt^{n-2}, a, b \in \mathbb{R}, t \in [0,\infty) \}$;
(ii) $\operatorname{Im} L = \{ z \in Z : Q_1 z = Q_2 z = 0 \}$.

Proof Obviously, (i) holds. Hence $\ker L$ is homeomorphic to $\mathbb{R}^2$. Thus $\dim \ker L = 2$. To prove (ii), let $z \in \operatorname{Im} L$ and consider the equation

$$\left(\phi_p \left(y^{(n-1)}(t) \right) \right)' = z(t) \tag{2.16}$$

with boundary conditions (1.2). Then

$$y^{(n-3)}(t) = -\int_0^t \int_0^s \phi_q \left(\int_v^\infty z(\tau)\, d\tau \right) dv\, ds + y^{(n-2)}(0) t + y^{(n-3)}(0),$$

$$y^{(n-2)}(t) = -\int_0^t \phi_q \left(\int_s^\infty z(\tau)\, d\tau \right) ds + y^{(n-2)}(0).$$

Hence from the boundary conditions we derive

$$y^{(n-3)}(0) + y^{(n-2)}(0) = -\sum_{j=1}^m \beta_j \int_0^{\eta_j} \int_0^s \phi_q \left(\int_v^\infty z(\tau)\, d\tau \right) dv\, ds + \sum_{j=1}^m \beta_j \eta_j y^{(n-2)}(0) + \sum_{j=1}^m \beta_j y^{(n-3)}(0).$$

Since $\sum_{j=1}^m \beta_j = \sum_{j=1}^m \beta_j \eta_j = 1$, we obtain

$$\sum_{j=1}^m \beta_j \int_0^{\eta_j} \int_0^s \phi_q \left(\int_v^\infty z(\tau)\, d\tau \right) dv\, ds = Q_1 z = 0.$$

Similarly,

$$y^{(n-2)}(\infty) = -\int_0^{\infty} \phi_q\left(\int_s^{\infty} z(\tau)\,d\tau\right) ds + y^{(n-2)}(0)$$
$$= -\sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} \phi_q\left(\int_s^{\infty} z(\tau)\,d\tau\right) ds + \sum_{i=1}^{m} \alpha_i y^{(n-2)}(0),$$

which implies

$$\sum_{i=1}^{m} \alpha_i \int_{\xi_i}^{\infty} \phi_q\left(\int_s^{\infty} z(\tau)\,d\tau\right) ds = Q_2 z = 0.$$

Thus L is a quasi-linear operator.

On the other hand, if $z \in Z$ satisfies $Q_1 z = Q_2 z = 0$, we take

$$y(t) = at^{n-3} + bt^{n-2} - \frac{1}{(n-2)!}\int_0^t (t-s)^{n-2}\phi_q\left(\int_s^{\infty} z(\tau)\,d\tau\right) ds, \tag{2.17}$$

where a, b are arbitrary constants. Then, for $y \in Y$, $(\phi_p(y^{(n-1)}(t)))' = z(t)$ satisfies (1.2). Thus $y \in \operatorname{dom} L$, that is, $z \in \operatorname{Im} L$. □

We define the projector $P : Y \to \ker L$ by

$$Py(t) = \frac{y^{(n-3)}(0)t^{n-3}}{(n-3)!} + \frac{y^{(n-2)}(0)}{(n-2)!}t^{n-2}, \tag{2.18}$$

and the operator $T_1, T_2 : Z \to Z_1$ by

$$T_1 z = \frac{e^{-t}}{\Delta}[c_{22}Q_1 z - c_{21}Q_2 z], \tag{2.19}$$

$$T_2 z = \frac{e^{-t}}{\Delta}[-c_{12}Q_1 z + c_{11}Q_2 z]. \tag{2.20}$$

Define the operator $Q : Z \to Z$ by

$$Qz = T_1 z(t)t^{n-3} + T_2 z(t)t^{n-2}.$$

Then we can calculate and obtain $T_1((T_1 z)t^{n-3}) = T_1 z$, $T_1((T_2 z)t^{n-2}) = 0$, $T_2((T_1 z)t^{n-3}) = 0$, $T_2((T_2 z)t^{n-2}) = T_2 z$. Hence, $Q^2 z = Qz$ and $Q(\sigma z) = \sigma Qz$. Thus Q is a semi-projector.

Lemma 2.2 *If h is an $L^1[0,\infty)$ Caratheodory's function, then $N_\sigma : \overline{W} \to Z$ is L-compact in $\overline{W}$ for $W \subset Y$ an open and bounded subset with $0 \in W$.*

Proof To prove (2.1) we have

$$Q(I-Q)N_\sigma(\overline{W}) = QN_\sigma(\overline{W}) - Q^2 N_\sigma(\overline{W}) = QN_\sigma(\overline{W}) - QN_\sigma(\overline{W}) = 0.$$

Thus, $(I-Q)N_\sigma(\overline{W}) \subset \operatorname{Im} L$. Also, for $z \in \operatorname{Im} L$, we have $Qz = 0$. Hence $z \in \ker Q$ i.e. $z \in (I-Q)z$. Hence, $\operatorname{Im} L \subset (I-Q)z$. Therefore,

$$(I-Q)N_\sigma(\overline{W}) \subset \operatorname{Im} L \subset (I-Q)Z.$$

To prove (2.2), suppose $QN_\sigma y = 0$ for $\sigma \in (0,1)$. Then

$$0 = QN_\sigma y = Q\bigl(\sigma h\bigl(t, y(t), \dots, y^{(n-1)}(t)\bigr)\bigr) = \sigma Qh\bigl(t, y(t), \dots, y^{(n-1)}(t)\bigr) = \sigma QNy.$$

Thus, $QNy = 0$. On the other hand, if $QNy = 0$, we have

$$\begin{aligned}
0 = QNy &= T_1(QN_\sigma y)t^{n-3} - T_2(QN_\sigma y)t^{n-2} \\
&= \frac{e^{-t}}{\Delta}\bigl[c_{22}Q_1(QN_\sigma y)t^{n-3} - c_{21}Q_2(QN_\sigma y)t^{n-3} \\
&\quad - c_{12}Q_1(QN_\sigma y)t^{n-2} + c_{11}Q_2(QN_\sigma y)t^{n-2}\bigr] \\
&= \frac{1}{\Delta}\bigl[(c_{11}c_{22} - c_{21}c_{12}) + (-c_{21}c_{12} + c_{11}c_{22})\bigr](QN_\sigma y) \\
&= 2QN_\sigma y.
\end{aligned}$$

Accordingly, $QN_\sigma y = 0$. To establish (2.3), (2.4), and (2.5) we define

$$R(y,\sigma)(t) = -\frac{1}{(n-2)!}\int_0^t (t-s)^{n-2}\phi_q\left(\int_s^\infty (I-Q)N_\sigma y(\tau)\,d\tau\right)ds. \tag{2.21}$$

Clearly, $R(y,0) = 0$. For $y \in \Omega_\sigma = \{y \in \overline{W} : Ly = N_\sigma y\}$,

$$\bigl(\phi_p\bigl(y^{(n-1)}(t)\bigr)\bigr)' = \sigma h\bigl(t, y(t), y'(t), \dots, y^{(n-1)}(t)\bigr) \in \operatorname{Im} L \subset \ker Q.$$

Hence

$$\begin{aligned}
R(y,\sigma)(t) &= -\frac{1}{(n-2)!}\int_0^t (t-s)^{n-2}\phi_q\left(\int_s^\infty (I-Q)N_\sigma y(\tau)\,d\tau\right)ds \\
&= \int_0^t (t-s)^{n-2} y^{(n-1)}(s)\,ds \\
&= y(t) - \frac{y^{(n-2)}(0)t^{n-2}}{(n-2)!} - \frac{y^{(n-3)}(0)t^{n-3}}{(n-3)!} \\
&= (I-P)y(t).
\end{aligned} \tag{2.22}$$

Similarly,

$$\begin{aligned}
L\bigl[P + R(y,\sigma)\bigr](t) = \Biggl\{\phi_p\Biggl[&\frac{y^{(n-3)}(0)t^{n-3}}{(n-3)!} + \frac{y^{(n-2)}(0)t^{n-2}}{(n-2)!} \\
&- \frac{1}{(n-2)!}\int_0^t (t-s)^{n-2}\phi_q\left(\int_s^\infty (I-Q)N_\sigma y(\tau)\,d\tau\right)ds)\Biggr]^{(n-1)}\Biggr\}'
\end{aligned} \tag{2.23}$$

$$= \left\{-\phi_p\left[\phi_q\left(\int_t^\infty (I-Q)N_\sigma y(\tau)\,d\tau\right)\right]\right\}'$$
$$= (I-Q)N_\sigma y(t).$$

This verifies (2.3) and (2.4). Next we show that R is relatively compact for $\sigma \in [0,1]$.

Let $W \subset Y$ be a bounded set, that is, there exists $r > 0$ such that $r = \sup\{\|y\| : y \in W\}$. Since L is $L^1[0,\infty)$ Caratheodory, there exists $\alpha_r \in L^1[0,\infty)$ such that for $y \in W$ and a.e. $t \in [0,\infty)$

$$\left|h\left(t, y(t), y'(t), \dots, y^{(n-1)}(t)\right)\right| \le \alpha(t). \tag{2.24}$$

Therefore, for $y \in W$,

$$\int_0^\infty \left|N_\sigma y(\tau)\right| d\tau + \int_0^\infty \left|QN_\sigma y(\tau)\right| d\tau \le \|\alpha_r\|_1 + \|QN_\sigma\|_1, \tag{2.25}$$

where $\|z\|_1 = \int_0^\infty |z(t)|\,dt$, $z \in Z$.

For $y \in W$ and setting

$$E_n = \max_{0\le i\le n-2}\left(\sup_{t\in[0,\infty)} e^{-t}t^{n-2-i}\right), \tag{2.26}$$

we have for $0 \le i \le n-2$

$$\begin{aligned} e^{-t}\left|R^{(i)}(y,\sigma)(t)\right| &= e^{-t}\left|-\frac{1}{(n-2-i)!}\int_0^t (t-s)^{n-2-i}\phi_q\left(\int_s^\infty (I-Q)N_\sigma y(\tau)\,d\tau\right)ds\right| \\ &\le \max_{0\le i\le n-2}\left(\sup_{t\in[0,\infty)} e^{-t}t^{n-2-i}\right)\phi_q\left(\|\alpha_r\|_1 + \|QN_\sigma\|_1\right) \\ &= E_n\phi_q\left(\|\alpha_r\|_1 + \|QN_\sigma\|_1\right). \end{aligned} \tag{2.27}$$

For $i = n-1$,

$$\begin{aligned} e^{-t}\left|R^{(n-1)}(y,\sigma)(t)\right| &= e^{-t}\left|\phi_q\left(\int_s^\infty (I-Q)N_\sigma y(\tau)\,d\tau\right)\right| \\ &\le \phi_q\left(\|\alpha_r\|_1 + \|QN_\sigma\|_1\right). \end{aligned} \tag{2.28}$$

Therefore from (2.27) and (2.28) we obtain

$$\left\|R(y,\sigma)\right\| \le \max(E_n, 1)\phi_q\left(\|\alpha_r\|_1 + \|QN_\sigma\|_1\right) = C. \tag{2.29}$$

Thus $R(y,\sigma)$ is uniformly bounded in Y. For $t_1, t_2 \in [0,D]$, $D \in (0,\infty)$ with $t_1 < t_2$, $y \in W$ and $0 \le i \le n-2$, we have

$$\begin{aligned} \left|e^{-t_2}R^{(i)}(y,\sigma)(t_2) - e^{-t_1}R^{(i)}(y,\sigma)(t_1)\right| &= \left|\int_{t_1}^{t_2}\left[e^{-\tau}R^{(i)}(y,\sigma)(\tau)\right]'\,d\tau\right| \\ &= \left|\int_{t_1}^{t_2}\left[-e^{-\tau}R^{(i)}(y,\sigma)(\tau) + e^{-\tau}R^{(i+1)}(y,\sigma)(\tau)\right]d\tau\right| \\ &\le 2(t_2-t_1)\left\|R(y,\sigma)\right\| \to 0 \quad \text{as } t_1 \to t_2. \end{aligned}$$

For $i = n-1$,

$$\begin{aligned}
&\left|e^{-t_2}\phi_p(R^{(n-1)}(y,\sigma)(t_2) - e^{-t_1}\phi_p(R^{(n-1)}(y,\sigma)(t_1)\right| \\
&\quad = \left|e^{-t_2}\int_{t_2}^{\infty}(I-Q)N_\sigma y(\tau)\,d\tau - e^{-t_1}\int_{t_1}^{\infty}(I-Q)N_\sigma y(\tau)\,d\tau\right| \\
&\quad \leq \left|e^{-t_2} - e^{-t_1}\right|\int_{t_2}^{\infty}\left|(I-Q)N_\sigma y(\tau)\right|d\tau + e^{-t_1}\int_{t_2}^{t_1}\left|(I-Q)N_\sigma y(\tau)\right|d\tau \\
&\quad \leq \left|e^{-t_2} - e^{-t_1}\right|\left[\|\alpha_r\|_1 + \|QN_\sigma\|_1\right] + e^{-t_1}\int_{t_2}^{t_1}\left[|\alpha_r| + |QN_\sigma|\right]d\tau \\
&\quad \to 0 \quad \text{as } t_1 \to t_2.
\end{aligned}$$

Thus

$$\left|e^{-t_2}R^{(n-1)}(y,\sigma)(t_2) - e^{-t_1}R^{(n-1)}(y,\sigma)(t_1)\right| \to 0 \quad \text{as } t_1 \to t_2.$$

We therefore conclude that $R(y,\sigma)$ is equicontinuous on every compact subset of $[0,\infty)$. We next show that $R(y,\sigma)(W)$ is equiconvergent a infinity.

For $y \in W$ and $0 \leq i \leq n-2$, we have

$$\begin{aligned}
e^{-t}\left|R^{(i)}(y,\sigma)(t)\right| &= e^{-t}\left|\frac{1}{(n-2-i)!}\int_0^t (t-s)^{n-2-i}\phi_q\left(\int_s^{\infty}(I-Q)N_\sigma y(\tau)\,d\tau\right)ds\right| \\
&\leq e^{-t}t^{n-2-i}\phi_q\left(\|\alpha_r\|_1 + \|QN_\sigma\|_1\right) \to 0 \quad \text{as } t \to \infty.
\end{aligned}$$

For $i = n-1$,

$$\begin{aligned}
e^{-t}\left|R^{(n-1)}(y,\sigma)(t)\right| &= e^{-t}\left|\phi_q\left(\int_t^{\infty}(I-Q)N_\sigma y(\tau)\,d\tau\right)\right| \\
&\leq \phi_q\left(\int_t^{\infty}\left(\left|\alpha_r(\tau)\right| + \left|QN_\sigma y(\tau)\right|\right)d\tau\right) \\
&\to 0 \quad \text{as } t \to \infty.
\end{aligned}$$

Therefore $R(y,\sigma)(W)$ is equiconvergent at infinity. Thus all the conditions of Theorem 2.1 are satisfied. The continuity of $R(y,\sigma)$ follows from the Lebesque convergence theorem. Hence, N_σ is compact in $\overline{W}$. □

3 Main result

We assume the following conditions:

(H_1) $\sum_{i=1}^{m}\alpha_i = \sum_{j=1}^{m}\beta_j = \sum_{j=1}^{m}\beta_j\eta_j = 1$.

(H_2) There exist functions $a_i, r \in L^1[0,\infty)$ such that for a.e. $t \in [0,\infty)$

$$\left|h(t,y_1,y_2,\ldots,y_n)\right| \leq \phi_p\left(e^{-t}\right)\left[\sum_{i=1}^{n}a_i(t)\left|y_i(t)\right|^{p-1}\right] + r(t). \tag{3.1}$$

(H_3) For $y \in \operatorname{dom} L$, there exist constants $D > 0$, $B_n > 0$ such that if $|y^{(n-3)}(t)| > B_n$ for $t \in [0, D]$ or $|y^{(n-2)}(t)| > B_n$ for every $t \in [0, \infty)$, then either

$$Q_1 Ny(t) \neq 0 \quad \text{or} \quad Q_2 Ny(t) \neq 0.$$

(H_4) There exists a constant $D_n > 0$ such that for $|y^{(n-3)}(0)| > D_n$ or $y^{(n-2)}(0) > D_n$ either

$$Q_1 N\bigl(at^{n-3} + bt^{n-2}\bigr) + Q_2\bigl(at^{n-3} + bt^{n-2}\bigr) < 0, \quad t \in (0, \infty)$$

or

$$Q_1 N\bigl(at^{n-3} + bt^{n-2}\bigr) + Q_2\bigl(at^{n-3} + bt^{n-2}\bigr) > 0, \quad t \in (0, \infty).$$

Theorem 3.1 *If conditions* (H_1)–(H_4) *are fulfilled, then boundary value problem* (1.1)–(1.2) *has at least one solution provided*

$$2^{q-2}\left(\sum_{i=1}^{n} \|a_i\|_i\right)^{q-1} E_n(1 + D) < 1 \quad \text{if } 1 < p \leq 2 \tag{3.2}$$

or

$$\left(\sum_{i=1}^{n} \|a_i\|_1\right)^{q-1} E_n(1 + D) < 1 \quad \text{if } p > 2. \tag{3.3}$$

Proof We construct an open bounded set $W \subset Y$ that satisfies the assumptions of Theorem 2.1. Let $U_1 = \{y \in \operatorname{dom} L : Ly = N_\sigma y, \sigma \in (0,1)\}$. For $y \in U_1$, then $QN_\sigma y = 0$. Therefore from (H_3) there exist $t_1 \in [0, D]$, $t_2 \in [0, \infty)$ such that $y^{(n-3)}(t_1) < B_n$, $y^{(n-3)}(t_2) < B_n$,

$$\bigl|y^{(n-2)}(t)\bigr| = \left|y^{(n-2)}(t_2) - \int_t^{t_2} y^{(n-1)}(s)\,ds\right| \leq B_n + \bigl\|y^{(n-1)}\bigr\|_1. \tag{3.4}$$

Hence

$$\bigl\|y^{(n-2)}\bigr\|_\infty \leq B_n + \bigl\|y^{(n-1)}\bigr\|_1, \tag{3.5}$$

$$\bigl|y^{(n-3)}(0)\bigr| = \left|y^{(n-3)}(t_1) - \int_0^{t_1} y^{(n-2)}(s)\,ds\right| \leq B_n + \bigl\|y^{(n-2)}\bigr\|_\infty D. \tag{3.6}$$

From (3.4) we obtain

$$\bigl|y^{(n-2)}(0)\bigr| \leq B_n + \bigl\|y^{(n-1)}\bigr\|_1. \tag{3.7}$$

From $y \in U_1$, $(I - P)y \in \operatorname{dom} L \cap \ker P$. Hence, from (2.22) and (2.29), we derive

$$\bigl\|(I - P)\bigr\| = \bigl\|R(y, \sigma)\bigr\| \leq C. \tag{3.8}$$

From the definition of P in (2.18) we obtain

$$(Py)^{(i)}(t) = \frac{y^{(n-3)}(0)t^{n-3-i}}{(n-3-i)!}\ 0 \leq i \leq n-3 + \frac{y^{(n-2)}(0)t^{n-2-i}}{(n-2-i)!}\ 0 \leq i \leq n-2,$$

$$\begin{aligned}\|Py\| &\le \max\Big[\max_{0\le i\le n-3}\Big(\big|y^{(n-3)}(0)\big|\sup_{t\in[0,\infty)} e^{-t}t^{n-3-i} + \big|y^{(n-2)}(0)\big|\sup_{t\in[0,\infty)} e^{-t}t^{n-2-i}\Big),\\ &\quad \sup_{t\in[0,\infty)} e^{-t}\big|y^{(n-2)}(0)\big|\Big]\\ &\le A_n\big[\big|y^{(n-3)}(0)\big| + \big|y^{(n-2)}(0)\big|\big],\end{aligned} \tag{3.9}$$

where

$$A_n = \max\Big[\max_{0\le i\le n-3}\Big(\sup_{t\in[0,\infty)} e^{-t}t^{n-3-i} + \sup_{t\in[0,\infty)} e^{-t}t^{n-2-i}\Big), 1\Big]. \tag{3.10}$$

Hence, from (3.6) and (3.7), we get

$$\begin{aligned}\|Py\| &\le A_n\big(B_n + \big\|y^{(n-1)}\big\|_1 + B_n + \big\|y^{(n-2)}\big\|_\infty D\big)\\ &\le A_n\big[B_n + \big\|y^{(n-1)}\big\|_1 + B_n + D\big(B_n + \big\|y^{(n-1)}\big\|_1\big)\big]\\ &= 2B_nA_n + A_nB_nD + A_n\big\|y^{(n-1)}\big\|_1 + A_nD\big\|y^{(n-1)}\big\|_1\\ &= B_nA_n(2+D) + \big\|y^{(n-1)}\big\|_1(A_n + A_nD),\end{aligned} \tag{3.11}$$

$$\begin{aligned}\|y\| &= \big\|Py + (I-P)y\big\| \le \|Py\| + \big\|(I-P)y\big\|\\ &\le B_nA_n(2+D) + \big\|y^{(n-1)}\big\|_1(A_n + A_nD) + C.\end{aligned} \tag{3.12}$$

If $p \le 2$, then from (2.6), (2.17), and (3.1), we obtain

$$\begin{aligned}\big\|y^{(n-1)}\big\|_1 &= \int_0^\infty \Big|\phi_q\Big(\int_t^\infty N_\sigma y(\tau)\,d\tau\Big)\Big|\,dt\\ &\le \phi_q\Big(\sum_{i=1}^n \|a_i\|_1\|y\|^{p-1} + \|r\|_1\Big)\\ &\le 2^{q-2}\Big[\Big(\sum_{i=1}^n \|a_i\|_1\Big)^{q-1}\|y\| + \|r\|_1^{q-1}\Big].\end{aligned} \tag{3.13}$$

Using (3.2) in (3.13), we derive

$$\big\|y^{(n-1)}\big\|_1 \le 2^{q-2}\Big\{\Big(\sum_{i=1}^n \|a_i\|_1\Big)^{q-1}\big[B_nA_n(2+D) + \big\|y^{(n-1)}\big\|_1(A_n + A_nD)\big] + C_n + \|r\|_1^{q-1}\Big\}$$

or

$$\begin{aligned}&\Big[1 - 2^{q-2}\Big(\sum_{i=1}^n \|a_i\|_1\Big)^{q-1} A_n(1+D)\Big]\big\|y^{(n-1)}\big\|_1\\ &\quad\le 2^{q-2}\Big(\sum_{i=1}^n \|a_i\|_1\Big)^{q-1}\big[B_nA_n(2+D) + C_n\big] + 2^{q-2}\|r\|_1^{q-1},\\ &\big\|y^{(n-1)}\big\|_1 \le \frac{2^{q-2}(\sum_{i=1}^n \|a_i\|_1)^{q-1}[B_nA_n(2+D) + C] + 2^{q-2}\|r\|_1^{q-1}}{1 - 2^{q-2}(\sum_{i=1}^n \|a_i\|_1)^{q-1}A_n(1+D)}.\end{aligned} \tag{3.14}$$

From (3.12) and (3.14), we obtain $C_n^* > 0$ such that $\|y\| \le C_n^*$. So U_1 is bounded.

Let $U_2 = \{y \in \ker L : N_\sigma y \in \operatorname{Im} L\}$. For $y \in U_2 = \{y \in \ker L : y(t) = at^{n-3} + bt^{n-2}, a, b \in \mathbb{R}, t \in (0,\infty)\}$, $Ny \in \operatorname{Im} L$ implies that $QNy = 0$, and hence

$$Q_1N\left(at^{n-3} + bt^{n-2}\right) = Q_2N\left(at^{n-3} + bt^{n-2}\right) = 0.$$

From (H_4) we get

$$|a| + |b| < 2D_n. \tag{3.15}$$

Thus U_2 is bounded. We choose $W_0 > 0$ large enough such that

$$W = \left\{y \in W : \|y\| < W_0\right\} \supset \overline{U}_1 \cup \overline{U}_2.$$

Then, from the above computations, $Ly \neq Ny$ for $y \in \partial W \cap \operatorname{dom} L$. Thus, the first part of Theorem 2.2 is verified. Let

$$H(y,\lambda) = -\lambda Jy + (1-\lambda)QNy, \quad \lambda \in [0,1], \tag{3.16}$$

where $J : \ker L \to \operatorname{Im} Q$ is the homeomorphism

$$J\left(at^{n-3} + bt^{n-2}\right) = \frac{e^{-t}}{\Delta}\left[\left(c_{11}|a| + C_{12}|b|\right)t^{n-3} + \left(c_{21}|a| + c_{22}|b|\right)t^{n-2}\right]. \tag{3.17}$$

For $y \in W \cap \ker L$, $y(t) = at^{n-3} + bt^{n-2} \neq 0$ and $H(y,0) = QNy \neq 0$ since $Ny \notin \operatorname{Im} L$. Hence, for $\lambda = 0$, $\lambda = 1$, $H(y,\lambda) \neq 0$. Assume $H(y,\lambda) = 0$ for $0 < \lambda < 1$, where $y(t) = at^{n-3} + bt^{n-2} \in \partial W \cap \ker L$. Then from (3.16), (3.17) we obtain

$$\begin{aligned}
&\lambda\left[c_{11}|a| + c_{12}|b|\right] = (1-\lambda)\left[c_{11}Q_1N\left(at^{n-3} + bt^{n-2}\right) + c_{12}Q_2N\left(at^{n-3} + bt^{n-2}\right)\right],\\
&\lambda\left[c_{21}|a| + c_{22}|b|\right] = (1-\lambda)\left[c_{21}Q_1N\left(at^{n-3} + bt^{n-2}\right) + c_{22}Q_2N\left(at^{n-3} + bt^{n-2}\right)\right],
\end{aligned}$$

or

$$\begin{aligned}
&c_{11}\left[\lambda|a| - (1-\lambda)Q_1N\left(at^{n-3} + bt^{n-2}\right)\right] + c_{12}\left[\lambda|b| - (1-\lambda)Q_2N\left(at^{n-3} + bt^{n-2}\right)\right] = 0,\\
&c_{21}\left[\lambda|a| - (1-\lambda)Q_1N\left(at^{n-3} + bt^{n-2}\right)\right] + c_{22}\left[\lambda|b| - (1-\lambda)Q_2N\left(at^{n-3} + bt^{n-2}\right)\right] = 0.
\end{aligned}$$

Since $\Delta = \left|\begin{smallmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{smallmatrix}\right| = c_{22}c_{11} - c_{21}c_{22} \neq 0$, then

$$\begin{aligned}
&\lambda|a| = (1-\lambda)Q_1N\left(at^{n-3} + bt^{n-2}\right),\\
&\lambda|b| = (1-\lambda)Q_2N9at^{n-3} + bt^{n-2}).
\end{aligned}$$

If $|a| > D_n$, $|b| > D_n$, then from (H_4) we obtain

$$\lambda\left(|a| + |b|\right) = (1-\lambda)\left[Q_1N\left(at^{n-3} + bt^{n-2}\right) + Q_2N\left(at^{n-3} + bt^{n-2}\right)\right] < 0,$$

which is a contradiction. If the second part of (H_4) holds, let

$$H(y,\lambda) = \lambda Jy + (1-\lambda)Qy, \quad \lambda \in [0,1].$$

Then, using a similar argument as above, we obtain a contradiction. Hence, $H(y,\lambda) \neq 0$ for $y \in \partial W \cap \ker L$, $\lambda \in [0,1]$. Therefore, by the invariance of the degree under a homotopy, we obtain

$$\begin{aligned}
\deg(QN|_{\ker L}, W \cap \ker L, 0) &= \deg\big((H\cdot, 0), W \cap \ker L, 0\big) \\
&= \deg\big(H(\cdot, 1), W \cap \ker L, 0\big) \\
&= \deg(\pm J, W \cap \ker L, 0) \\
&= \operatorname{sgn}\left\{\pm \begin{vmatrix} \frac{c_{11}}{\Delta} & \frac{c_{21}}{\Delta} \\ \frac{c_{21}}{\Delta} & \frac{c_{22}}{\Delta} \end{vmatrix}\right\} \\
&= \operatorname{sgn}\left(\frac{\pm 1}{\Delta}\right) = \pm 1 \neq 0.
\end{aligned}$$

Thus from Theorem 2.2 we conclude that $Ly = Ny$ has at least one solution in $\operatorname{dom} L \cap W$, which in turn implies that (1.1)–(1.2) has at least one solution in Y. □

4 Example

Consider the third order boundary value problem

$$\big(\phi_p\big(y''(t)\big)\big)' = h\big(t, y(t), y'(t), y''(t)\big), \quad t \in (0, \infty), \tag{4.1}$$

$$y'(\infty) = \sum_{i=1}^{2} \alpha_i y'(\xi_i), \qquad y(0) + y'(0) = \sum_{j=1}^{2} \beta_j y(\xi_j), \qquad y''(\infty) = 0 \tag{4.2}$$

corresponding to problem (1.1)–(1.2), we have $m = 2$, $n = 3$, $\beta_1 = -1$, $\beta_2 = 2$, $\eta_1 = 1/2$, $\eta_2 = 3/4$, $\alpha_1 = \alpha_2 = 1/2$, $\xi_1 = 1$ $\xi_2 = 2$, $p = 4/3$, $q = 4$. Then $\sum_{j=1}^{2} \beta_j \eta_j = \sum_{i=1}^{2} \alpha_i = \sum_{j=1}^{2} \beta_j = 1$. Hence condition (H_1) is satisfied.

$$\begin{aligned}
h\big(t, y, y', y''\big) &= e^{-t}\left[\frac{\sin^{\frac{1}{3}}}{24} + \frac{y'^{\frac{1}{3}}}{24} + \frac{\sin^{\frac{1}{3}} y'}{24} + \frac{y''^{\frac{1}{3}}}{48} + \frac{\sin^{\frac{1}{3}} y''}{48} - \frac{1}{24}\right], \\
\big|h\big(t, y, y', y''\big)\big| &\le e^{-\frac{t}{3}}\left[\frac{e^{-\frac{2}{3}t}|y|^{\frac{1}{3}}}{24} + \frac{e^{-\frac{2}{3}t}|y'|^{\frac{1}{3}}}{2} + \frac{e^{-\frac{2}{3}t}|y''|^{\frac{1}{3}}}{24}\right] - \frac{e^{-t}}{24}.
\end{aligned}$$

Thus condition (H_2) is verified. To verify conditions (H_3) and (H_4), we have

$$\Delta = c_{11}c_{22} - c_{12}c_{22} = 072(0.076) - 0.018(0622) = 0.497 \neq 0.$$

$a_1(t) = \frac{e^{-\frac{2}{3}t}}{24}$, $a_2(t) = \frac{e^{-\frac{2}{3}t}}{12}$, $a_3(t) = \frac{e^{-\frac{2}{3}t}}{24}$, $r(t) = -\frac{e^{-t}}{24}$. We set $B_n = 5^3$. Let $|y'(t)| > B_n$, then $y'(t) > B_n$ or $y'(t) < -B_n$. If $y'(t) > B_n$, then

$$\begin{aligned}
Q_2Ny &= \frac{1}{2}\int_{\frac{1}{2}}^{\infty}\left(\int_s^{\infty} e^{-t}\left[\frac{\sin^{\frac{1}{3}}}{24} + \frac{y'^{\frac{1}{3}}}{24} + \frac{\sin^{\frac{1}{3}} y'}{24} + \frac{y''^{\frac{1}{3}}}{48} + \frac{\sin^{\frac{1}{3}} y''}{48} - \frac{1}{24}\right] dt\right)^3 ds \\
&\quad + \frac{1}{2}\int_{\frac{3}{4}}^{\infty}\left(\int_s^{\infty} e^{-t}\left[\frac{\sin^{\frac{1}{3}}}{24} + \frac{y'^{\frac{1}{3}}}{24} + \frac{\sin^{\frac{1}{3}} y'}{24} + \frac{y''^{\frac{1}{3}}}{48} + \frac{\sin^{\frac{1}{3}} y''}{46} - \frac{1}{24}\right] dt\right)^3 ds \\
&> \frac{1}{2}\int_{\frac{1}{2}}^{\infty}\left(\int_s^{\infty} e^{-t}\left[-\frac{1}{24} + \frac{B_n^{\frac{1}{3}}}{24} - \frac{1}{24} - \frac{1}{48} - \frac{1}{24}\right] dt\right)^3 ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_{\frac{3}{4}}^{\infty}\left(\int_{s}^{\infty}e^{-t}\left[-\frac{1}{24}+\frac{B_n^{\frac{1}{3}}}{24}-\frac{1}{24}-\frac{1}{48}-\frac{1}{24}\right]dt\right)^3 ds \\
&=\frac{1}{2}\left(\frac{2B_n^{\frac{1}{3}}-7}{48}\right)^3\int_{\frac{1}{2}}^{\infty}\left(\int_{5}^{\infty}e^{-t}\right)^3 dt\,ds+\frac{1}{2}\left(\frac{2B_n^{\frac{1}{3}}-7}{48}\right)^3\int_{\frac{3}{4}}^{\infty}\left(\int_{5}^{\infty}e^{-t}\right)^3 dt\,ds \\
&>0.
\end{aligned}
$$

If $y'(t) < -B_n$, then

$$
\begin{aligned}
Q_2Ny &\leq \frac{1}{2}\int_{\frac{1}{2}}^{\infty}\left(\int_{s}^{\infty}e^{-t}\left[\frac{1}{24}-\frac{B_n^{\frac{1}{3}}}{24}+\frac{1}{24}+\frac{1}{48}-\frac{1}{24}\right]dt\right)^3 ds \\
&\quad+\frac{1}{2}\int_{\frac{3}{4}}^{\infty}\left(\int_{s}^{\infty}e^{-t}\left[\frac{1}{24}-\frac{B_n^{\frac{1}{3}}}{24}+\frac{1}{24}+\frac{1}{48}-\frac{1}{24}\right]dt\right)^3 ds \\
&=\frac{1}{2}\left(\frac{3-2B_n^{\frac{1}{3}}}{48}\right)^3\int_{\frac{1}{2}}^{\infty}\left(\int_{s}^{\infty}e^{-t}\right)^3 dt\,ds<0.
\end{aligned}
$$

Thus condition (H_3) is verified. Taking $D_n = 6^3$ then for $|b| > D_n$, that is, $b > D_n$ or $b < -D_n$. If $b > D_n$, then we can verify that

$$
Q_1(a+bt)+Q_2(a+bt)>0.
$$

Similarly, if $b < -D_n$, then

$$
Q_1(a+bt)+Q_2(a+bt)<0,
$$

which verifies (H_4). Finally, $\|a_1\|_1 = \frac{1}{16}$, $\|a_2\|_1 = \frac{1}{8}$, $\|a_3\|_1 = \frac{1}{16}$,

$$
\begin{aligned}
A_n &= \max\left\{\sup_{t\in[0,\infty)} e^{-t}+\sup_{t\in[0,\infty)} te^{-t},1\right\} \\
&= \max\left[1+e^{-1},1\right]=1+e^{-1}.
\end{aligned}
$$

Taking $D = 1$, we have for $P \leq 2$

$$
2^{q-2}\left(\sum_{i=1}^{3}\|a_i\|_1\right)^{q-1}A_n(1+D)=2^2\left(\frac{1}{4}\right)^3 2\left(1+e^{-1}\right)=\frac{1+e^{-1}}{8}<1.
$$

Hence, all the conditions of Theorem 3.1 are verified. Thus (4.1)–(4.2) has at least one solution.

Acknowledgements
The authors are grateful to Covenant University for its support.

Funding
Authors received no specific funding.

Availability of data and materials
Not applicable.

Declarations

Ethics approval and consent to participate
The work is original and has not been submitted elsewhere.

Competing interests
The authors declare no competing interests.

Author contributions
All authors contributed to the manuscript and read and approved the final manuscript.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 30 March 2022 Accepted: 21 June 2022 Published online: 11 July 2022

References

1. Agarwal, R.P.: Boundary Value Problems for Higher Order Differential Equations. World Scientific, Singapore (1986)
2. Agarwal, R.P., O'Regan, D.: Infinite Interval Problems for Differential, Difference and Integral Equations. Kluwer Academic, Dordrecht (2001)
3. Cabada, A., Liz, E.: Boundary value problems for higher order ordinary differential equations with impulses. Nonlinear Anal., Theory Methods Appl. **32**, 775–786 (1998)
4. Du, Z., Lin, X., Ge, W.: Some higher order multipoint boundary value problems at resonance. J. Comput. Appl. Math. **177**(1), 55–65 (2005)
5. Frioui, A., Guezane-Lakoud, A., Khaldi, R.: Higher order boundary value problems at resonance on an unbounded domain. Electron. J. Differ. Equ. **2016**, 29 (2016)
6. Ge, W., Ren, J.: An extension of Mawhin's continuation theorem and its application to boundary value problems with a p-Laplacian. Nonlinear Anal. **58**, 477–488 (2004)
7. Iyase, S.A., Imaga, O.F.: Higher order boundary value problems with integral boundary conditions on the half-line. J. Niger. Math. Soc. **38**(2), 165–183 (2019)
8. Iyase, S.A., Opanuga, A.A.: Higher order nonlocal boundary value problems at resonance on the half-line. Eur. J. Pure Appl. Math. **13**, 33–44 (2020)
9. Jeong, J., Kim, C.G., Lee, E.K.: Solvability for nonlocal boundary value problems on a half-line with $\dim\ker L = 2$. Bound. Value Probl. **2014**, 167 (2014)
10. Jiang, W., Kosmatov, N.: Resonant p-Laplacian problems with functional boundary conditions. Bound. Value Probl. **2018**, 72 (2018)
11. Jiang, W., Wang, B., Wang, Z.: Solvability of a second order multipoint boundary value problems at resonance on a half-line with $\dim\ker L = 2$. Electron. J. Differ. Equ. **2011**, 120 (2011)
12. Lin, X., Du, Z., Ge, W.: Solvability of multi-point boundary value problems at resonance for higher order ordinary differential equations. Comput. Math. Appl. **49**, 1–11 (2005)
13. Liu, Y.J.: Nonhomogeneous boundary value problems of higher order differential equations with p-Laplacian. Electron. J. Differ. Equ. **2008**, 20 (2008)
14. Mawhin, J.: Topological Degree Methods in Nonlinear Boundary Value Problems. NSF-CBM Regional Conference Series in Math, vol. 40. Am. Math. Soc., Providence (1979)
15. Royden, H.L.: Real Analysis, 3rd edn. Prentice Hall, Englewood Cliffs (1988)

Submit your manuscript to a SpringerOpen journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

Submit your next manuscript at ► springeropen.com