STABILITY ANALYSIS OF DELAY DIFFERENTIAL EQUATION MODELS IN MATHEMATICAL BIOLOGY

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A THESIS SUBMITTED TO THE SCHOOL OF POSTGRADUATE STUDIES IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF DOCTOR OF PHILOSOPHY (Ph. D) IN INDUSTRIAL MATHEMATICS IN THE DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE AND TECHNOLOGY, COVENANT UNIVERSITY, OTA, OGUN STATE, NIGERIA

MAY, 2024

ACCEPTANCE

This is to attest that this thesis is accepted in partial fulfilment of the requirements for the award of the degree of Doctor of Philosophy (Ph.D.) in Industrial Mathematics in the Department of Mathematics, College of Science and Technology, Covenant University, Ota, Ogun State, Nigeria.

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DECLARATION

I, **EZEKIEL IMEKELA DONALDSON (14PCD00857),** declare that this research work was carried out by me under the supervision of Prof. Samuel A. Iyase and Prof. Timothy A. Anake of the Department of Mathematics, College of Science and Technology, Covenant University, Ota, Ogun State, Nigeria. I attest that this thesis has not been presented either wholly or partially for the award of any degree elsewhere. All the sources of materials and scholarly publications used in the thesis are duly acknowledged accordingly.

EZEKIEL IMEKELA DONALDSON

Signature and Date

CERTIFICATION

We certify that this thesis titled "**STABILITY ANALYSIS OF DELAY DIFFERENTIAL EQUATION MODELS IN MATHEMATICAL BIOLOGY**" is an original work carried out by **EZEKIEL, Imekela Donaldson** (14PCD00857) in the Department of Mathematics, College of Science and Technology, Covenant University, Ota, Ogun State, Nigeria, under the supervision of Prof. Samuel A. Iyase and Prof. Timothy A. Anake. We have examined and found the work acceptable as part of the requirements for the award of Doctor of Philosophy (Ph. D) degree in Industrial Mathematics.

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DEDICATION

To the glory of God and my dear children:

Oshukunuofayo Francis, Shemaze Enoch and Ashimije Clement

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TABLE OF CONTENTS

LIST OF FIGURES

FIGURES **LIST OF FIGURES** **PAGES**

LIST OF SYMBOLS

$\mathbb{R}^n$	n-tuples of real numbers
$C\ ([-\tau, 0], \mathbb{R}^n)$	Space of continuous functions
Dim	Dimension
Ker	Kernel
Im	Image
Deg	Degree
$\subset$	Proper subset of
$\cap$	Intersection
$\cup$	Union
$\overline{\Omega}$	Closure of Ω
$\partial\Omega$	Boundary of Ω
Ind	Index
Coker	Co kernel
$\in$	is a member of or belongs to
$\forall$	for all
$\exists$	There exists
$\Vert \cdot \Vert$	Norm
$L^1[0, \infty)$	L^1 function space
$\mathbb{Z}$	The set of integers
$\mathbb{N}$	The set of natural numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$Re(\lambda)$	The real part of λ for $\lambda \in \mathbb{C}$
$Im(\lambda)$	The imaginary part of λ for $\lambda \in \mathbb{C}$
$\mathbb{C}_0$	The set $\{\lambda \in \mathbb{C}: Re(\lambda) = 0\}$

$\mathbb{C}_+$	The set $\{\lambda \in \mathbb{C}: Re(\lambda) > 0\}$
$\mathbb{C}_-$	The set $\{\lambda \in \mathbb{C}: Re(\lambda) < 0\}$

ABBREVIATIONS

a.e.	almost every
BVP	Boundary Value Problem
CMT	Center Manifold Theorem
DDEs	Delay Differential Equations
DIEs	Differential Integral Equations
DFEs	Delay Functional Equations
FDEs	Functional Differential Equations
FDDEs	Functional Delay Differential Equations
ivc	Initial value conditions
O_pDE	Operator differential equation
ODEs	Ordinary Differential Equations
RFDEs	Retarded Functional Differential Equations
n.d.	negative definite
n.s.d.	negative semi definite
p.d.	positive definite
p.s.d.	positive semi definite

ABSTRACT

This thesis studied differential equation models in mathematical biology with finite discrete delays for the stability of the linearised systems. It investigated the dynamical behaviour of the more general single species delay independent and delay dependent models with linear harvesting functions of delayed estimate of the true population. The study equally investigated the dynamical behaviour of susceptible (S), infectious (I) and recovered (R) (SIR) disease epidemic model with intra cellular delay. The study employed the characteristics and bifurcation techniques for the analysis of the linearised system for determining conditions for stability and instability of the systems. The normal form theory and the center manifold theorem (CMT) were employed for stability analysis where linearisation method does not apply. The popular reproduction number R_0 for the disease free equilibrium (DFE) was further used to determine conditions under which the center manifold can be applied while using the threshold theorem of epidemiology to determine conditions for stability analysis. The results of the single species delay population models showed that a sequence of Hopf bifurcations occurred when the bifurcation parameters crossed some critical values. The critical positive delay periodic solutions guaranteed the existence of the critical value, $\bar{\tau}$ from which the threshold theorem of epidemiology can be determined. If $\tau \in [0, \bar{\tau})$, the positive equilibrium of the models keeps a steady state. As the delay $\bar{\tau}$ varies, some $\bar{\tau} = \tau_k$, $k = 1,2,\cdots$ of the positive equilibrium become unstable and Hopf bifurcations occur. The conditions of Hopf bifurcations were sufficient and chaotic phenomenon appeared when model maturation delays are large. However, conditions for local stability of equilibria were found in the disease epidemic equation models using the reproduction number, R_0. It was observed that the positive equilibrium E^* is locally stable when $R_0 < 1$, which indicates that the infection is no longer present in the population. It is also noted that E^* is unstable when $R_0 > 1$ and this indicates that the infection exists in the system. Using the CMT, it was observed that when $R_0 = 1$ the decoupled disease epidemic system undergoes forward bifurcation. In particular, the results of the disease epidemic models showed that the conditions for bifurcations obtained from the behavior of the dynamical systems are sufficient but not necessary as the models were unable to stabilise the unstable interior non hyperbolic equilibrium due to Hopf bifurcations. Specifically, the direction of Hopf bifurcations, stability and the period of the bifurcating periodic solutions of the non hyperbolic disease reduced model were explicitly determined by applying the normal form concept and the center manifold reduction theorem to the perturbed operator differential equation (OPDE) where the linearised equation has at least one characteristic root with zero real part while every other eigenvalue has negative real part. To ensure model relevance, the study is recommended to ecologists, biologists and public health workers. Finally, numerical simulations to verify the analytically findings in support of stability for both population and disease epidemic models were performed using MATLAB software and Pic Wish Version 1.5.6.

Keywords: ***Center Manifold theorem; Characteristic equation; Hopf bifurcation; Jacobian matrix; Lyapunov functional; Stability analysis; Time delay***.