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polynomial collocation method for initial value problem of mixed integro-differential equations

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Abstract: This paper presents the development and implementation of a numerical method for the solution of one dimensional Mixed Fredholm Volterra Intergro-Differential Equations (MFVIDEs). The new technique transformed MFVIDEs into an integral equation which is then approximated using a polynomial collocation method. Standard collocation points are then used to convert the problem into a system of algebraic equations. Some numerical examples are used to test the efficiency of the method. The results show that the new method is efficient, accurate and easy to implement.

Keywords: One dimensional Fredholm-Volterra equations, Integro-differential equations, Polynomial collocation, Standard collocation points, Integral equations
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1 Introduction

The integro differential equation, which is a hybrid of integral and differential equations, was discovered as a result of Vito Volterra's findings on hereditary influences while examining a population growth model [19]. This arises in nano-dynamics and desert wind ripple [18, 22], fluid dynamics and chemical kinetics [15, 13], biological modeling and ecology [6, 1]. Some of the methods developed to solve integro differential equations include: iterative method [13, 6, 3], moving least method [7], collocation method [1, 16, 14, 10, 5]. The aim of this paper is to develop a new polynomial collocation method different from the methods discussed earlier for the solution of MVIDEs of the form

$$\sum_{m=0}^{M} P_m\left(x\right) u^{(m)}(x) = f(x) + \int_0^x \int_0^1 k(r,t)u(t)dtdr, x \in [0,1] \tag{1.1}$$

with the initial condition

$$u^m(0) = b_m,\ m = 0, 1, ..., (M-1) \tag{1.2}$$

where $u^{(m)}(x)$ is the mth derivative of the unknown function $u(x)$ to be determined, $u^{(0)}\left(x\right) = u\left(x\right)$. $k \in C[0,1]$, f, $u \in C[0,1]$, $P_M\left(x\right) = 1$, $K(r,t)$ is the Kernel function of (1.1)

The remaining paper is organized as follows: preliminary studies are presented in Section 2. Section 3 discusses methodology which includes the theoretical background, method of solution, and convergence of

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the method of solution. Section 4 deals with numerical examples and discussion of results. This paper is concluded in Section 5.

2 Preliminary Study

In this section, we present definition of terms, results that are used in this study

Definition 2.1. *[11] (Lipschitzian) Let (X, d) denotes a metric space. A mapping $T : X \to X$ is called a contraction on X if there exist a positive constant $k < 1$ such that*

$$d\left(T(x), T(y)\right) \leq kd(x, y),\ \forall x, y \in X. \tag{2.1}$$

Definition 2.2. *[4] (Strict contraction): Let (X, d) be a metric space, A mapping $T : X \to X$ is strict contraction if T is $\alpha-$Lipschitzian with $\alpha \in [0, 1)$ i.e.*

$$d\left(T\left(x\right), T\left(y\right)\right) \leq \alpha d(x, y), \alpha \in [0, 1),\ \forall x, y \in X \tag{2.2}$$

Lemma 2.3. *[20] Let $\alpha, \beta \in \mathbb{R}^+$, then*

$$\int_0^x (x-t)^{\alpha-1} t^{\beta-1} dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} x^{\alpha+\beta-1}. \tag{2.3}$$

Definition 2.4. *[21] (Convergence) A sequence $\{u_n\}$ in a metric space (X, d) converges to $u \in X$ if $\lim_{n\to\infty} d(u_n, u) = 0$ i.e. $\lim_{n\to\infty} u_n = u$.*

Theorem 2.5. *[8] (Continuity): Let (X, d) be a metric space and $T : X \to X$ be a mapping, then let $u(x)$, $u_n(x) \in X$, and the $\lim_{n\to\infty} u_n(x) = u(x)$, then T is continuous if $d\left(Tu_n, Tu\right) \to 0$ as $n \to \infty$.*

Theorem 2.6. *[4] Let (X, d) denotes a metric space and $T : X \to X$ be a self-mapping of x. Then, T is called a Lipschitz continuous mapping if*

$$d\left(T^n\left(y_1\right), T^n\left(y_2\right)\right) \leq k^n d\left(y_1, y_2\right),\ L > 0, \qquad \forall y_1, y_2 \in X, \tag{2.4}$$

Theorem 2.7. *[4] (Contraction mapping principle) Let (X, d) be a complete metric space and $T : X \to X$ is a given contraction on X, then T has a unique fixed point p and $T^n\left(x\right) \to p$ (as $n \to \infty$) for each $x \in X$*

Theorem 2.8. *[4] (Banach fixed point) Let (X, d) be a complete metric space and $T : X \to X$ is a contraction on X, then T has a unique fixed point $x \in X$ such that $Tx = x$, moreover the iterative scheme*

$$x_n = Tx_{n-1} \tag{2.5}$$

converges to x.

3 Methodology

In this section, the uniqueness of the solution is proven. In order to obtain the uniqueness of the solution, the following (i) strict contraction, and (ii) Lipschitzian continuity were established, hence we can conclude the existence of a convergence sequence in T. Therefore, Banach contraction principle can be used to establish the uniqueness of solution. Expressing (1.1) in the form

$$u^{(M)}\left(x\right) = f\left(x\right) - \sum_{m=0}^{M-1} P_m\left(x\right) u^{(m)}\left(x\right) + \int_0^x \int_0^1 k(r, t) u(t) dt dr \tag{3.1}$$

On integrating 3.1 M^{th} times and applying (1.2) to get

$$\begin{aligned} u(x) = G(x) \quad & -\frac{1}{\Gamma(M)} \sum_{m=0}^{M-1} \int_0^x (x-t)^{M-1} P_m(t) u^{(m)}(t) dt \\ & + \frac{1}{\Gamma(M)} \int_0^x (x-t)^{M-1} \left[\int_0^t \int_0^1 k(r,s) u(s) ds dr \right] dt \end{aligned} \tag{3.2}$$

where $G(x) = \sum\limits_{k=0}^{M-1} \dfrac{x^k}{k!} u^{(k)}(0) + \dfrac{1}{\Gamma(M)} \int_0^x (x-t)^{M-1} f(t)\, dt$

In order to establish the uniqueness of solution, we use the following hypothesis

H_1 :Let $T : X \to X$ be a mapping, for $u_1, u_2 \in X$, $L > 0$. Define $u^{(m)}(t) = \dfrac{d^m u}{dt^m}$, then $\left| u_1^{(m)}(t) - u_2^{(m)}(t) \right| \leq L_m \left| u_1(t) - u_2(t) \right|$

H_2 : Let $P : [0,1] \to [0,1]$ be a continuous function, then $P^* = \max\limits_{x \in J} \sum_{m=0}^{M-1} |P(t)| L_m,\ J = [0,1]$

H_3 : Let $k : [0,1] \to [0,1]^2$ be a continuous function, then $k^* = \max\limits_{x \in J} \int_0^x \int_0^1 |k(r,s)|\, ds dr$

Theorem 3.1. *Let $T : X \to X$ be a mapping defined by* (3.2)*, then T is a strict contraction if*

$$\frac{P^* + k^*}{\Gamma(M+1)} < 1. \tag{3.3}$$

Proof. Let $u_1(x), u_2(x) \in X$, using the Banach fixed point in (3.2)

$$\begin{aligned} |(Tu_1)(x) - (Tu_2)(x)| \quad \leq \quad & \frac{1}{\Gamma(M)} \sum_{j=0}^{M-1} \int_0^x (x-t)^{M-1} \left| P_j(t) \left(u_1^{(j)}(t) - u_2^{(j)}(t) \right) \right| dt \\ & + \frac{1}{\Gamma(M)} \int_0^x (x-t)^{M-1} \left[\int_0^t \int_0^1 \left| k(r,s) \left(u_1(s) - u_2(s) \right) \right| ds dr \right] dt \end{aligned}$$

$$\begin{aligned} \leq \quad & \frac{1}{\Gamma(M)} \int_0^x (x-t)^{M-1} \sum_{j=0}^{M-1} \left| P_j(t) \left(u_1^{(j)}(t) - u_2^{(j)}(t) \right) \right| dt \\ & + \frac{1}{\Gamma(M)} \int_0^x (x-t)^{M-1} \left[\int_0^t \int_0^1 |k(r,s)| \left| u_1(s) - u_2(s) \right| ds dr \right] dt \end{aligned}$$

Using H_1

$$\begin{aligned} \leq \quad & \frac{1}{\Gamma(M)} \int_0^x (x-t)^{M-1} \sum_{j=0}^{M-1} |P_j(t)| L_j \left| u_1(t) - u_2(t) \right| dt \\ & + \frac{1}{\Gamma(M)} \int_0^x (x-t)^{M-1} \left[\int_0^t \int_0^1 |k(r,s)| \left| u_1(s) - u_2(s) \right| ds dr \right] dt \end{aligned}$$

Taking maximum of both sides

$$\begin{aligned} \leq \quad & \frac{1}{\Gamma(M)} \int_0^x (x-t)^{M-1} \max_{x \in J} \sum_{j=0}^{M-1} |P_j(t)| L_j \max_{x \in J} |u_1(t) - u_2(t)| dt \\ & + \frac{1}{\Gamma(M)} \int_0^x (x-t)^{M-1} \left[\max_{x \in J} \int_0^t \int_0^1 |k(r,s)| \max_{x \in J} |u_1(s) - u_2(s)| ds dr \right] dt \end{aligned}$$

Using $H_2 - H_3$

$$d\left(Tu_1, Tu_2\right) \leq P^* \frac{1}{\Gamma(M)} \int_0^x (x-t)^{M-1}\, dt\ d\left(u_1, u_2\right) + k^* \frac{1}{\Gamma(M)} \int_0^x (x-t)^{M-1} dt\ d\left(u_1, u_2\right)$$

$$d\left(Tu_1, Tu_2\right) \leq \left(\frac{P^* + k^*}{\Gamma(M+1)}\right)\ d\left(u_1, u_2\right)$$

under the condition (2.2), then T is a strict contraction mapping $\square$

Theorem 3.2. (*Continuity*)*Let $T : X \to X$ be a mapping defined by (3), Let $u(x), u_N\left(x\right) \in X$ be the exact and approximate solutions of (4), respectively. If $\lim_{n\to\infty} u_n(x) = u(x)$, then T is continuous.*

Proof.

$$\begin{aligned} \left|Tu_n\left(x\right) - Tu\left(x\right)\right| &\leq \frac{1}{\Gamma(M)} \int_0^x (x-t)^{M-1} \sum_{j=0}^{M-1} \left|P_j\left(t\right)\left(u_n^{(j)}\left(t\right) - u^{(j)}\left(t\right)\right)\right| dt \\ &\quad + \frac{1}{\Gamma(M)} \int_0^x (x-t)^{M-1} \left[\int_0^t \int_0^1 \left|k(r,s)\left(u_n\left(s\right) - u\left(s\right)\right)\right| dsdr\right] dt \\ &\leq \frac{1}{\Gamma(M)} \int_0^x (x-t)^{M-1} \sum_{j=0}^{M-1} \left|P\left(t\right)\right| L_j \left|u_n(t) - u(t)\right| dt \\ &\quad + \frac{1}{\Gamma(M)} \int_0^x (x-t)^{M-1} \left[\int_0^t \int_0^1 \left|k(r,s)\right| \left|u_n(s) - u(s)\right| dsdr\right] dt \\ &\leq \frac{1}{\Gamma(M)} \int_0^x (x-t)^{M-1} \max_{x\in[0,1]} \sum_{j=0}^{M-1} \left|P\left(t\right)\right| L_j \max_{x\in[0,1]} \left|u_n(t) - u(t)\right| dt \\ &\quad + \frac{1}{\Gamma(M)} \int_0^x (x-t)^{M-1} \left[\max_{x\in[0,1]} \int_0^t \int_0^1 \left|k(r,s)\right| \max_{x\in[0,1]} \left|u_n(s) - u(s)\right| dsdr\right] dt \end{aligned}$$

$$d\left(Tu_n, Tu\right) \leq \left(\frac{P^* + k^*}{\Gamma(M+1)}\right)\ d\left(u_n, u\right).$$

Since $d\left(u_n, u\right) \to 0$ as $n \to \infty$, then $d\left(Tu_n, Tu\right) \to 0$ as $n \to \infty$, hence T is continuous. $\square$

Theorem 3.3. *(Lipschitizian continuity) Let X be a metric space and let $T : X \to X$ be a mapping defined by (2.1), Let $u \in X$ be the solution to (2.2), then*

$$d\left(T^n u_1, T^n u_2\right) \leq \left(\frac{P^* + k^*}{\Gamma(M+1)}\right)^n d\left(Tu_1, Tu_2\right)$$

Proof. **:** We have shown that T is continuous and Lipschitzian. If $k = \dfrac{P^* + k^*}{\Gamma(M+1)}$ in Theorem (2.4), the result is obtained. $\square$

Remark 3.4. *Since T is Lipschitzian continuous, hence there exist a convergence sequence in X, therefore, X is a complete metric space. Moreover, since X is a complete metrix space and a contraction, therefore using theorem 2.3, there exist a unique fixed point $x \in X$*

3.1 Method of solution

Let the solution to (1.1) and (1.2) be approximated by

$$u_N(x)=\sum_{n=0}^{N}a_nx^n \tag{3.4}$$

therefore

$$u_N^{(L)}(x)=\sum_{n=L}^{N}\frac{a_n\Gamma(n+1)}{\Gamma(n-L+1)}x^{n-L}, n\geq L \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.2) , we obtain,

$$\begin{aligned}\sum_{n=0}^{N}a_nx^n &= G(x)-\frac{1}{\Gamma(M)}\sum_{n=m}^{N}\sum_{m=0}^{M-1}\frac{a_n\Gamma(n+1)}{\Gamma(n-m+1)}\int_0^x(x-t)^{M-1}P_m(t)\,t^{n-m}dt \\ &+\frac{1}{\Gamma(M)}\sum_{n=0}^{N}a_n\int_0^x(x-t)^{M-1}\left[\int_0^t\int_0^1k(r,s)\,s^ndsdr\right]dt\end{aligned} \tag{3.6}$$

Let $P_m(t)=\sum_{r=0}^{R}\frac{p_r}{\Gamma(r+1)}s^r, p_r=\left[\frac{d^r}{ds^r}P(s)\right]_{s=0}$ and $k(r,s)=\sum\limits_{i=0}^{I}\sum\limits_{j=0}^{J}k_{ij}r^is^j$, where $k_{ij}=\frac{1}{i!j!}\left[\frac{\partial^{i+j}}{\partial r^i\partial s^j}k(r,s)\right]_{(r,s)=(0,0)}$.
Hence

$$\begin{aligned}\sum_{n=0}^{N}a_nx^n &= G(x)-\frac{1}{\Gamma(M)}\sum_{n=m}^{N}\sum_{m=0}^{M-1}\sum_{r=0}^{R}\frac{a_np_r\Gamma(n+1)}{\Gamma(n-m+1)}\int_0^x(x-t)^{M-1}t^{n-m+r}dt \\ &+\frac{1}{\Gamma(M)}\sum_{n=m}^{N}\sum_{i=0}^{I}\sum_{j=0}^{J}a_nk_{ij}\int_0^x(x-t)^{M-1}\left[\int_0^t\int_0^1r^is^{n-L+j}dsdr\right]dt\end{aligned} \tag{3.7}$$

$$\begin{aligned}\sum_{n=0}^{N}a_nx^n &= G(x)-\frac{1}{\Gamma(M)}\sum_{n=m}^{N}\sum_{m=0}^{M-1}\sum_{r=0}^{R}\frac{a_np_r\Gamma(n+1)}{\Gamma(n-m+1)}\int_0^x(x-t)^{M-1}t^{n-m+r}dt \\ &+\frac{1}{\Gamma(M)}\sum_{n=0}^{N}\sum_{i=0}^{I}\sum_{j=0}^{J}\frac{a_nk_{ij}}{(n+j+1)(i+1)}\int_0^x(x-t)^{M-1}t^{i+1}dt\end{aligned} \tag{3.8}$$

Applying (2.3) to (3.8)

$$\sum_{n=0}^{N}a_nx^n=G(x)-\sum_{n=m}^{N}\sum_{m=0}^{M-1}\sum_{r=0}^{R}H_n(m,r)\,x^{M+n-m+r}+\sum_{n=m}^{N}\sum_{i=0}^{I}\sum_{j=0}^{J}Q_n(i,j)\,x^{M+i+1} \tag{3.9}$$

where $H_n(m,r)=\frac{a_np_r\Gamma(n+1)\Gamma(n-m+r+1)}{\Gamma(n-m+1)\Gamma(M+n-m+r+1)}$,
$Q_n(i,j)=\frac{a_nk_{ij}\Gamma(i+2)}{(n+j+1)(i+1)\Gamma(M+i+2)}$. We then collocate (3.9) at x_i

$$\sum_{n=0}^{N}a_nx_i^n+\sum_{n=m}^{N}\sum_{m=0}^{M-1}\sum_{r=0}^{R}H_n(m,r)\,x_i^{M+n-m+r}-\sum_{n=0}^{N}\sum_{i=0}^{I}\sum_{j=0}^{J}Q_n(i,j)\,x_i^{M+i+1}-G(x_i)=0 \tag{3.10}$$

We then solve (3.10) , the system of linear equations for the unknown constants and substitute the results into (3.4)

3.2 Convergence of solution

Lemma 3.5. *(Convergence) Let* (X,d) *be a metric space and* $T : X \to X$ *be a mapping. If* $u_N(t)$ *and* $u_{N-1}(t) \in X$ *are convergent approximate solutions, then*

$$\lim_{N\to\infty}\left(u_N(t) - \lim u_{N-1}(t)\right) = 0. \tag{3.11}$$

Proof. Let $u(t) \in X$ be the exact solution, since $u_N(t)$ and $u_{N-1}(t)$ are convergent, then

$$\lim_{N\to\infty} u_N(t) = u(t)$$

Moreover

$$\lim_{N\to\infty} u_{N-1}(t) = u(t)$$

hence

$$\lim_{N\to\infty}\left(u_N(t) - \lim u_{N-1}(t)\right) = 0$$

which implies that the solution converges to a unique fixed point in X □

Theorem 3.6. *(Convergence of method of solution) Let* (X,d) *denotes a metrix space and* $T : X \to X$ *a Lipschitz continuous mapping,* $u_N(t), u_{N-1}(t) \in X$ *are approximate solutions of* (3.1)*. Let* $\triangle_N(t) = |u_N(t) - u_{N-1}(t)|$, *if* $\lim_{N\to 0}(\triangle_N(t)) \to 0$, *then the method converges to the exact solution.*

Proof. Let $u_1(x)$ and $u_2(x)$ be approximate by $u_N(x) = \sum\limits_{n=0}^{N} a_n x^n$ and $u_{N-1}(x) = \sum\limits_{n=0}^{N-1} b_n x^n$ respectively. Applying the fixed point theorem on (3.5) and substituting the approximate solution

$$\begin{aligned} Tu_N(x) &= G(x) - \frac{1}{\Gamma(M)} \sum_{n=m}^{N} \sum_{m=0}^{M-1} \frac{a_n \Gamma(n+1)}{\Gamma(n-m+1)} \int_0^x (x-t)^{M-1} P_m(t)\, t^{n-m} dt \\ &+ \frac{1}{\Gamma(M)} \sum_{n=0}^{N} a_n \int_0^x (x-t)^{M-1} \left[\int_0^t \int_0^1 k(r,s)\, s^n ds dr\right] dt. \end{aligned} \tag{3.12}$$

Following the approach in the method of solution, then

$$\begin{aligned} Tu_N(x) &= G(x) - \sum_{n=m}^{N} \sum_{m=0}^{M-1} \sum_{r=0}^{R} \frac{a_n p_r \Gamma(n+1)\Gamma(n-m+r+1)}{\Gamma(n-m+1)\Gamma(M+n-m+r+1)} x^{M+n-m+r} \\ &+ \sum_{n=0}^{N} \sum_{i=0}^{I} \sum_{j=0}^{J} \frac{a_n k_{ij} \Gamma(i+2)}{(n+j+1)(i+1)\Gamma(M+i+2)} x^{M+i+1} \end{aligned} \tag{3.13}$$

similarly

$$\begin{aligned} Tu_{N-1}(x) &= G(x) - \sum_{n=m}^{N-1} \sum_{m=0}^{M-1} \sum_{r=0}^{R} \frac{b_n p_r \Gamma(n+1)\Gamma(n-m+r+1)}{\Gamma(n-m+1)\Gamma(M+n-m+r+1)} x^{M+n-m+r} \\ &+ \sum_{n=0}^{N-1} \sum_{i=0}^{I} \sum_{j=0}^{J} \frac{b_n k_{ij} \Gamma(i+2)}{(n+j+1)(i+1)\Gamma(M+i+2)} x^{M+i+1} \end{aligned} \tag{3.14}$$

similarly

$$\begin{aligned} |Tu_N(x) - Tu_{N-1}(x)| &= \sum_{m=0}^{M-1}\sum_{r=0}^{R} \frac{a_N p_r \Gamma(N+1)\Gamma(N-m+r+1)}{\Gamma(N-m+1)\Gamma(M+N-m+r+1)} x^{M+N-m+r} && (3.15) \\ &+ \sum_{n=m}^{N-1}\sum_{m=0}^{M-1}\sum_{r=0}^{R} \frac{p_r \Gamma(n+1)\Gamma(n-m+r+1)}{\Gamma(n-m+1)\Gamma(M+n-m+r+1)} |a_n - b_n| x^{M+n-m+r} \\ &+ \sum_{i=0}^{I}\sum_{j=0}^{J} \frac{a_N k_{ij}\Gamma(i+2)}{(N+j+1)(i+1)\Gamma(M+i+2)} x^{M+i+1} && (3.16) \\ &+ \sum_{n=0}^{N-1}\sum_{i=0}^{I}\sum_{j=0}^{J} \frac{k_{ij}\Gamma(i+2)}{(n+j+1)(i+1)\Gamma(M+i+2)} |a_n - b_n| x^{M+i+1} && (3.17) \end{aligned}$$

Since $x \in [0,1]$, and $|a_n - b_n| \neq 0$, obviously

$$\lim_{N\to 0} \triangle_N(t) \to 0.$$

Therefore the method converges to the exact solution $\square$

4 Numerical Examples

Five numerical examples were considered to test the efficiency and accuracy of the new method. All computations were done with the aid of program written in MATLAB (2015a) and ran on a PC. The standard collocation $x_i = a + \frac{b-a}{N} i,\ [a,b] = [0,1]$. Results are presented in Tables whenever the exact solution is not a polynomial.

Example 4.1. *[12] Consider the first order mixed problem*

$$u'(x) = \frac{1}{4}x^2 - e^x + \int_0^x \int_0^1 rtu(t)dtdr,\ \ x \in [0,1] \tag{4.1}$$

with initial condition $u(0) = 0$. The exact solution is $u(x) = 1 - e^x$. Comparing with (3.2), $M = 1, m = 0$, $k(r,t) = rt,\ P_0(x) = 0,\ P_1(x) = 1,\ g(x) = \frac{1}{4}x^2 - e^x$. Expressing in the integral form

$$u(x) = \int_0^x \left(\frac{1}{4}t^2 - e^t\right) dt + \int_0^x \left(\int_0^t \int_0^1 rsu(s)\, dsdr\right) dt$$

hence, $k^* = \max_{x\in[0,1]} \int_0^t \int_0^1 rsu(s)\, dsdr = \frac{1}{4}$, $P^* = 0$, hence, $\dfrac{P^* + k^*}{\Gamma(M+1)} = \frac{1}{4} < 1$. It shows that T is strict contraction, hence the solution converges to a unique solution in X. we take $I = J = N$. Taking $N = 3$ for illustration $u_N(x) = \sum_{n=0}^{3} a_n x^n$, $G(x) = \frac{x^3}{12} - e^x + 1$, $u_3(x_i) = \sum_{n=0}^{3} a_n x_i^n = \begin{bmatrix} a_0 \\ a_0 + \frac{a_1}{3} + \frac{a_2}{9} + \frac{a_3}{27} \\ a_0 + \frac{2a_1}{3} + \frac{4a_2}{9} + \frac{8a_3}{27} \\ a_0 + a_1 + a_2 + a_3 \end{bmatrix}$,

$\sum_{n=0}^{3}\sum_{i=0}^{3}\sum_{j=0}^{3} Q_n(i,j)\, x_i^{M+i+1} = \begin{bmatrix} 0 \\ \frac{a_0}{324} + \frac{a_1}{486} + \frac{a_2}{648} + \frac{a_3}{810} \\ \frac{2a_0}{81} + \frac{4a_1}{243} + \frac{a_2}{81} + \frac{4a_3}{405} \\ \frac{a_0}{12} + \frac{a_1}{18} + \frac{a_2}{24} + \frac{a_3}{30} \end{bmatrix}$, $G(x_i) = \begin{bmatrix} 0 \\ \frac{325}{324} - \exp\left(\frac{1}{3}\right) \\ \frac{83}{81} - \exp\left(\frac{2}{3}\right) \\ \frac{13}{12} - \exp(1) \end{bmatrix}$, the unknown constants give $A = \begin{bmatrix} 0 & -\frac{1522}{1501} & -\frac{272}{639} & -\frac{497}{1719} \end{bmatrix}^T$, hence

$$u_3(x) = -0.278650264607189x^3 - 0.425664946481447x^2 - 1.01399046948781x$$

$$\begin{aligned} u_6(x) &= -0.00230584431051832x^6 - 0.00694954373010421x^5 - 0.0426743917496825x^4 \\ &\quad -0.166287194703431x^3 - 0.500069630985783x^2 - 0.99999522453344x \end{aligned}$$

$$\begin{aligned} u_8(x) &= -0.0000411044057060689x^8 - 0.000164845682255676x^7 - 0.00142516954681341x^6 \\ &\quad -0.00831036062507852x^5 - 0.0416754020405732x^4 - 0.166664730572359x^3 \\ &\quad -0.500000226029704x^2 - 0.999999989558325x \end{aligned}$$

$$\begin{aligned} u_{10}(x) &= -0.000000456240272157352x^{10} - 0.00000228500272953853x^9 \\ &\quad -0.0000254753361496324x^8 - 0.000197813024715949x^7 - 0.00138923705835441x^6 \\ &\quad -0.00833320004282125x^5 - 0.0416666997204681x^4 - 0.166666661626246x^3 \\ &\quad -0.500000000421728x^2 - 0.999999999985562x \end{aligned}$$

Table 1: *Exact solution and numerical solution for* (4.1)

	Exact	*Present method*		
x_i	$1-e^x$	$N=6$	$N=8$	$N=10$
0.2	*-0.22140275816017*	*-0.221402778158576*	*-0.2214027581822*	*-0.22140275816017*
0.4	*-0.49182469764127*	*-0.491824683727005*	*-0.491824697645566*	*-0.491824697641271*
0.6	*-0.822118800390509*	*-0.82211881509425*	*-0.822118800386624*	*-0.822118800390509*
0.8	*-1.22554092849247*	*-1.2255409077469*	*-1.22554092847*	*-1.22554092849247*
1.0	*-1.71828182845905*	*-1.71828183001296*	*-1.71828182846081*	*-1.71828182845905*

Table 2: *Errors for* (4.1)

x_i	Err_6	Err_8	Err_{10}	$\triangle_8$	$\triangle_{10}$
0.2	*1.9998e-08*	*2.2055e-11*	*0.0000*	*1.9976e-08*	*2.203e-11*
0.4	*1.3914e-08*	*4.3201e-12*	*0.0000*	*1.3919e-08*	*4.295e-12*
0.6	*1.4704e-08*	*4.0927e-12*	*0.0000*	*1.4708e-08*	*3.885e-12*
0.8	*2.0746e-08*	*2.2737e-11*	*0.0000*	*2.0723e-08*	*2.247e-11*
1.0	*1.5539e-09*	*1.8190e-12*	*0.0000*	*1.5522e-09*	*1.760e-12*

Table I shows the exact and numerical results at different values of N. Table 2 displays the errors at different values of N. it shows clearly that as $N \to \infty$, $err_N \to 0$, which shows that the solution convergence. Moreover, as $N \to \infty$, $\triangle_N \to 0$, which affirms the theorem on convergence.

Example 4.2. *[23] Consider the Volterra-Fredholm integro-differential equation*

$$u''(x) = 2 + 6x - \frac{77}{120}x^2 + \int_0^x \int_0^1 rtu(t)dtdr \tag{4.2}$$

with initial condition $u(0) = 1$, $u'(0) = 1$.The exact solution is $u(x) = 1 + x + x^2 + x^3$. writing in the integral form

$$u(x) = 1 + x + \int_0^x (x-t)\left(2 + 6t - \frac{77}{120}t^2\right)dt + \int_0^x (x-t)\left[\int_0^t \int_0^1 rsu(s)\,dsdr\right]dt \tag{4.3}$$

hence, $g(x) = 1 + x + \int_0^x (x-t)\left(2 + 6t = \frac{77}{120}t^2\right) dt,\ M = 2,\ P_2(x) = 1,\ P_1(x) = 0,\ P_0(x) = 0.$ *using* $N = 3,$ *we obtain* $u_3(x_i) = \sum_{n=0}^{3} a_n x_i^n = \begin{bmatrix} a_0 \\ a_0 + \frac{a_1}{3} + \frac{a_2}{9} + \frac{a_3}{27} \\ a_0 + \frac{2a_1}{3} + \frac{4a_2}{9} + \frac{8a_3}{27} \\ a_0 + a_1 + a_2 + a_3 \end{bmatrix},$

$\sum_{n=0}^{3}\sum_{i=0}^{3}\sum_{j=0}^{3} Q_n(i,j)\, x_i^{M+i+1} = \begin{bmatrix} 0 \\ \frac{a_0}{3888} + \frac{a_1}{5832} + \frac{a_2}{7776} + \frac{a_3}{9720} \\ \frac{a_0}{243} + \frac{2a_1}{729} + \frac{a_2}{486} + \frac{2a_3}{1215} \\ \frac{a_0}{148} + \frac{a_1}{72} + \frac{a_2}{96} + \frac{a_3}{120} \end{bmatrix},\ G(x_i) = \begin{bmatrix} 1 \\ \frac{172723}{116640} \\ \frac{17473}{7290} \\ \frac{5683}{1440} \end{bmatrix},$ *the unknown constants give* $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T,$ *hence* $u_3(x) = x^3 + x^2 + x + 1,$ *which is the exact result*

Example 4.3. *[23] Consider the Volterra-Fredholm integro-differential equation*

$$u'(x) = 8x + \frac{5}{4}x^2 + \int_0^x \int_0^1 (1 - rt)\, u(t) dt dr \tag{4.4}$$

with initial condition $u(0) = 2$. *The exact solution is* $u(x) = 2 + 6x^2$. *writing in the integral form*

$$u(x) = 2 + \int_0^x \left(2 + 6t^2\right) dt + \int_0^x \left[\int_0^t \int_0^1 (1 - rs)\, u(s)\, ds dr\right] dt \tag{4.5}$$

hence, $g(x) = 2 + \int_0^x \left(2 + 6t^2\right) dt,\ M = 1,\ P_1(x) = 1,\ P_0(x) = 0.$ *using* $N = 2,$ *we obtain* $u_2(x_i) = \sum_{n=0}^{2} a_n x_i^n = \begin{bmatrix} a_0 \\ a_0 + \frac{a_1}{2} + \frac{a_2}{4} \\ a_0 + a_1 + a_2 \end{bmatrix},\ \sum_{n=0}^{2}\sum_{i=0}^{2}\sum_{j=0}^{2} Q_n(i,j)\, x_i^{M+i+1} = \begin{bmatrix} 0 \\ \frac{11a_0}{96} + \frac{a_1}{18} + \frac{7a_2}{192} \\ \frac{5a_0}{12} + \frac{7a_1}{36} + \frac{a_2}{8} \end{bmatrix},\ G(x_i) = \begin{bmatrix} 2 \\ \frac{293}{96} \\ \frac{77}{12} \end{bmatrix},$ *the unknown constants give* $A = \begin{bmatrix} 2 & 0 & 6 \end{bmatrix}^T,$ *hence* $u_3(x) = 2 + 6x^2,$ *which is the exact result*

Example 4.4. *[23] Consider the Volterra-Fredholm integro-differential equation*

$$u'(x) = e^x (1 + x) - \frac{1}{2}x^2 + \int_0^x \int_0^1 rtu(t) dt dr \tag{4.6}$$

with initial condition $u(0) = 2$. *The exact solution is* $u(x) = xe^x$. *writing in the integral form*

$$u(x) = 2 + \int_0^x \left(2 + 6t^2\right) dt + \int_0^x \left[\int_0^t \int_0^1 (1 - rs)\, u(s)\, ds dr\right] dt \tag{4.7}$$

hence, $g(x) = 2 + \int_0^x \left(e^t (1 + t) - \frac{1}{2}t^2\right) dt,\ M = 1,\ P_1(x) = 1,\ P_0(x) = 0.$ *using* $N = 3,$ *we obtain* $u_3(x_i) = \sum_{n=0}^{3} a_n x_i^n = \begin{bmatrix} a_0 \\ a_0 + \frac{a_1}{3} + \frac{a_2}{9} + \frac{a_3}{27} \\ a_0 + \frac{2a_1}{3} + \frac{4a_2}{9} + \frac{8a_3}{27} \\ a_0 + a_1 + a_2 + a_3 \end{bmatrix},\ \sum_{n=0}^{3}\sum_{i=0}^{3}\sum_{j=0}^{3} Q_n(i,j)\, x_i^{M+i+1}$

$= \begin{bmatrix} 0 \\ \frac{a_0}{160} + \frac{a_1}{324} + \frac{a_2}{486} + \frac{a_3}{648} \\ \frac{4a_0}{81} + \frac{2a_1}{81} + \frac{4a_2}{243} + \frac{a_3}{81} \\ \frac{a_0}{6} + \frac{a_1}{12} + \frac{a_2}{18} + \frac{a_3}{24} \end{bmatrix},\ G(x_i) = \begin{bmatrix} 0 \\ \frac{e^{\frac{1}{3}}}{3}\frac{1}{162} \\ \frac{2e^{\frac{2}{3}}}{3}\frac{4}{81} \\ e^1 - \frac{1}{6} \end{bmatrix},$ the unknown constants give $A = \begin{bmatrix} 0 & \frac{1252}{1179} & \frac{629}{934} & \frac{4019}{4088} \end{bmatrix}^T,$

$$u_3(x) = 0.983121300496991x^3 + 0.673447076444735x^2 + 1.06191698055329x$$

The following results are obtained

$$\begin{aligned} u_6(x) &= 0.0150199988169183x^6 + 0.0314654263513296x^5 + 0.174133468928364x^4 \\ &+0.497180210361634x^3 + 1.00051830958536x^2 + 0.999964415982862x \end{aligned}$$

$$\begin{aligned} u_8(x) &= 0.000349815507932986x^8 + 0.00107449758563351x^7 + 0.00867451996938073x^6 \\ &+0.0414501205943585x^5 + 0.166749132085284x^4 + 0.499981704086224x^3 \\ &+1.00000213741656x^2 + 0.999999901215456x \end{aligned}$$

$$\begin{aligned} u_{10}(x) &= 0.00000479432422640396x^{10} + 0.0000194529645571553x^9 + 0.000206093785655605x^8 \\ &+0.0013820388187085x^7 + 0.00833731558100826x^6 + 0.0416651407793949x^5 \\ &+0.166667045301502x^4 + 0.499999942234779x^3 + 1.00000000483478x^2 \\ &+0.99999999983444x \end{aligned}$$

Table 3: Exact solution and numerical solution for (4.7)

	Exact	Present method		
x_i	xe^x	$N=6$	$N=8$	$N=10$
0.2	0.244280551632034	0.244280701029522	0.244280551840745	0.244280551632034
0.4	0.596729879056508	0.596729774075505	0.59672987909635	0.596729879056509
0.6	1.09327128023431	1.09327138866956	1.09327128019408	1.09327128023431
0.8	1.78043274279397	1.7804325809758	1.78043274257189	1.78043274279398
1.0	2.71828182845905	2.71828183002647	2.71828182846083	2.71828182845905

Table 4: Errors for (4.7)

x_i	Err_6	Err_8	Err_{10}	$\triangle_8$	$\triangle_{10}$
0.2	1.4940e-07	2.0873e-10	2.8422e-14	1.4919e-07	2.0871e-10
0.4	1.0498e-07	3.9790e-11	0.0000	1.0502e-07	3.9841e-11
0.6	1.0844e-07	4.0245e-11	0.0000	1.0848e-07	4.0230e-11
0.8	1.6182e-07	2.2192e-10	0.0000	1.616e-07	2.2209e-10
1.0	1.5675e-09	1.8190e-12	0.0000	1.5656e-9	1.7799e-12

The results affirm the convergence of the method with good accuracy.

Example 4.5. *[23] Consider the Volterra-Fredholm integro-differential equation*

$$u'(x) = 6 + 29x - \frac{7}{2}x^2 + \int_0^x \int_0^1 (r-t)\, u(t) dt dr \tag{4.8}$$

with initial condition $u(0)=0$. *The exact solution is* $u(x) = 6x + 12x^2$. *writing in the integral form*

$$u(x) = \int_0^x \left(6 + 29t - \frac{7}{2}t^2\right) dt + \int_0^x \left[\int_0^t \int_0^1 (r-s)\, u(s)\, ds dr\right] dt \tag{4.9}$$

hence, $g(x) = \int_0^x \left(6 + 29t - \frac{7}{2}t^2\right) dt$, $M = 1$, $P_1(x) = 1$, $P_0(x) = 0$. *using* $N = 2$, *we obtain* $u_2(x_i) = \sum_{n=0}^{2} a_n x_i^n = \begin{bmatrix} a_0 \\ a_0 + \frac{a_1}{2} + \frac{a_2}{4} \\ a_0 + a_1 + a_2 \end{bmatrix}$, $\sum_{n=0}^{2}\sum_{i=0}^{2}\sum_{j=0}^{2} Q_n(i,j)\, x_i^{M+i+1} = \begin{bmatrix} 0 \\ -\frac{a_0}{24} - \frac{a_1}{32} - \frac{7a_2}{288} \\ -\frac{a_0}{12} - \frac{a_1}{12}\frac{5a_2}{72} \end{bmatrix}$, $G(x_i) = \begin{bmatrix} 2 \\ \frac{311}{48} \\ \frac{58}{3} \end{bmatrix}$, *the unknown constants give* $A = \begin{bmatrix} 0 & 6 & 12 \end{bmatrix}^T$, *hence* $u_3(x) = 6x + 12x^2$, *which is the exact result*

5 Conclusion

The numerical method for the solution of one dimensional mixed integro differential equations has been presented and discussed. The implementation of this method is carried out with the aid of a MATLAB code, this makes it easy and flexible. The results show that the method is accurate with fast convergence; exact solution is almost reached at $N = 10$.

Conflict of interest

The authors declare that there are no conflict of interest regarding the publication of this paper

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