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HIGHER FRACTIONAL ORDER p-LAPLACIAN BOUNDARY VALUE PROBLEM AT RESONANCE ON AN UNBOUNDED DOMAIN

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Abstract

In this work, we use the Ge and Ren extension of Mawhin's coincidence degree theory to investigate the solvability of the p-Laplacian fractional order boundary value problem of the form

$$(\phi_p(D_{0+}^{\alpha}x(t)))'$$
$$= f(t, x(t), D_{0+}^{\alpha-3}x(t), D_{0+}^{\alpha-2}x(t), D_{0+}^{\alpha-1}x(t), D_{0+}^{\alpha}x(t)), \quad t \in (0, +\infty),$$

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$$x(0) = 0 = D_{0+}^{\alpha-3}x(0), \quad D_{0+}^{\alpha-2}x(0) = \int_0^1 D_{0+}^{\alpha-2}x(t)\,dA(t),$$

$$\lim_{t\to+\infty} D_{0+}^{\alpha-1}x(t) = \sum_{i=1}^{m}\mu_i D_{0+}^{\alpha-1}x(\xi_i), \quad D_{0+}^{\alpha}x(\infty) = 0,$$

where $3 < \alpha \le 4$. The conditions $\int_0^1 dA(t) = 1$, $\int_0^1 t dA(t) = 0$, $\sum_{i=1}^{m}\mu_i = 1$ and $\sum_{i=1}^{m}\mu_i\xi_i^{-1} = 0$ are critical for resonance.

1. Introduction

The aim of this paper is to establish the existence of at least one solution for the p-Laplacian boundary value problem

$$(\phi_p(D_{0+}^{\alpha}x(t)))' = f(t, x(t), D_{0+}^{\alpha-3}x(t), D_{0+}^{\alpha-2}x(t), D_{0+}^{\alpha-1}x(t), D_{0+}^{\alpha}x(t)), \quad t \in (0, +\infty), \tag{1.1}$$

$$x(0) = 0 = D_{0+}^{\alpha-3}x(0), \quad D_{0+}^{\alpha-2}x(0) = \int_0^1 D_{0+}^{\alpha-2}x(t)\,dA(t),$$

$$\lim_{t\to+\infty} D_{0+}^{\alpha-1}x(t) = \sum_{i=1}^{m}\mu_i D_{0+}^{\alpha-1}x(\xi_i), \quad D_{0+}^{\alpha}x(\infty) = 0, \tag{1.2}$$

where $f : (0, +\infty)\times\mathbb{R}^5 \to \mathbb{R}$ is an L^1-Carathéodory function, D_{0+}^{α} is the fractional derivative of order α of Riemann-Liouville type; $3 < \alpha \le 4$, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < +\infty$, $\mu_i \in \mathbb{R}$ for $i = 1, 2, ..., m$, $A(t)$ is a function of bounded variation and the p-Laplacian operator is defined as $\phi_p(s) = |s|^{p-2}s$, $p > 2$, $\phi_p^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$.

When $p = 2$, the differential operator in BVP (1.1)-(1.2) is linear and the results for such are obtained by using Mawhin's coincidence degree argument. The fractional order p-Laplacian BVP (1.1)-(1.2) is nonlinear, since $p > 2$ and Mawhin's continuation theorem can no longer be applied directly.

In [12], the authors obtained existence results for the following fractional order boundary value problem with linear differential operator on the half-line by using Mawhin's coincidence degree arguments:

$$D_{0+}^{\alpha}x(t) = f(t, x(t), D_{0+}^{\alpha-3}x(t), D_{0+}^{\alpha-2}x(t), D_{0+}^{\alpha-1}x(t)), \quad t \in (0, +\infty), \quad (1.3)$$

$$x(0) = 0 = D_{0+}^{\alpha-3}x(0), \quad D_{0+}^{\alpha-2}x(0) = \sum_{i=1}^{m} \mu_i D_{0+}^{\alpha-2}x(\xi_i),$$

$$D_{0+}^{\alpha-1}x(+\infty) = \int_0^{\eta} D_{0+}^{\alpha-2}x(t)dA(t), \quad (1.4)$$

under resonant conditions $\sum_{i=1}^{m}\mu_i = 1$, $\sum_{i=1}^{m}\mu_i\xi_i = 0$, $\int_0^{\eta} tdA(t) = 1$ and $\int_0^{\eta} dA(t) = 0$, where $\mu_i \in \mathbb{R}$, $3 < \alpha \leq 4$, $\dim \ker L = 2$, $0 < \xi_1 < \xi_2 < \xi_3 < \cdots < \xi_m < \infty$, $\eta \in (0, +\infty)$, $A(t)$ is a continuous and bounded variation function on $(0, +\infty)$, f is a Carathéodory function, and D_{0+}^{α} is the standard Riemann-Liouville fractional derivative.

In this paper, the differential operator is nonlinear, hence we will utilize Ge and Ren extension of the coincidence degree theory for the existence result.

Fractional order boundary value problems have wide applications in modelling processes in viscoelastic media, steady flow of gas, blood flow, electromagnetics, acoustics, control theory, electrochemistry, finance, and material science [6, 11, 13].

For some recent works on fractional order boundary value problems with a p-Laplacian operator on the half-line, where Ge and Ren extension of the coincidence degree was applied to establish the existence results, we refer to [4-9, 13].

The contribution of this paper is to extend the results in a linear case (1.3)-(1.4) to a p-Laplacian boundary value problem with a nonlinear differential operator.

2. Materials and Methods

We state and prove some relevant lemmas. Let $(X, \|\cdot\|_X)$ and $(W, \|\cdot\|_W)$ be any two Banach spaces such that

$$X = \{x(t) \in \mathbb{C}^2[0, +\infty) : (x(t), D_{0+}^{\alpha-3}x(t), D_{0+}^{\alpha-2}x(t),$$

$$D_{0+}^{\alpha-1}x(t), D_{0+}^{\alpha}x(t)) \in \mathbb{C}[0, +\infty), \sup_{t\geq 0}\frac{|x(t)|}{1+t^{\alpha}} < +\infty,$$

$$\sup_{t\geq 0}\frac{|D_{0+}^{\alpha-3}x(t)|}{1+t^3} < +\infty, \sup_{t\geq 0}\frac{|D_{0+}^{\alpha-2}x(t)|}{1+t^{\alpha-1}} < +\infty,$$

$$\sup_{t\geq 0}\frac{|D_{0+}^{\alpha-1}x(t)|}{1+t^{\alpha-2}} < +\infty, \sup_{t\geq 0}|D_{0+}^{\alpha}x(t)| < +\infty\}$$

and $W = L^1(0, +\infty)$ with norms

$$\|x\|_X = \max\{\|x(t)\|_0, \|D_{0+}^{\alpha-3}x(t)\|_0,$$

$$\|D_{0+}^{\alpha-2}x(t)\|_0, \|D_{0+}^{\alpha-1}x(t)\|_0, \|D_{0+}^{\alpha}x(t)\|_\infty\},$$

$\|w\|_W = \|w\|_{L^1}$, where

$$\|x(t)\|_0 = \sup_{t\geq 0}\frac{|x(t)|}{1+t^{\alpha}}, \quad \|D_{0+}^{\alpha-3}x(t)\|_0 = \sup_{t\geq 0}\frac{|D^{\alpha-3}x(t)|}{1+t^3},$$

$$\|D^{\alpha-2}x(t)\|_0 = \sup_{t\geq 0}\frac{|D^{\alpha-2}x(t)|}{1+t^{\alpha-1}}, \quad \|D_{0+}^{\alpha-1}x(t)\|_0 = \sup_{t\geq 0}\frac{|D^{\alpha-1}x(t)|}{1+t^{\alpha-2}},$$

$$\|D_{0+}^{\alpha}x(t)\|_\infty = \sup_{t\geq 0}|D_{0+}^{\alpha}x(t)| \quad \text{and} \quad \|w\|_{L^1} = \int_0^{+\infty}|w(t)|dt.$$

Let M be a continuous operator such that

$$M : dom\, M \subset X \to W,$$

where

$$dom\, M = \{x \in X : x(0) = 0 = D_{0+}^{\alpha-3}x(0),$$
$$D_{0+}^{\alpha-2}x(0) = \int_0^1 D_{0+}^{\alpha-2}x(t)\,dA(t),$$
$$\lim_{t\to\infty} D_{0+}^{\alpha-1}x(t) = \sum_{i=1}^{M} \mu_i D_{0+}^{\alpha-1}x(\xi_i),\ D_{0+}^{\alpha}x(+\infty) = 0\}.$$

The fractional order differential equation (1.1) can be written in abstract form as

$$Mx(t) = Nx(t), \tag{2.1}$$

where

$$Mx(t) = (\phi_p(D_{0+}^{\alpha}x(t)))',$$

$$N_\lambda x(t) = \lambda f(t,\ x(t),\ D_{0+}^{\alpha-3}x(t),\ D_{0+}^{\alpha-2}x(t),\ D_{0+}^{\alpha-1}x(t),\ D_{0+}^{\alpha}x(t)),\ \text{and}\ \lambda = 1.$$

Definition 1 [3]**.** An operator $M : X \cap dom\, M \to W$ is said to be *quasilinear* if

(i) $\operatorname{Im} M = M(X \cap dom\, M)$ is a closed subset of W;

(ii) $\ker M = \{x \in X \cap dom\, M : Mx = 0\}$ is linearly homeomorphic to $\mathbb{R}^n$.

Definition 2. An operator $T : X \to W$ is said to be *bounded* if $T(U) \subset W$ is bounded for any bounded subset $U \subset X$. Let $X_1 = \ker M$ and X_2 be the complement space of X_1 in X such that $X = X_1 \oplus X_2$. Also, let $W_1 \subset W$ be a subspace of W and W_2 be the complement space of W_1 in W such that $W = W_1 \oplus W_2$. Let $P : X \to X_1$ be a projector, $Q : W \to W_1$ a semi-projector, and $\Omega \subset X$ an open bounded set with the origin $0 \in \Omega$.

Definition 3. If W_1 is a subset of a Banach space W, then the mapping $Q : W \to W_1$ is a *semi-projector* if

$$Q^2 w = Qw \quad \text{and} \quad Q(kw) = kQw, \text{ for any } w \in W \text{ and } k \in \mathbb{R}.$$

Definition 4. A linear operator $T : V \to V$, where V is a vector space, is a *projector* if $T^2 v = Tv$ for any $v \in V$.

Lemma 5 [2]**.** *Let* $\phi_p(s) = |s|^{p-2}s$. *Then* $\phi_p : \mathbb{R} \to \mathbb{R}$ *satisfies the following properties*:

(i) ϕ_p *is continuous*, *monotonically increasing*, *and invertible* $(\phi_p^{-1} = \phi_q, \frac{1}{p} + \frac{1}{q} = 1$, *for* $q > 1)$;

(ii) $\phi_p(x + y) \leq \phi_p(x) + \phi_p(y)$, *if* $1 < p \leq 2$;

(iii) $\phi_p(x + y) \leq 2^{p-2}(\phi_p(x) + \phi_p(y))$, *if* $p > 2$.

Definition 6 [3]**.** An operator N_λ is said to be *M-compact* in $\overline{\Omega}$ if there exists a vector subspace $W_1 \subset W$, such that $\dim W_1 = \dim X_1$ and a compact and continuous operator, $T : \overline{\Omega} \times [0, 1] \to X_2$ such that for $\lambda \in [0, 1]$:

(i) $(I - Q)N_\lambda(\overline{\Omega}) \subset \operatorname{Im} M \subset (I - Q)W$;

(ii) $QN_\lambda x = 0 \Leftrightarrow QNx = 0, \lambda \in (0, 1)$;

(iii) $T(\cdot, 0)$ is the zero operator;

(iv) $T(\cdot, \lambda)|X_\lambda = (I - P)|X_\lambda$, where $X_\lambda = \{x \in \overline{\Omega} : Mx = N_\lambda x\}$ and

(v) $M[P + T(\cdot, \lambda)] = (I - Q)N_\lambda$.

Theorem 7 [3]**.** *Let* $(X, \|\cdot\|_X)$ *and* $(W, \|\cdot\|_W)$ *be any two Banach spaces and* $\Omega \subset X$ *be an open bounded set of X. Then the following* *properties hold*:

(i) *the operator* $M : X \cap dom\, M \to W$ *is quasilinear*;

(ii) *the operator* $N_\lambda : \overline{\Omega} \to W,\ \lambda \in [0, 1]$ *is M-compact*;

(iii) $QNx \neq 0,\ \forall x \in \ker M \cap \partial\Omega$;

(iv) $Mx \neq N_\lambda x,\ \lambda \in [0, 1],\ x \in \partial\Omega$;

(v) $\deg\{JQN, \Omega \cap \ker M, 0\} \neq 0$, *where the operator* $J : W_1 \to X_1$ *is a homeomorphism such that* $J(\theta) = \theta$ *and the degree is the Brouwer degree*,

then the operator equation $Mx(t) = Nx(t)$ *has at least one solution in* $\overline{\Omega} \cap dom\, M$.

Lemma 8 [1]**.** *If* $\alpha \in (0, +\infty)$, *then the general solution of the fractional differential equation* $D_{0+}^{\alpha} x(t) = 0$ *is* $x(t) = b_1 t^{\alpha-1} + b_2 t^{\alpha-2} + \cdots + b_n t^{\alpha-n}$, *where* $b_i \in \mathbb{R},\ i = 1, 2, ..., n$ *and* $n = [\alpha] + 1$ *is the smallest integer greater than or equal to* α.

Lemma 9. *Let* $n > 0$ *and* $m = [n] + 1$. *Assume that* $I_{0+}^{m-n} f \in A^m[0, b]$. *Then*

$$I_{0+}^{n} D_{0+}^{n} f(x) = f(x) - \sum_{k=0}^{m-1} \frac{x^{n-k-1}}{\Gamma(n-k)} \lim_{z \to 0+} D_{0+}^{m-k-1} I_{0+}^{m-n} f(z). \tag{2.2}$$

Lemma 10. *If* $\sum_{i=1}^{m} \mu_i = 1,\ \sum_{i=1}^{m} \mu_i \xi_i^{-1} = 0,\ \int_0^1 dA(t) = 1$ *and* $\int_0^1 t dA(t) = 0$, *then* $\ker M = \{x \in dom\, M : x(t) = b_1 t^{\alpha-1} + b_2 t^{\alpha-2},\ b_1, b_2 \in \mathbb{R},\ t \in (0, \infty)\}$ *and* $\dim \ker M = 2$.

Proof. The corresponding homogeneous equation of (1.1)-(1.2) is

$$(\phi_p(D_{0+}^{\alpha} x(t)))' = 0. \tag{2.3}$$

Integrating (2.3) with respect to t from t to $+\infty$, we have

$$\phi_p(D_{0+}^{\alpha}x(+\infty)) - \phi_p(D_{0+}^{\alpha}x(t)) = 0$$

and

$$\phi_p(D_{0+}^{\alpha}x(t)) = \phi_p(D_{0+}^{\alpha}x(+\infty)).$$

Hence,

$$D_{0+}^{\alpha}x(t) = D_{0+}^{\alpha}x(+\infty) = 0.$$

Thus,

$$D_{0+}^{\alpha}x(t) = 0. \tag{2.4}$$

Given that $3 < \alpha \leq 4$, by Lemma 8, the solution of (2.4) is

$$x(t) = b_1 t^{\alpha-1} + b_2 t^{\alpha-2} + b_3 t^{\alpha-3} + b_4 t^{\alpha-4}, \quad b_1, b_2, b_3, b_4 \in \mathbb{R}.$$

By applying the boundary condition of (1.2), we obtain

$$x(0) = 0 \Rightarrow b_4 = 0.$$

Then $x(t) = b_1 t^{\alpha-1} + b_2 t^{\alpha-2} + b_3 t^{\alpha-3}$. Also,

$$D_{0+}^{\alpha-3}x(t) = D_{0+}^{\alpha-3}(b_1 t^{\alpha-1} + b_2 t^{\alpha-2} + b_3 t^{\alpha-3}),$$

$$D_{0+}^{\alpha-3}x(0) = 0 \Rightarrow b_3 = 0.$$

Thus,

$$x(t) = b_1 t^{\alpha-1} + b_2 t^{\alpha-2}. \tag{2.5}$$

By applying the boundary conditions (1.2) to (2.5), we have

$$\int_0^1 dA(t) = 1, \quad \int_0^1 t dA(t) = 0 \tag{2.6}$$

and

$$\sum_{i=1}^{m} \mu_i = 1, \quad \sum_{i=1}^{m} \mu_i \xi_i^{-1} = 0. \tag{2.7}$$

Conditions expressed in equations (2.6) and (2.7) are critical for the resonance of the BVP (1.1)-(1.2), where

$$\ker M = \{x(t) | x(t) = b_1 t^{\alpha-1} + b_2 t^{\alpha-2},\ b_1,\ b_2 \in \mathbb{R},\ t \in (0, +\infty)\}.$$

Since $\ker M$ depends on two coefficients, $\dim \ker M = 2$. □

In this paper, we assume (H_1):

$$\int_0^1 dA(t) = 1, \quad \int_0^1 t dA(t) = 0, \quad \sum_{i=1}^m \mu_i = 1, \quad \sum_{i=1}^m \mu_i \xi_i^{-1} = 0.$$

Lemma 11. *The following statement holds*:

$$\operatorname{Im} M = \{w \in W : Q_1 w = Q_2 w = 0\},$$

where

$$Q_1 w = \int_0^1 \int_0^t (t-r)\phi_q\left(-\int_t^\infty w(s)ds\right) dr dA(t),$$

$$Q_2 w = \sum_{i=1}^m \mu_i \int_0^{\xi_i} \phi_q\left(-\int_{\xi_i}^\infty w(s)ds\right) dr$$

and the operator $M : dom\, M \subset X \to W$ *is quasilinear*.

Proof. If $w \in \operatorname{Im} M$, then there exists $x(t) \in dom\, M$ such that

$$(\phi_p(D_{0+}^{\alpha} x(t)))' = w(t), \quad t \in (0, +\infty). \tag{2.8}$$

Integrating (2.8) with respect to t from t to $+\infty$,

$$\phi_p(D_{0+}^{\alpha} x(t))|_t^{+\infty} = \int_t^{+\infty} w(s)ds,$$

$$\phi_p(D_{0+}^{\alpha} x(+\infty)) - \phi_p(D_{0+}^{\alpha} x(t)) = \int_t^{+\infty} w(s)ds.$$

By the boundary condition (1.2), $\phi_p(D_{0+}^{\alpha} x(+\infty)) = 0$ since $D_{0+}^{\alpha} x(+\infty) = 0$ and $\phi_p(0) = 0$. Thus,

$$\phi_p(D_{0+}^{\alpha}x(t)) = -\int_t^{+\infty} w(s)ds,$$

$$D_{0+}^{\alpha}x(t) = \phi_q\left(-\int_t^{+\infty} w(s)ds\right). \tag{2.9}$$

Hence,

$$I_{0+}^{\alpha}D_{0+}^{\alpha}x(t) = I_{0+}^{\alpha}\phi_q\left(-\int_t^{+\infty} w(s)ds\right) + b_1t^{\alpha-1} + b_2t^{\alpha-2},$$

$$x(t) = I_{0+}^{\alpha}\phi_q\left(-\int_t^{\infty} w(s)ds\right) + b_1t^{\alpha-1} + b_2t^{\alpha-2}. \tag{2.10}$$

Applying the boundary conditions (1.2) to (2.10), we obtain

$$b_2\Gamma(\alpha-1) = \int_0^1\int_0^t (t-r)\phi_q\left(-\int_t^{\infty} w(s)ds\right)drdA(t) + b_2\Gamma(\alpha-1).$$

Thus,

$$\int_0^1\int_0^t (t-r)\phi_q\left(-\int_t^{\infty} w(s)ds\right)drdA(t) = 0 = Q_1w. \tag{2.11}$$

By applying the boundary condition

$$\lim_{t\to\infty} D_{0+}^{\alpha-1}x(t) = \sum_{i=1}^{m}\mu_i D_{0+}^{\alpha-1}x(\xi_i)$$

with (2.10), we have

$$b_1 = \sum_{i=1}^{m}\mu_i\int_0^{\xi_i}\phi_q\left(-\int_{\xi_i}^{\infty} w(s)ds\right)dr + b_1.$$

Hence,

$$\sum_{i=1}^{m}\mu_i\int_0^{\xi_i}\phi_q\left(-\int_{\xi_i}^{\infty} w(s)ds\right)dr = 0 = Q_2w. \tag{2.12}$$

We have shown that $\operatorname{Im} M = \{w \in W : Q_1 w = Q_2 w = 0\}$ for $x \in dom\, M$. It is observed that the $\dim \ker M = 2$ and the $\operatorname{Im} M$ is a closed subset of W, hence, M is a quasilinear operator. □

Throughout this study, we assume (H_2):

$$D = (Q_1 t^{\alpha-1} e^{-t} \cdot Q_2 t^{\alpha-2} e^{-t}) - (Q_2 t^{\alpha-1} e^{-t} \cdot Q_1 t^{\alpha-2} e^{-t}) := d_{11} d_{22} - d_{12} d_{21} \neq 0,$$

where

$$Q_1 w = \int_0^1 \int_0^t (t-r) \varphi_q\left(-\int_t^{+\infty} w(s) ds\right) dr dA(t)$$

and

$$Q_2 w = \sum_{i=1}^{m} \mu_i \int_0^{\xi_i} \phi_q\left(-\int_{\xi_i}^{+\infty} w(s) ds\right) dr.$$

Lemma 12. *If operators $P : X \to X_1$ and $Q : W \to W_1$ are defined as*

$$Px(t) = \frac{D_{0+}^{\alpha-1} x(+\infty)}{\Gamma(\alpha)} t^{\alpha-1} + \frac{D_{0+}^{\alpha-2} x(0)}{\Gamma(\alpha-1)} t^{\alpha-2} \tag{2.13}$$

and

$$Qw = \psi_1 w(t) \cdot t^{\alpha-1} + \psi_2 w(t) \cdot t^{\alpha-2}, \tag{2.14}$$

where

$$\psi_1 w = \frac{1}{D}(d_{22} Q_1 w - d_{21} Q_2 w) e^{-t}, \quad \psi_2 w = \frac{1}{D}(-d_{12} Q_1 w + d_{11} Q_2 w), \tag{2.15}$$

then Q is a semi-projector.

Proof. We make the following computations to prove the lemma:

$$\begin{aligned} \psi_1(\psi_1 w) t^{\alpha-1} &= \frac{1}{D}(\psi_1(d_{22} Q_1 w t^{\alpha-1} - d_{21} Q_2 w t^{\alpha-1})) e^{-t} \\ &= \frac{1}{D} \cdot D(\psi_1 w) = \psi_1 w, \end{aligned} \tag{2.16}$$

$$\psi_1(\psi_2 w)t^{\alpha-2} = \frac{1}{D}(d_{22}Q_1(\psi_2 w)t^{\alpha-2} - d_{21}Q_2(\psi_2 w)t^{\alpha-2})e^{-t}$$

$$= \frac{1}{D}\cdot 0(\psi_2 w) = 0, \tag{2.17}$$

$$\psi_2(\psi_1 w)t^{\alpha-1} = \frac{1}{D}(-d_{12}Q_1(\psi_1 w)t^{\alpha-1} + d_{11}Q_2(\psi_1 w)t^{\alpha-1})e^{-t}$$

$$= \frac{1}{D}\cdot 0(\psi_1 w) = 0, \tag{2.18}$$

$$\psi_2(\psi_2 w)t^{\alpha-2} = \frac{1}{D}(-d_{12}Q_1(\psi_2 w)t^{\alpha-2} + d_{11}Q_2(\psi_2 w)t^{\alpha-2})e^{-t}$$

$$= \frac{1}{D}\cdot D(\psi_2 w) = \psi_2 w. \tag{2.19}$$

From results in (2.16)-(2.19), we have

$$Q^2 w = Q((\psi_1 w)t^{\alpha-1} + (\psi_2 w)t^{\alpha-2})$$

$$= (\psi_1 w)\cdot t^{\alpha-1} + (\psi_2 w)t^{\alpha-2} = Qw. \tag{2.20}$$

Next, for any $k \in \mathbb{R}$,

$$\psi_1 kw = \frac{1}{D}(d_{22}Q_1 kw - d_{21}Q_2 kw)e^{-t}$$

$$= \frac{k}{D}(d_{22}Q_1 w - d_{21}Q_2 w)e^{-t} = k\psi_1 w, \tag{2.21}$$

$$\psi_2 kw = \frac{1}{D}(-d_{12}\phi_1 kw + d_{11}\phi_2 kw)e^{-t}$$

$$= \frac{k}{D}(-d_{12}\phi_1 w + d_{11}\phi_2 w)e^{-t} = k\psi_2 w. \tag{2.22}$$

Hence,

$$Qkw = \psi_1 kw\cdot t^{\alpha-1} + \psi_2 kw\cdot t^{\alpha-2}$$

$$= (\psi_1 w\cdot t^{\alpha-1} + \psi_2 w\cdot t^{\alpha-2}) = kQw. \tag{2.23}$$

From the results in (2.20)-(2.23) and by Definition 3, we see that Q is a semi-projector. □

Lemma 13. *If* f *is an* $L^1(0, +\infty)$ *Carathéodory function, then* $N_\lambda : \overline{\Omega} \to W$ *is M-compact in* $\overline{\Omega}$ *for an open bounded subset* $\Omega \subset X$ *containing the origin.*

Proof. To show that N_λ is M-compact, we verify all conditions of Definition 6.

(i)

$$Q(I - Q)N_\lambda(x) = QN_\lambda(x) - Q^2 N_\lambda(x) = QN_\lambda(x) - QN_\lambda(x) = 0.$$

Hence, $(I - Q)N_\lambda(x) \subset \operatorname{Im} M$. Also, for $w \in \operatorname{Im} M$, $Qw = 0$. Thus, $w \in \ker Q$, i.e., $w \in (I - Q)w$. Hence, $\operatorname{Im} M \in (I - Q)w$. Therefore,

$$(I - Q)N_\lambda(x) \subset \operatorname{Im} M \subset (I - Q)w.$$

(ii) Let $QN_\lambda x = 0$ for $\lambda \in (0, 1)$. Then

$$\begin{aligned} QN_\lambda x = 0 &= Q(\lambda f(t, x(t), D_{0+}^{\alpha-3}x(t), D_{0+}^{\alpha-2}x(t), D_{0+}^{\alpha-1}x(t), D_{0+}^{\alpha}x(t))) \\ &= \lambda QNx. \end{aligned}$$

Hence,

$$QNx = 0. \tag{2.24}$$

Conversely, if $QNx = 0$, then

$$\begin{aligned} QNx = 0 &= \psi_1(QN_\lambda x)t^{\alpha-1} + \psi_2(QN_\lambda x)t^{\alpha-2} \\ &= \frac{e^{-t}}{D}(d_{22}Q_1(QN_\lambda x)t^{\alpha-1} - d_{21}Q_2(QN_\lambda x)t^{\alpha-1} \\ &\qquad - d_{12}Q_1(QN_\lambda x)t^{\alpha-2} + d_{11}Q_2(QN_\lambda x)t^{\alpha-2}) \\ &= \frac{1}{D}[d_{22}Q_1 t^{\alpha-1}e^{-t} - d_{21}Q_2 t^{\alpha-1}e^{-t} \\ &\qquad - d_{12}Q_1 t^{\alpha-2}e^{-t} + d_{11}Q_2 t^{\alpha-2}e^{-t}]QN_\lambda x \\ &= 2QN_\lambda x. \end{aligned}$$

Hence, $QN_{\lambda}x = 0.$

(iii) Let $T : X \times [0, 1] \to X_2$ be defined as

$$T(x, \lambda)(t)$$

$$= I_{0+}^{\alpha}\left[\phi_q\left(\phi_p(I_{0+}^{-1}D_{0+}^{\alpha-1}x(+\infty) + I_{0+}^{-2}D_{0+}^{\alpha-2}x(0)) - \int_t^{+\infty}(I - Q)N_{\lambda}x(\tau)d\tau\right)\right.$$

$$\left. - (I_{0+}^{-1}D_{0+}^{\alpha-1}x(+\infty) + I_{0+}^{-1}D_{0+}^{\alpha-2}x(0))\right], \tag{2.25}$$

where $\lambda \in [0, 1]$ and X_2 is the complement space of X_1 in X.

From Definition 6(ii), $QN_{\lambda}x = 0.$

It is easy to show that $T(x, 0) = 0,$ hence, (6)(iii) is satisfied.

(iv) For $x \in X_{\lambda} = \{x \in \overline{\Omega} : Mx = N_{\lambda}x\},$

$$(\phi_p(D_{0+}^{\alpha}x(t)))' = \lambda f(t, x(t), D_{0+}^{\alpha-3}x(t), D_{0+}^{\alpha-2}, D_{0+}^{\alpha-1}x(t), D_{0+}^{\alpha}x(t))$$

$$= N_{\lambda}x(t) \in \operatorname{Im} M \subset \ker Q,$$

$$T(x, \lambda)(t) = I_{0+}^{\alpha}\left[\phi_q\left(\phi_p(I_{0+}^{-1}D_{0+}^{\alpha-1}x(+\infty) + I_{0+}^{-2}D_{0+}^{\alpha-2}x(0))\right.\right.$$

$$\left. - \int_t^{+\infty}(I - Q)N_{\lambda}x(\tau)d\tau\right)$$

$$\left. - (I_{0+}^{-1}D_{0+}^{\alpha-1}x(+\infty) + I_{0+}^{-1}D_{0+}^{\alpha-2}x(0))\right]$$

$$= I_{0+}^{\alpha-1}D_{0+}^{\alpha-1}x(+\infty) + I_{0+}^{\alpha-2}D_{0+}^{\alpha-2}x(0)$$

$$+ I_{0+}^{\alpha}\left(\phi_q\left(-\int_t^{\infty}(\phi_p(D_{0+}^{\alpha}x(0)))'\right)\right)$$

$$- (I_{0+}^{\alpha-1}D_{0+}^{\alpha-1}x(+\infty) + I_{0+}^{\alpha-2}D_{0+}^{\alpha-2}x(0))$$

$$= I_{0+}^{\alpha-1}D_{0+}^{\alpha-1}x(+\infty) + I_{0+}^{\alpha-2}D_{0+}^{\alpha-2}x(0)$$

$$+ I_{0+}^{\alpha}\phi_q(-\phi_p(D_{0+}^{\alpha}x(\infty)) + \phi_p(D_{0+}^{\alpha}x(t)))$$

$$- \frac{D_{0+}^{\alpha-1}x(+\infty)}{\Gamma(\alpha)}t^{\alpha-1} - \frac{D_{0+}^{\alpha-2}x(0)}{\Gamma(\alpha-1)}t^{\alpha-2}.$$

Since $\phi_p(D_{0+}^{\alpha}x(\infty)) = 0$,

$$T(x, \lambda)(t) = I_{0+}^{\alpha-1}D_{0+}^{\alpha-1}x(+\infty) + I_{0+}^{\alpha-2}D_{0+}^{\alpha-2}x(0) + I_{0+}^{\alpha}D_{0+}^{\alpha}x(t) - Px(t)$$

$$= b_1t^{\alpha-1} + b_2t^{\alpha-2} + I_{0+}^{\alpha}D_{0+}^{\alpha}x(t) - (Px)(t).$$

By Lemma 9, $I_{0+}^{\alpha}D_{0+}^{\alpha}x(t) = x(t) - b_1t^{\alpha-1} - b_2t^{\alpha-2}$.

Thus,

$$T(x, \lambda)(t) = x(t) - (Px)(t) = [(I - P)x](t). \tag{2.26}$$

For any $x \in \overline{\Omega}$, we have

$$M[Px + T(x, \lambda)](t)$$

$$= [\phi_P(D_{0+}^{\alpha}(Px + T(x, \lambda))(t))]'$$

$$= \phi_P\left(D_{0+}^{1}D_{0+}^{\alpha-1}x(+\infty) + D_{0+}^{2}D_{0+}^{\alpha-2}x(0) + \phi_q\left(-\int_t^{+\infty}(I - Q)N_\lambda x(\tau)d\tau\right)\right)$$

$$= \left[\int_t^{+\infty}(I - Q)N_\lambda x(\tau)d\tau\right]'$$

$$= (I - Q)N_\lambda x(t). \tag{2.27}$$

Definitions 6(iv) and (v) are satisfied.

Next, we show that T is relatively compact for $\lambda \in [0, 1]$ in the following steps:

Step 1: $T(x, \lambda)(t)$ is uniformly bounded in X.

Step 2: $T(x, \lambda)(t)$ is equicontinuous on every compact subset of $\lambda \in [0, +\infty)$.

Step 3: $T(x, \lambda)(t)$ is equiconvergent at infinity.

Step 1. If $\Omega \subset X$ is an open bounded set, then for any $x \in \overline{\Omega}$, there exists a constant $k > 0$ such that $\| x \|_X < k$. Since f is an L^1-Carathéodory function, for a.e. $t \in [0, +\infty)$ and $\lambda \in [0, 1]$, there exists $\Theta_k : [0, +\infty) \to [0, +\infty)$ such that

$$\int_0^{+\infty} \Theta_k(t) dt < +\infty, \ | f(t, x(t), D_{0+}^{\alpha-3} x(t), D_{0+}^{\alpha-2} x(t), D_{0+}^{\alpha-1} x(t), D_{0+}^{\alpha} x(t)) | \leq \Theta_k(t)$$

and

$$\int_0^{+\infty} | (I - Q) N_\lambda x(s) | ds \leq \| \Theta_k(s) \|_W + \| QNx(s) \|_W .$$

Thus, for any $x \in \overline{\Omega}$,

$$\begin{aligned}
\| T(x, \lambda) \|_0 &= \sup_{t \geq 0} \frac{| T(x, \lambda)(t) |}{1 + t^\alpha} \\
&= \sup_{t \geq 0} \frac{\left| I_{0+}^{\alpha} \phi_q \left(- \int_t^{+\infty} (I - Q) N_\lambda x(s) ds \right) \right|}{1 + t^\alpha} \\
&\leq \sup_{t \geq 0} \frac{[I_{0+}^{\alpha} \phi_q (\| \Theta_k(s) \|_W + \| QNx \|_W)]}{1 + t^\alpha} \\
&\leq \frac{1}{\Gamma(\alpha + 1)} (\phi_q (\| \Theta_k(s) \|_W + \| QNx \|_W)),
\end{aligned}$$

$$\begin{aligned}
\| D_{0+}^{\alpha-3} T(x, \lambda) \|_0 &= \sup_{t \geq 0} \frac{| D_{0+}^{\alpha-3} T(x, \lambda)(t) |}{1 + t^3} \\
&= \sup_{t \geq 0} \frac{| I_{0+}^{3-\alpha} I_{0+}^{\alpha} \phi_q (\| \Theta_k(s) \|_W + \| QNx \|_W) |}{1 + t^3}
\end{aligned}$$

$$= \sup_{t \geq 0} \frac{t^3}{1+t^3} \left(\phi_q(\| \Theta_k(s) \|_W + \| QNx \|_W) \frac{1}{3\Gamma(3)} \right)$$

$$\leq \phi_q(\| \Theta_k(s) \|_W + \| QNx \|_W),$$

$$\| D_{0+}^{\alpha-2} T(x, \lambda) \|_0 = \sup_{t \geq 0} \frac{| D_{0+}^{\alpha-2} T(x, \lambda)(t) |}{1 + t^{\alpha-1}}$$

$$= \sup_{t \geq 0} \frac{1}{1 + t^{\alpha-1}} \left[\phi_q(\| \Theta_k(s) \|_W + \| QNx \|_W) \times \frac{1}{\Gamma(2)} \int_0^t (t-s) ds \right]$$

$$\leq \phi_q(\| \Theta_k(s) \|_W + \| QNx \|_W).$$

Similarly,

$$\| D_{0+}^{\alpha-1} T(x, \lambda) \|_0 = \sup_{t \geq 0} \frac{| D_{0+}^{\alpha-1} T(x, \lambda)(t) |}{1 + t^{\alpha-2}}$$

$$\leq \phi_q(\| \Theta_k(s) \|_W + \| QNx \|_W)$$

and

$$\| D_{0+}^{\alpha} T(x, \lambda) \|_\infty = \sup_{t \geq 0} \| D_{0+}^{\alpha} T(x, \lambda)(t) \|$$

$$\leq \phi_q(\| \Theta_k(s) \|_W + \| QNx \|_W).$$

Hence, $T(x, \lambda)$ is uniformly bounded in X.

Step 2. We show that $T(x, \lambda)$ is equicontinuous on every compact subset of $(0, +\infty)$. Let $A > 0$ and $t_1, t_2 \in [0, A]$ such that, for $t_1 < t_2$ and $\lambda \in [0, 1]$, we have

$$\left|\frac{T(x,\lambda)(t_2)}{1+t_2^{\alpha}}-\frac{T(x,\lambda)(t_1)}{1+t_1^{\alpha}}\right|$$

$$=\phi_q(\|\Theta_k(s)\|_W+\|\phi Nx\|_W)$$

$$\times\left(\frac{1}{\Gamma(\alpha)}\left(\int_0^{t_2}\left|\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha}}-\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha}}\right|ds\right)\right)$$

$$\leq\frac{\phi_q(\|\Theta_k(s)\|_W+\|\phi Nx\|_W)}{\Gamma(\alpha)}$$

$$\times\left(\int_{t_1}^{t_2}\left|\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha}}\right|ds+\int_0^{t_1}\left|\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha}}-\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha}}\right|ds\right)\to 0$$

$$\text{as } t_1\to t_2,$$

$$\left|\frac{D_{0+}^{\alpha-3}T(x,\lambda)(t_2)}{1+t_2^{3}}-\frac{D_{0+}^{\alpha-3}T(x,\lambda)(t_1)}{1+t_1^{3}}\right|$$

$$=\frac{\phi_q(\|\Theta_k(s)\|_W+\|QNx\|_W)}{\Gamma(3)}\left(\left|\int_0^{t_2}\frac{(t_2-s)^{2}}{1+t_2^{3}}-\int_0^{t_1}\frac{(t_1-s)^{2}}{1+t_1^{3}}\right|ds\right)$$

$$\leq\frac{\phi_q(\|\Theta_k(s)\|_W+\|QNx\|_W)}{2}$$

$$\times\left(\int_{t_1}^{t_2}\left|\frac{(t_2-s)^{2}}{1+t_2^{3}}\right|ds+\int_0^{t_1}\left|\frac{(t_2-s)^{2}}{1+t_2^{3}}-\frac{(t_1-s)^{2}}{1+t_1^{3}}\right|ds\right)\to 0 \text{ as } t_1\to t_2,$$

$$\left|\frac{D_{0+}^{\alpha-2}T(x,\lambda)(t_2)}{1+t_2^{\alpha-1}}-\frac{D_{0+}^{\alpha-2}T(x,\lambda)(t_1)}{1+t_1^{\alpha-1}}\right|$$

$$=\phi_q(\|\Theta_k(s)\|_W+\|QNx\|_W)$$

$$\times\left(\left|\int_0^{t_2}\frac{t_2-s}{1+t_2^{\alpha-1}}-\int_0^{t_1}\frac{t_1-s}{1+t_1^{\alpha-1}}\right|ds\right)$$

$$\leq \phi_q(\|\Theta_k(s)\|_W + \|QNx\|_W)$$

$$\times\left(\int_{t_1}^{t_2}\left|\frac{t_2 - s}{1 + t_2^{\alpha-1}}\right| ds + \int_0^{t_1}\left|\frac{t_2 - s}{1 + t_2^{\alpha-1}} - \frac{t_1 - s}{1 + t_1^{\alpha-1}}\right| ds\right) \to 0 \text{ as } t_1 \to t_2,$$

$$\left|\frac{D_{0+}^{\alpha-1}T(x, \lambda)(t_2)}{1 + t_2^{\alpha-2}} - \frac{D_{0+}^{\alpha-1}T(x, \lambda)(t_1)}{1 + t_1^{\alpha-2}}\right|$$

$$= \phi_q(\|\Theta_k(s)\|_W + \|QNx\|_W)$$

$$\times\left(\left|\int_0^{t_2}\frac{1}{1 + t_2^{\alpha-2}} - \int_0^{t_1}\frac{1}{1 + t_1^{\alpha-2}}\right| ds\right)$$

$$\leq \phi_q(\|\Theta_k(s)\|_W + \|QNx\|_W)$$

$$\times\left(\int_{t_1}^{t_2}\left|\frac{1}{1 + t_2^{\alpha-2}}\right| ds + \int_0^{t_1}\left|\frac{1}{1 + t_2^{\alpha-2}} - \frac{1}{1 + t_1^{\alpha-2}}\right| ds\right)$$

$$= \phi_q(\|\Theta_k(s)\|_W + \|QNx\|_W)$$

$$\times\left(\frac{t_2}{1 + t_2^{\alpha-2}} - \frac{t_1}{1 + t_2^{\alpha-2}} + \frac{t_1}{1 + t_2^{\alpha-2}} - \frac{t_1}{1 + t_1^{\alpha-2}}\right) \to 0 \text{ as } t_1 \to t_2,$$

$$|D_{0+}^{\alpha}T(x, \lambda)(t_2) - D_{0+}^{\alpha}T(x, \lambda)(t_1)|$$

$$= \left| D_{0+}^{\alpha}I_{0+}^{\alpha}\phi_q\left(-\int_{t_2}^{+\infty}(I - Q)N_\lambda x(s)ds\right)\right.$$

$$\left. - D_{0+}^{\alpha}I_{0+}^{\alpha}\phi_q\left(-\int_{t_1}^{+\infty}(I - Q)N_\lambda x(s)ds\right)\right|$$

$$= \left|\phi_q\left(-\int_{t_2}^{+\infty}(I - Q)N_\lambda x(s)ds\right) - \phi_q\left(-\int_{t_1}^{t_2}(I - Q)N_\lambda x(s)ds\right)\right.$$

$$\left. - \phi_q\left(-\int_{t_2}^{+\infty}(I - Q)N_\lambda x(s)ds\right)\right|$$

$$\leq \left|-\phi_q\left(-\int_{t_1}^{t_2}(I - Q)N_\lambda x(s)ds\right)\right| \to 0 \text{ as } t_1 \to t_2. \tag{2.28}$$

Hence, $T(x, \lambda)\overline{\Omega}$ is equicontinuous on every compact subset $[0, A]$ of $[0, +\infty)$.

Step 3. We now show that $T(x, \lambda)\overline{\Omega}$ is equiconvergent at $+\infty$. Let $x \in \overline{\Omega}$. If

$$\int_t^{+\infty} |(I - Q)N_\lambda x(s)| ds \leq \| \Theta_k(s) \|_W + \| QNx(s) \|_W,$$

then $\phi_q(x)$ is uniformly continuous on $[-h, h]$, where $h = \| \Theta_k(s) \|_W + \| QNx(s) \|_W$. Hence, given $\varepsilon > 0$, and for any $x \in \overline{\Omega}$, there exists $j > 0$ such that, for $s \geq j$,

$$\left| \phi_q\left(\int_s^{+\infty} (Q - I) N_\lambda x(r) dr \right) \right| < \varepsilon.$$

Since $\lim_{t \to +\infty} \frac{t^{\alpha-1}}{1 + t^\alpha} = 0$, $\lim_{t \to +\infty} \frac{t^2}{1 + t^3} = 0$, $\lim_{t \to +\infty} \frac{t}{1 + t^{\alpha-1}} = 0$ and $\lim_{t \to +\infty} \frac{1}{1 + t^{\alpha-2}} = 0$, for given $\varepsilon > 0$, there exists $j_1 < j > 0$ such that for any $t_1, t_2 \geq j_1$ and $s \in [0, j]$, we have

$$\left| \frac{(t_2 - s)^{\alpha-1}}{1 + t_2^\alpha} - \frac{(t_1 - s)^{\alpha-1}}{1 + t_1^\alpha} \right| < \varepsilon, \quad \left| \frac{(t_2 - s)^2}{1 + t_2^3} - \frac{(t_1 - s)^2}{1 + t_1^3} \right| < \varepsilon,$$

$$\left| \frac{(t_2 - s)}{1 + t_2^{\alpha-1}} - \frac{(t_1 - s)}{1 + t_1^{\alpha-1}} \right| < \varepsilon \quad \text{and} \quad \left| \frac{1}{1 + t_2^{\alpha-2}} - \frac{1}{1 + t_1^{\alpha-2}} \right| < \varepsilon.$$

Thus, for $t_1, t_2 \geq j_1$, we have

$$\left| \frac{T(x, \lambda)(t_2)}{1 + t_2^\alpha} - \frac{T(x, \lambda)(t_2)}{1 + t_1^\alpha} \right|$$

$$= \left| \frac{I_{0+}^\alpha \phi_q\left(-\int_{t_2}^{+\infty} (I - Q) N_\lambda x(s) ds \right)}{1 + t_2^\alpha} - \frac{I_{0+}^\alpha \phi_q\left(-\int_{t_1}^{+\infty} (I - Q) N_\lambda x(s) ds \right)}{1 + t_1^\alpha} \right|$$

$$\leq \frac{\phi_q(\| \Theta_k(s) \|_W + \| QNx \|_W)}{\Gamma(\alpha)} \left[\int_0^j \left| \frac{(t_2 - s)^{\alpha-1}}{1 + t_2^\alpha} - \frac{(t_1 - s)^{\alpha-1}}{1 + t_1^\alpha} \right| ds \right.$$

$$\left. + \left| \int_{j_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{1 + t_2^\alpha} ds - \int_{j_1}^{t_1} \frac{(t_2 - s)^{\alpha-1}}{1 + t_1^\alpha} ds \right| \right]$$

$$\leq \frac{\phi_q(\| \Theta_k(s) \|_W + \| QNx \|_W)}{\Gamma(\alpha)} \left(j_1 + \frac{1}{\alpha} \right) \varepsilon,$$

$$\left| \frac{D_{0+}^{\alpha-3} T(x, \lambda)(t_2)}{1 + t_2^3} - \frac{D_{0+}^{\alpha-3} T(x, \lambda)(t_1)}{1 + t_1^3} \right|$$

$$= \left| \frac{I_{0+}\phi_q\left(-\int_{t_2}^{+\infty} (I - Q) N_\lambda x(s) ds \right)}{1 + t_2^3} - \frac{I_{0+}\phi_q\left(-\int_{t_1}^{+\infty} (I - Q) N_\lambda x(s) ds \right)}{1 + t_1^3} \right|$$

$$\leq \frac{\phi_q(\| \Theta_k(s) \|_W + \| QNx \|_W)}{2} \left(j_1 + \frac{1}{3} \right) \varepsilon.$$

Similarly,

$$\left| \frac{D_{0+}^{\alpha-2} T(x, \lambda)(t_2)}{1 + t_2^{\alpha-1}} - \frac{D_{0+}^{\alpha-2} T(x, \lambda)(t_1)}{1 + t_1^{\alpha-1}} \right|$$

$$\leq \phi_q(\| \Theta_k(s) \|_W + \| QNx \|_W) \left(j_1 + \frac{1}{2} \right) \varepsilon,$$

$$\left| \frac{D_{0+}^{\alpha-1} T(x, \lambda)(t_2)}{1 + t_2^{\alpha-2}} - \frac{D_{0+}^{\alpha-1} T(x, \lambda)(t_1)}{1 + t_1^{\alpha-2}} \right|$$

$$\leq \phi_q(\| \Theta_k(s) \|_W + \| QNx \|_W)(j_1 + 1)\varepsilon$$

and

$$| D_{0+}^{\alpha} T(x, \lambda)(t_2) - D_{0+}^{\alpha-1} T(x, \lambda)(t_1) |$$

$$= \left| \phi_q\left(-\int_{t_2}^{+\infty} (I - Q) N_\lambda x(s) ds \right) - \phi_q\left(-\int_{t_1}^{+\infty} (I - Q) N_\lambda x(s) ds \right) \right|$$

$$\leq \left|\phi_q\left(-\int_{t_2}^{+\infty}(I-Q)N_\lambda x(s)ds\right)\right| + \left|\phi_q\left(-\int_{t_1}^{+\infty}(I-Q)N_\lambda x(s)ds\right)\right|$$

$$< \varepsilon + \varepsilon = 2\varepsilon.$$

Hence, $T(x, \lambda)\overline{\Omega}$ is equiconvergent at $+\infty$. Since $T(x, \lambda)\overline{\Omega}$ is bounded, equicontinuous and equiconvergent at infinity, $T : X \times [0, 1] \to X_2$ is relatively compact. □

3. Results and Discussion

In this section, we establish the existence of a solution of the BVP (1.1)-(1.2).

Theorem 14. *If f is an L^1-Carathéodory function and the following assumptions hold*:

(H_1) $\int_0^1 dA(t) = 1$, $\int_0^1 t dA(t) = 0$, $\sum_{i=1}^{m}\mu_i = 1$, $\sum_{i=1}^{m}\mu_i\xi_i^{-1} = 0$,

(H_2) $D = d_{11}d_{22} - d_{12}d_{21} \neq 0$,

(H_3) *there exist nonnegative functions* $g_i(t) \in W$, $i = 1, 2, ..., 6$ *such that* $\forall t \in (0, +\infty)$,

$$|f(t, w_1, w_2, w_3, w_4, w_5)|$$

$$\leq \frac{g_1(t)|w_1|^{p-1}}{(1+t^\alpha)^{p-1}} + \frac{g_2(t)|w_2|^{p-1}}{(1+t^3)^{p-1}} + \frac{g_3(t)|w_3|^{p-1}}{(1+t^{\alpha-1})^{p-1}} + \frac{g_4(t)|w_4|^{p-1}}{(1+t^{\alpha-2})^{p-1}}$$

$$+ g_5(t)|w_5|^{p-1} + g_6(t), \quad (w_1, w_2, w_3, w_4, w_5) \in \mathbb{R}^5,$$

(H_4) *there exists a constant* $E > 0$ *such that for any* $t \in [0, +\infty)$, *if* $|D_{0+}^{\alpha-2}x(t)| > E$ *or* $|D_{0+}^{\alpha-1}x(t)| > E$, *then either* $Q_1Nx(t) \neq 0$ *or* $Q_2Nx(t) \neq 0$,

(H_5) *there exists* $A > 0$ *such that for* $|b_1| > A$ *or* $|b_2| > A$, *where* $b_1, b_2 \in \mathbb{R}$, *either*

$$b_1 Q_1 N(b_1 t^{\alpha-1} + b_2 t^{\alpha-2}) + b_2 Q_2 N(b_1 t^{\alpha-1} + b_2 t^{\alpha-2}) < 0 \qquad (H_{5a})$$

or

$$b_1 Q_1 N(b_1 t^{\alpha-1} + b_2 t^{\alpha-2}) + b_2 Q_2 N(b_1 t^{\alpha-1} + b_2 t^{\alpha-2}) > 0, \qquad (H_{5b})$$

then the BVP (1.1)-(1.2) *has at least one solution in X provided*

$$2^{2q-4}\left(\frac{1}{\Gamma(\alpha+1)}\| g_1 \|_1^{q-1} + \frac{1}{6}\| g_2 \|_1^{q-1} + \frac{1}{2}\| g_3 \|_1^{q-1} + \| g_4 \|_1^{q-1} + \| g_5 \|_1^{q-1}\right) < 1,$$

if $1 < p < 2$, *or*

$$\frac{1}{\Gamma(\alpha+1)}\| g_1 \|_1^{q-1} + \frac{1}{6}\| g_2 \|_1^{q-1} + \frac{1}{2}\| g_3 \|_1^{q-1} + \| g_4 \|_1^{q-1} + \| g_5 \|_1^{q-1} < 1,$$

if $p \geq 2$.

Before proving the theorem, two lemmas are stated and proved.

Lemma 15. *If* (H_3) *and* (H_4) *are satisfied, then the set* $\Omega_1 = \{x \in dom\, M \setminus \ker M : Mx = N_\lambda x\}$, $\lambda \in [0, 1]$ *is bounded.*

Proof. Suppose $x \in \Omega_1$. Then $Mx = N_\lambda x$ and $QN_\lambda x = 0$, since $QN_\lambda x \in \operatorname{Im} M$. By Lemma 11,

$$x(t) = I_{0+}^{\alpha}\phi_q\left(-\int_t^{\infty} w(s)ds\right) + b_1 t^{\alpha-1} + b_2 t^{\alpha-2},$$

$$x(t) = b_1 t^{\alpha-1} + b_2 t^{\alpha-2} + I_{0+}^{\alpha} D_{0+}^{\alpha} x(t),$$

$$|x(t)| = \left| b_1 t^{\alpha-1} + b_2 t^{\alpha-2} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} D_{0+}^{\alpha} x(s)ds \right|.$$

Thus,

$$\| x \|_0 = \sup_{t \geq 0} \frac{|x(t)|}{1 + t^{\alpha}}$$

$$\leq \sup_{t \geq 0} \frac{\left| b_1 t^{\alpha-1} + b_2 t^{\alpha-2} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} D_{0+}^{\alpha} x(s) ds \right|}{1 + t^{\alpha}}$$

$$\leq m_1 + \frac{1}{\Gamma(\alpha+1)} \| D_{0+}^{\alpha} x \|_{\infty}, \text{ where } m_1 = |b_1| + |b_2|,$$

$$\| D_{0+}^{\alpha-3} x(t) \|_0 = \sup_{t \geq 0} \frac{|D_{0+}^{\alpha-3} x(t)|}{1 + t^3}$$

$$= \sup_{t \geq 0} \frac{|D_{0+}^{\alpha-3}(b_1 t^{\alpha-1} + b_2 t^{\alpha-2}) + I_{0+}^{\alpha} D_{0+}^{\alpha} x(t)|}{1 + t^3}$$

$$\leq m_2 + \frac{1}{6} \| D_{0+}^{\alpha} x \|_{\infty}, \text{ where } \frac{\Gamma(\alpha)}{2} |b_1| + \Gamma(\alpha - 1) |b_2| = m_2,$$

$$\| D_{0+}^{\alpha-2} x(t) \|_0 = \sup_{t \geq 0} \frac{|D_{0+}^{\alpha-2} x(t)|}{1 + t^{\alpha-1}}$$

$$= \sup_{t \geq 0} \frac{|D_{0+}^{\alpha-2}(b_1 t^{\alpha-1} + b_2 t^{\alpha-2}) + I_{0+}^{\alpha} D_{0+}^{\alpha} x(t)|}{1 + t^{\alpha-1}}$$

$$\leq m_3 + \frac{1}{2} \| D_{0+}^{\alpha} x \|_{\infty}, \text{ where } \Gamma(\alpha) |b_1| + \Gamma(\alpha - 1) |b_2| = m_3,$$

$$\| D_{0+}^{\alpha-1} x(t) \|_0 = \sup_{t \geq 0} \frac{|D_{0+}^{\alpha-1} x(t)|}{1 + t^{\alpha-2}}$$

$$= \sup_{t \geq 0} \frac{|D_{0+}^{\alpha-1}(b_1 t^{\alpha-1} + b_2 t^{\alpha-2} + I_{0+}^{\alpha} D_{0+}^{\alpha} x(t))|}{1 + t^{\alpha-2}}$$

$$\leq m_3 + \| D_{0+}^{\alpha} x \|_{\infty},$$

$$\| D_{0+}^{\alpha} x(t) \|_{\infty} = \sup_{t \geq 0} | D_{0+}^{\alpha} x(t) |. \tag{3.1}$$

Thus, by the assumption (H_4) of Theorem 14, there exists a positive constant E for any $t_0 \in [0, +\infty)$, such that $|D_{0+}^{\alpha}x(t_0)| < E$. By (H_3) of Theorem 14 and from the fractional order BVP (1.1)-(1.2),

$$(\phi_p(D_{0+}^{\alpha}x(t)))' = f(t, x(t), D_{0+}^{\alpha-3}x(t), D_{0+}^{\alpha-2}x(t), D_{0+}^{\alpha-1}x(t), D_{0+}^{\alpha}x(t)),$$

we have

$$|\phi_p D_{0+}^{\alpha}x(t)|$$

$$= \left|\phi_p(D_{0+}^{\alpha}x(t_0)) - \int_{t_0}^{t} \lambda f(s, x(s), D_{0+}^{\alpha-3}x(s), D_{0+}^{\alpha-2}x(s), D_{0+}^{\alpha-1}x(s), D_{0+}^{\alpha}x(s))ds\right|$$

$$\leq \phi_p(E) + \int_0^{+\infty}\left[g_1(s)\frac{|x|^{p-1}}{(1+s^{\alpha})^{p-1}} + g_2(s)\frac{|D_{0+}^{\alpha-3}x(s)|^{p-1}}{(1+t^3)^{p-1}} + g_3(s)\frac{|D_{0+}^{\alpha-2}x(s)|^{p-1}}{(1+t^{\alpha-1})^{p-1}}\right.$$

$$\left. + g_4(s)\frac{|D_{0+}^{\alpha-1}x(s)|^{p-1}}{(1+t^{\alpha-2})^{p-1}} + g_5(s)|D_{0+}^{\alpha}x(s)|^{p-1} + g_6(s)\right]ds$$

$$\leq \phi_p(E) + \| g_6 \|_1 + \| g_1 \|_1 \phi_p\left(m_1 + \frac{1}{\Gamma(\alpha+1)}\| D_{0+}^{\alpha}x \|_{\infty}\right)$$

$$+ \| g_2 \|_1 \phi_p\left(m_2 + \frac{1}{6}\| D_{0+}^{\alpha}x \|_{\infty}\right) + \| g_3 \|_1 \phi_p\left(m_3 + \frac{1}{2}\| D_{0+}^{\alpha}x \|_{\infty}\right)$$

$$+ \| g_4 \|_1 \phi_p(m_3 + \| D_{0+}^{\alpha}x \|_{\infty}) + \| g_5 \|_1 \phi_p(\| D_{0+}^{\alpha}x \|_{\infty}).$$

If $1 < p < 2$, then

$$\| D_{0+}^{\alpha}x \|_{\infty} \leq 2^{2q-4}\left[(E + \| g_6 \|_1^{q-1} + \| g_1 \|_1^{q-1} m_1 + \| g_2 \|_1^{q-1} m_2\right.$$

$$+ (\| g_3 \|_1^{q-1} + \| g_4 \|_1^{q-1})m_3) + \left(\frac{1}{\Gamma(\alpha+1)}\| g_1 \|_1^{q-1} + \frac{1}{6}\| g_2 \|_1^{q-1}\right.$$

$$\left.\left. + \frac{1}{2}\| g_3 \|_1^{q-1} + \| g_4 \|_1^{q-1} + \| g_5 \|_1^{q-1}\right)\| D_{0+}^{\alpha}x \|_{\infty}\right].$$

Hence,

$$\| D_{0+}^{\alpha}x \|_{\infty}$$

$$\leq \frac{2^{2q-4}(E + \| g_6 \|_1^{q-1} + \| g_1 \|_1^{q-1} m_1 + \| g_2 \|_1^{q-1} m_2 + (\| g_3 \|_1^{q-1} + \| g_4 \|_1^{q-1}) m_3)}{1 - 2^{2q-4}\left(\frac{1}{\Gamma(\alpha+1)}\| g_1 \|_1^{q-1} + \frac{1}{6}\| g_2 \|_1^{q-1} + \frac{1}{2}\| g_3 \|_1^{q-1} + \| g_4 \|_1^{q-1} + \| g_5 \|_1^{q-1}\right)}. \tag{3.2}$$

Similarly, if $p > 2$, then

$$\| D_{0+}^{\alpha}x \|_{\infty}$$

$$\leq \frac{E + \| g_6 \|_1^{q-1} + \| g_1 \|_1^{q-1} m_1 + \| g_2 \|_1^{q-1} m_2 + (\| g_3 \|_1^{q-1} + \| g_4 \|_1^{q-1}) m_3}{1 - \left(\frac{1}{\Gamma(\alpha+1)}\| g_1 \|_1^{q-1} + \frac{1}{6}\| g_2 \|_1^{q-1} + \frac{1}{2}\| g_3 \|_1^{q-1} + \| g_4 \|_1^{q-1} + \| g_5 \|_1^{q-1}\right)}. \tag{3.3}$$

Hence,

$$\| x \|_X = \max\{\| x \|_0, \| D_{0+}^{\alpha-3}x(t) \|_0, \| D_{0+}^{\alpha-2}x(t) \|_0, \| D_{0+}^{\alpha-1}x(t) \|_0, \| D_{0+}^{\alpha}x(t) \|_{\infty}\}$$

$$\leq \Gamma(\alpha)| b_1 | + \Gamma(\alpha - 1)| b_2 | + \| D_{0+}^{\alpha}x \|_{\infty}. \tag{3.4}$$

Therefore, Ω_1 is bounded in *X*. □

Lemma 16. *Suppose* (H_5) *of Theorem* 14 *holds. Then* $\Omega_2 = \{x \in \ker M : Nx \in \operatorname{Im} M\}$ *is bounded.*

Proof. Let $x \in \Omega_2$, where $x = b_1 t^{\alpha-2} + b_2 t^{\alpha-2}$, $b_1, b_2 \in \mathbb{R}$. Since $Nx \in \operatorname{Im} M = \ker Q$, $Q_1 Nx = 0 = Q_2 Nx$. By (H_5), it follows that $| b_1 | < A$ and $| b_2 | < A$. Hence,

$$
\begin{aligned}
\| x \|_X &= \max\{\| x \|_0, \| D_{0+}^{\alpha-3} x(t) \|_0, \| D_{0+}^{\alpha-2} x(t) \|_0, \| D_{0+}^{\alpha-1} x(t) \|_0, \| D_{0+}^{\alpha} x(t) \|_\infty\} \\
&= \max\left\{\sup_{t\geq 0} \frac{|x|}{1+t^{\alpha}}, \sup_{t\geq 0} \frac{|D_{0+}^{\alpha-3}x(t)|}{1+t^3}, \sup_{t\geq 0} \frac{|D_{0+}^{\alpha-2}x(t)|}{1+t^{\alpha-1}}, \right. \\
&\qquad \left. \sup_{t\geq 0} \frac{|D_{0+}^{\alpha-1}x(t)|}{1+t^{\alpha-2}}, \sup_{t\geq 0} |D_{0+}^{\alpha}x(t)| \right\} \\
&\leq \max\left\{|b_1| + |b_2|, \frac{\Gamma(\alpha)}{2}|b_1| + \Gamma(\alpha-1)|b_2|, \Gamma(\alpha)|b_1| \right. \\
&\qquad \left. + \Gamma(\alpha-1)|b_2|, \Gamma(\alpha)|b_1| + \Gamma(\alpha-1)|b_2|, 0\right\} \\
&\leq \frac{2+5\Gamma(\alpha)}{2}|b_1| + (1+3\Gamma(\alpha-1))|b_2|. \qquad (3.5)
\end{aligned}
$$

Therefore, Ω_2 is bounded. □

Proof of Theorem 14. We have proved (i)-(iv) of Theorem 7. Lastly, we prove that the condition (v) of Theorem 7 holds by using the condition (H_5) of Theorem 14. Suppose

$$H(x, \lambda) = -\lambda Jx + (1-\lambda)QNx, \quad \lambda \in [0, 1] \qquad (3.6)$$

and

$$J : \ker M \to \operatorname{Im} Q$$

is a homeomorphism such that

$$J(b_1 t^{\alpha-1} + b_2 t^{\alpha-2}) = \frac{e^{-t}}{D}[(d_{11}|b_1| + d_{12}|b_2|) + (d_{21}|b_1| + d_{22}|b_2|)t]. \qquad (3.7)$$

Now, if $x \in \partial\Omega \cap \ker M$, where $x(t) = b_1 t^{\alpha-1} + b_2 t^{\alpha-2} \neq 0$, then $H(x, 0) = QNx \neq 0$ and $H(x, 1) = -Jx \neq 0$. In other words, for $\lambda = 0$ or 1, $H(x, \lambda) \neq 0$.

Suppose $H(x, \lambda) = 0$ for $\lambda \in (0, 1)$, with

$$x(t) = b_1 t^{\alpha-1} + b_2 t^{\alpha-2} \in \partial\Omega \cap \ker M.$$

Then, from equations (3.6) and (3.7), we have

$$\lambda(d_{11}|b_1| + d_{12}|b_2|)$$
$$= (I - \lambda)[d_{11}Q_1 N(b_1 t^{\alpha-1} + b_2 t^{\alpha-2}) + d_{12}Q_2 N(b_1 t^{\alpha-1} + b_2 t^{\alpha-2})],$$
$$\lambda(d_{21}|b_1| + d_{22}|b_2|)$$
$$= (I - \lambda)[d_{21}Q_1 N(b_1 t^{\alpha-1} + b_2 t^{\alpha-2}) + d_{22}Q_2 N(b_1 t^{\alpha-1} + b_2 t^{\alpha-2})].$$

If $D = \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix} = d_{11}d_{22} - d_{12}d_{21} \neq 0,$ then

$$\lambda|b_1| = (I - \lambda)Q_1 N(b_1 t^{\alpha-1} + b_2 t^{\alpha-2}),$$
$$\lambda|b_2| = (I - \lambda)Q_2 N(b_1 t^{\alpha-1} + b_2 t^{\alpha-2}).$$

If $|b_1| > A, |b_2| > A,$ then from $(H_{5a}),$ we obtain

$$\lambda(|b_1| + |b_2|) = (I - \lambda)[Q_1 N(b_1 t^{\alpha-1} + b_2 t^{\alpha-2}) + Q_2 N(b_1 t^{\alpha-1} + b_2 t^{\alpha-2})] < 0$$

which is a contradiction.

If (H_{5b}) holds, then define

$$H(x, \lambda) = -\lambda Jx - (1 - \lambda)QNx, \quad \lambda \in [0, 1].$$

By using a similar argument as above, we get a contradiction. Thus, $H(x, \lambda) \neq 0$ for $x \in \partial\Omega \cap \ker M$, $\lambda \in [0, 1]$. Therefore, by the homotopy property of Brouwer degree, we obtain

$$\begin{aligned} \deg(QN, \partial\Omega \cap \ker M, 0) &= \deg(H(\cdot, 0), \Omega \cap \ker M, 0) \\ &= \deg(H(\cdot, 1), \Omega \cap \ker M, 0) \\ &= \deg(\pm j, \Omega \cap \ker M, 0) \end{aligned}$$

$$= \operatorname{sgn}\left\{ \pm \begin{vmatrix} \frac{d_{11}}{\Delta} & \frac{d_{12}}{\Delta} \\ \frac{d_{21}}{\Delta} & \frac{d_{22}}{\Delta} \end{vmatrix} \right.$$

$$= \operatorname{sgn}\left(\frac{\pm 1}{\Delta}\right) = \pm 1 \neq 0.$$

Therefore, we conclude that the fractional order BVP (1.1)-(1.2) has at least one solution in X. □

4. Conclusions

The study has investigated a higher fractional-order p-Laplacian boundary value problem with two-dimensional kernel on the half-line. By using Ge and Ren extension of coincidence degree theory and some assumed conditions, we proved that the fractional order BVP that is investigated has at least a solution. The result is new and an example was provided for illustration.

5. Examples

$$(\phi_{3/2}(D_{0+}^{7/2}x(t)))' = (t,\ x(t),\ D_{0+}^{1/2}x(t),\ D_{0+}^{3/2}x(t),\ D_{0+}^{5/2}x(t),\ D_{0+}^{7/2}x(t)),$$

$$t \in [0, +\infty) \qquad (5.1)$$

subject to

$$x(0) = 0 = D_{0+}^{1/2}x(0), \quad D_{0+}^{7/2}x(+\infty) = 0,$$

$$D_{0+}^{3/2}x(0) = \int_0^1 D_{0+}^{3/2}x(t)dA(t), \quad A(t) = 3t - 2t^3,$$

$$\lim_{t \to +\infty} D_{0+}^{5/2}x(t) = -2D_{0+}^{5/2}x(2) + 3D_{0+}^{5/2}x(3),$$

$$f(x,\ x(t),\ D_{0+}^{1/2}x(t),\ D_{0+}^{3/2}x(t),\ D_{0+}^{5/2}x(t),\ D_{0+}^{7/2}x(t))$$

$$= \frac{e^{-4t}\sin x(t)}{20(1+t^{7/2})} + \frac{e^{-4t}\sin D_{0+}^{1/2}x(t)}{5(1+t^{3})} + \frac{e^{-2t}D_{0+}^{3/2}x(t)}{10(1+t^{5/2})} + \frac{e^{-4t}\sin D_{0+}^{5/2}x(t)}{15(1+t^{3/2})}.$$

Here,

$$m = 2,\ \ \mu_1 = 2,\ \ \mu_2 = 3,\ \ p = 3/2,\ \ q = 3,\ \ \alpha = 7/2,\ \ \xi_1 = 2,\ \ \xi_2 = 3,$$

$$\sum_{i=1}^{2}\mu_i = -2+3 = 1; \quad \sum_{i=1}^{2}\mu_i\xi_i^{-1} = -\frac{2}{2}+\frac{3}{3} = 0,$$

$$\int_0^1 dA(t) = \int_0^1 (3-6t^2)dt = 1; \quad \int_0^1 tdA(t) = \int_0^1 (3t-6t^3)dt = 0.$$

The BVP satisfies the condition (H_1).

Next, we compute the determinant of the matrix of coefficients, $D = d_{11}d_{22} - d_{12}d_{21}$,

$$d_{11} = \int_0^1\int_0^t (t-r)\phi_q\left(-\int_r^{+\infty} s^{\alpha-1}e^{-s}ds\right)drdA(t) = -1.09312,$$

$$d_{12} = \sum_{i=1}^{2}\mu_i\int_0^{\xi_i}\phi_3\left(-\int_{\xi_i}^{+\infty} s^{5/2}e^{-s}ds\right)dr = 2.0958,$$

$$d_{21} = \int_0^1\int_0^t (t-r)(3-6t^2)\phi_3\left(-\int_r^{+\infty} e^{-s}s^{3/2}ds\right)drdt = -0.1658,$$

$$d_{22} = \sum_{i=1}^{2}\mu_i\int_0^{\xi_i}\phi_3\left(-\int_{\xi_i}^{+\infty} e^{-s}s^{3/2}ds\right)dr = -0.6423,$$

$$D = d_{11}d_{22} - d_{12}d_{21} = 1.0496 \neq 0.$$

Hence, the assumption (H_2) holds.

Next,

$$\| g_1 \|_{L^1} = \frac{1}{20}\int_0^{+\infty} | e^{-4t} | dt = \frac{1}{80}; \quad \| g_2 \|_{L^1} = \frac{1}{5}\int_0^{+\infty} | e^{-4t} | dt = \frac{1}{20};$$

$$\| g_3 \|_{L^1} = \frac{1}{10}\int_0^{+\infty} | e^{-2t} | dt = \frac{1}{20}; \quad \| g_4 \|_{L^1} = \frac{1}{15}\int_0^{+\infty} | e^{-4t} | dt = \frac{1}{60};$$

$$\| g_5 \|_{L^1} = 0, \quad \| g_6 \|_{L^1} = 0.$$

Since $p < 2$, we have

$$2^{2q-4} = 2^{6-4}\left(\frac{1}{\Gamma\left(\frac{7}{2}+1\right)}\left(\frac{1}{80}\right)^2 + \frac{1}{6}\left(\frac{1}{20}\right)^2 + \frac{1}{2}\left(\frac{1}{20}\right)^2 + \left(\frac{1}{60}\right)^2\right)$$

$$= 4(0.003768) = 0.01507 < 1.$$

Assume $| D_{0+}^{3/2}x(t) | > E$ for $t \in [0, +\infty)$ and any $x \in dom\, M$, from the continuity of $D_{0+}^{3/2}(x)$, either $| D_{0+}^{3/2}x(t) | > E$ or $| D_{0+}^{3/2}x(t) | < -E$ if $E = 25$.

If $| D_{0+}^{3/2}x(t) | > E$ holds for any $x \in dom\, M$, then

$$Q_1 Nx(t)$$

$$= \int_0^1\int_0^t (t-r)\phi_q\left(-\int_r^{+\infty} Nx(s)ds\right)drdA(t)$$

$$> \int_0^t (t-r)(3-6t^2)\phi_3\left(-\int_r^{+\infty}\left(-\frac{1}{20}-\frac{1}{5}-\frac{1}{15}\right)e^{-4s} + 10e^{-2s}E\right)dsdrdt$$

$$= \int_0^1\int_0^t (t-r)(3-6t^2)\left[\left(\frac{19}{240}e^{-4r}\right)^2 - \frac{25}{16}e^{-4r}\right]drdt = 0.039643 > 0.$$

Similarly, if $| D_{0+}^{5/2}x(t) | > E$ for $t \in [0, +\infty)$, then either $D_{0+}^{5/2}x(t) > E$ or $D_{0+}^{5/2}x(t) < -E$. Suppose that $D_{0+}^{5/2}x(t) > E$, for $t \in [0, +\infty)$. Then

$$Q_2Nx(t) = -2\int_0^2 \phi_3\left(-\int_2^{+\infty} Nx(s)ds\right)dr + 3\int_0^3 \phi_3\left(-\int_3^{+\infty} Nx(s)ds\right)dr$$

$$> -2\int_0^3 \phi_3\left(\int_2^{+\infty}\left(\frac{19}{60}e^{-4s} - \frac{1}{10}e^{-2s}E\right)ds\right)dr$$

$$+ 3\int_0^3 \phi_3\left(\int_0^{+\infty}\left(\frac{19}{60}e^{-4s} - \frac{1}{10}e^{-2s}E\right)ds\right)dr$$

$$= 0.002005 > 0.$$

Since $Q_1Nx(t) \neq 0$ and $Q_2Nx(t) \neq 0$, condition (H_4) is satisfied. Lastly, let $A = 18$, for any $d_1, d_2 \in \mathbb{R}$. If $|b_1|$ or $|b_1| > A$, $\alpha = 7/2$, then

$$Q_1N(b_1t^{\alpha-1} + b_2t^{\alpha-2}) + Q_2N(b_1t^{\alpha-1} + b_2t^{\alpha-2})$$

$$= Q_1N(b_1t^{5/2} + b_2t^{3/2}) + Q_2N(b_1t^{5/2} + b_2t^{3/2})$$

$$= \int_0^1\int_0^t (3-6t^2)(t-r)\phi_q\left(-\int_t^{+\infty} Nx(s)ds\right)drdt$$

$$+ \sum_{i=1}^{2}\mu_i\int_0^{\xi_i}\phi_q\left(-\int_{\xi_i}^{+\infty} Nx(s)ds\right)dr$$

$$> \int_0^1\int_0^t (3-6t^2)(t-r)\phi_3\left[\int_r^{+\infty}\left(\frac{19}{60}e^{-4s} - \frac{e^{-2s}}{10}(A\Gamma(7/2) + A\Gamma(5/2))\right)ds\right]drdt$$

$$+ \sum_{i=1}^{2}\mu_i\int_0^{\xi_i}\phi_3\left(\int_0^{+\infty}\frac{19}{60}e^{-4s} - \frac{e^{-2s}}{10}(A\Gamma(7/2)s + A\Gamma(5/2))drdt\right.$$

$$= \int_0^1\int_0^t (3-6t^2)(t-r)\left(\frac{475}{6000}e^{-4s} + \frac{e^{-2r}}{1000}(2991r + 2692)\right)$$

$$\times\left(\frac{475}{6000}e^{-4r} - \frac{e^{-2r}}{1000}(2991r + 2692)\right)drdt$$

$$- 2\int_0^2 0.025231ds + 3\int_0^3 0.000836ds$$

$$= 19.576 - 4 \times 0.025231 + 9 \times 0.00836$$

$$= 19.55 > 0.$$

Thus, $Q_1 N(b_1 t^{5/2} + b_2 t^{3/2}) + Q_2 N(b_1 t^{5/2} + b_2 t^{3/2}) > 0.$ Hence, the condition (H_{5b}) holds. Consider the BVP (5.1). Since all conditions of Theorem 14 are satisfied, the fractional boundary value problem has at least one solution in X.

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