Journal of Natural Sciences, Engineering and Technology

FOUR STEPS IMPLICIT METHOD FOR THE SOLUTION OF GENERAL SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

Four steps implicit scheme for the solution of second order ordinary differential equation was derived through interpolation and collocation method. Newton polynomial approximation method was used to generate the unknown parameters in the corrector. The method was tested with numerical examples and it was found to be efficient in solving second order ordinary differential equations.

Keyword: collocation, interpolation, continuous, implicit, Newton's polynomial, corrector.

INTRODUCTION

The second order initial value ordinary differential equation of the form

$$y'' = f(x, y, y') \quad , \quad y(a) = \eta_0, y'(a) = \eta_1 \qquad (1)$$

is considered in this paper. Differential equation is often used to model scientific and technological problems which most times, these equations do not have analytic solution, hence an approximate numerical method is required to solve these problems. Block methods for numerical solution of ordinary differential equations have been proposed by several researchers such as Milne (1953), Shampine and Watt (1969), Worland (1976) and Omar (1999). Rosser (1967) introduced the 3 point implicit block method based on integration formulae which is basically Newton's cote type. Zanariah *et al.* (2006) proposed 3 points implicit block method based on Newton's backward divided difference formula. Adesanya *et al*. (2009) proposed a two step method for the general solution of second order which is self stating and adopted Newton's polynomial to generate the starting value. Awoyemi *et al.* (2009) recently proposed a self starting Numerov's method. This method solves both initial and boundary value problems of ordinary differential equation. Yahaya (2007) constructed a Numerov method from a quadratic continuous polynomial solution. This process led to method which is applied to both initial and boundary value problems.

In this research work, we propose a block method for step length of four. This method adopts Newton's polynomial approximation to generate the starting value and solves general second order ordinary differential equation directly.

METHODOLOGY

We consider an approximate solution to (1) in power series

$$y(x) = \sum_{j=0}^{k} a_j \phi_j(x) \qquad (2.2)$$

$\phi_j = \phi^j, a_j, j = 0(1)2k-1$ are constants to be determined. We Consider a linear multistep method of the form

$$y(x) = \sum_{r=0}^{t-1} \phi_r(x) y_{n+r} + h^2 \sum_{r=0}^{m-1} \varphi_r(x) f_{n+r} \qquad (2)$$

where $x = [x_n, x_{n+r}]$, m=the distinct collocation point, t is the interpolation point, for our method, the step length k=4

$$\phi_r(x) = \sum_{i=0}^{i+m-1} \phi_{i+1,r} p_i(x), \quad r = 0,1,2,....,m-1 \qquad (3)$$

$$\varphi(x) = \sum_{i=0}^{t+m-1} \varphi_{i+1,r}(x) p_i(x), \quad \text{r=0, 1, 2…, m-1} \qquad (4)$$

$$y(x_{n+r}) = y_{n+r}, \quad r \in [0,1,2...,t-1] \quad y''(x) = f_{n+r}, \quad \text{r=0, 1, m-1} \qquad (5)$$

To get $\phi_j(x)$ and $\varphi_j(x)$, According to Yahaya (2007), he derived a matrix of the form

$$DC = I$$

where: I is an identity matrix of dimension $(t+m)\times(t+m)$.

$$D = \begin{pmatrix} 1 & x_n & x_n^{\ 2} & . & x_n^{t+i-1} \\ 1 & x_{n+1} & x_{n+1}^{\ 2} & . & x_{n+1}^{t+i-1} \\ . & . & . & . & . \\ 1 & x_{n+t-1} & x_{n+t-1}^{\ 2} & . & x_{n+t+1}^{t+m-1} \\ 0 & 0 & 2 & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 2 & . & (t+m-1)(t+m-2)x_{m-2}^{t+m-2} \end{pmatrix}. \qquad (6)$$

$$c = \begin{pmatrix} \alpha_{1,0} & \alpha_{1,1} & . & \alpha_{1,t-1} & h^2\varphi_{1,0} & . & h^2\varphi_{1,m-1} \\ \alpha_{2,0} & \alpha_{2,1} & . & \alpha_{2,t-1} & h^2\varphi_{2,0} & . & h^2\varphi_{2,m-1} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \alpha_{t+m,0} & \alpha_{t+m,1} & . & \alpha_{t+m,t-1} & h^2\varphi_{t+m,0} & . & h^2\varphi_{t+m,m-1} \end{pmatrix} \quad (7)$$

Development value for the unknown

Theorem (3.0):

Assuming that $f \in c^{n+1}[a,b]$ and $x_k \in [a,b]$ for k=0, 1, n are distinct values, then $f(x) = y(x) + R_n(x)$, where $y(x)$ is a polynomial that can be used to approximate $f(x)$

For Newton's polynomial

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + ... + a_n(x - x_0)(x - x_1)...(x - x_{n-1}) \quad (8)$$

$f(x) \cong y(x)$, $R_n(x)$ is the remainder and has the form

$$R_n(x) = \frac{f^{n+1}}{(x_{n+1})!}(x - x_0)(x - x_1)....(x - x_{n-1})(x - x_n) \quad (9)$$

(Awoyemi *et al.,*2009)

Development of four steps method

In developing the method with step length k=4, we consider

$$D = \begin{bmatrix} 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 \\ 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 \\ 0 & 0 & 2 & 6x_{n+4} & 12x_{n+4}^2 & 20x_{n+4}^3 & 30x_{n+4}^4 \end{bmatrix} \quad (10)$$

This gives a continuous scheme

$$\alpha_2 = -t$$

$$\alpha_3 = (t+1)$$

$$\beta_0(t) = \frac{h^2}{1440}\left(2t^6 + 6t^5 - 5t^4 - 20t^3 + 11t\right)$$

$$\beta_1(t) = \frac{h^2}{360}\left(-2t^6 - 9t^5 + 5t^4 + 30t^3 - 18t\right)$$

$$\beta_2(t) = \frac{h^2}{240}\left(2t^6 + 12t^5 + 5t^4 - 60t^3 + 55t\right)$$

$$\beta_3(t) = \frac{h^2}{360}\left(-2t^6 - 15t^5 - 25t^4 + 50t^3 + 180t^2 + 118t\right)$$

$$\beta_4(t) = \frac{h^2}{1440}\left(2t^6 + 18t^5 + 55t^4 + 60t^3 - 21t\right) \qquad (11)$$

where: $t = \dfrac{x - x_{n+3}}{h}$

Evaluating (11) at $x = x_{n+4}$ i.e. when $t = 1, -2, -3$, gives

$$240y_{n+4} - 480y_{n+3} + 240y_{n+2} = h^2(19f_{n+4} + 204f_{n+3} + 14f_{n+2} + 4f_{n+1} - f_n) \qquad (12)$$

$$240y_{n+3} - 480y_{n+2} + 240y_{n+1} = h^2(-f_{n+4} + 24f_{n+3} + 194f_{n+2} + 24f_{n+1} - f_n) \qquad (13)$$

$$480y_{n+3} - 720y_{n+2} + 240y_{n+1} = h^2(-3f_{n+4} + 52f_{n+3} + 402f_{n+2} + 252f_{n+1} + 17f_n) \qquad (14)$$

Evaluating the first derivative of (4,1) at t = -3, gives

$$1440hy_n' - 1440y_{n+3} + 1440y_{n+2} = h^2(33f_{n+4} - 284f_{n+3} - 966f_{n+2} - 1908f_{n+1} - 475f_n) \qquad (15)$$

Solving (12)-(16) using block method gives

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\\y_{n+4}\end{bmatrix}+\begin{bmatrix}0&0&0&-1\\0&0&0&-1\\0&0&0&-1\\0&0&0&-1\end{bmatrix}\begin{bmatrix}y_{n-1}\\y_{n-2}\\y_{n-3}\\y_n\end{bmatrix}=h^2\begin{bmatrix}\frac{3}{8}&\frac{-47}{240}&\frac{29}{360}&\frac{-7}{480}\\\frac{8}{5}&\frac{-1}{3}&\frac{8}{45}&\frac{-1}{30}\\\frac{117}{40}&\frac{27}{80}&\frac{3}{8}&\frac{-9}{160}\\\frac{49}{12}&\frac{191}{120}&\frac{577}{45}&\frac{57}{80}\end{bmatrix}\begin{bmatrix}f_{n+1}\\f_{n+2}\\f_{n+3}\\f_{n+4}\end{bmatrix}+$$

$$h^2\begin{bmatrix}0&0&0&\frac{367}{1440}\\0&0&0&\frac{53}{90}\\0&0&0&\frac{147}{160}\\0&0&0&\frac{869}{720}\end{bmatrix}\begin{bmatrix}f_{n-1}\\f_{n-2}\\f_{n-3}\\f_n\end{bmatrix}+h\begin{bmatrix}0&0&0&1\\0&0&0&2\\0&0&0&3\\0&0&0&4\end{bmatrix}\begin{bmatrix}y'_{n-1}\\y'_{n-2}\\y'_{n-3}\\y'_n\end{bmatrix} \tag{16}$$

Hence, from (16)

$$y_{n+1}=y_n+\frac{h^2}{1440}\left(540f_{n+1}+282f_{n+2}+116f_{n+3}-21f_{n+4}+367f\right)+hy'_n \tag{17}$$

$$y_{n+2}=y_n+\frac{h^2}{90}\left(144f_{n+1}-30f_{n+2}+16f_{n+3}+3f_{n+4}+53f\right)+2hy'_n \tag{18}$$

$$y_{n+3}=y_n+\frac{h^2}{160}\left(468f_{n+1}+54f_{n+2}+60f_{n+3}-9f_{n+4}+147f_n\right)+3hy'_n \tag{19}$$

$$y_{n+4}=y_n+\frac{h^2}{45}\left(192f_{n+1}+48f_{n+2}+64f_{n+3}+56f_n\right)+4hy'_n \tag{20}$$

Developing the unknown for k=4

We evaluate the first derivative of (8), and neglecting a_6 and higher values.

$$60hy_n' = 12y_{n+5} - 75y_{n+4} + 120y_{n+3} - 220y_{n+2} + 300y_{n+1} - 137y_n \quad (21)$$

$$60hy_{n+1}' = -3y_{n+5} + 20y_{n+4} - 60y_{n+3} + 120y_{n+2} - 65y_{n+1} - 12y_n \quad (22)$$

$$60hy_{n+2}' = 2y_{n+5} - 15y_{n+4} + 60y_{n+3} - 20y_{n+2} + 30y_{n+1} + 3y_n \quad (23)$$

$$60hy_{n+3}' = -3y_{n+5} + 30y_{n+4} + 20y_{n+3} - 60y_{n+2} + 15y_{n+1} - 2y_n$$

$$60hy_{n+4}' = 15y_{n+5} + 65y_{n+4} - 120y_{n+3} + 60y_{n+2} - 20y_{n+1} + 3y_n \quad (24)$$

Solving (24)-(28) for y_{n+i}, i = 1(1)4 gives,

$$y_{n+1} = y_n + \frac{h}{6348}(5386y_{n+1}' - 420y_{n+2}' + 2382y_{n+3}' - 247y_{n+4}' + 2259y_n') \quad (25)$$

$$y_{n+2} = y_n + \frac{h}{529}(705y_{n+1}' + 288y_{n+2}' + 135y_{n+3}' - 12y_{n+4}' + 174y_n') \quad (26)$$

$$y_{n+3} = y_n + \frac{h}{2116}(2598y_{n+1}' + 2520y_{n+2}' + 1578y_{n+3}' - 105y_{n+4}' + 729y_n') \quad (27)$$

$$y_{n+4} = y_n + \frac{h}{1587}(2188y_{n+1}' + 1272y_{n+2}' + 2580y_{n+3}' + 476y_{n+4}' + 504y_n') \quad (28)$$

Substituting for y_{n+2} in (15) and substituting in (16)-(19) and solving for f_{n+i}, $i = 1(1)4$ gives,

$$f_{n+1} = \frac{-1}{4}f_n + \frac{1}{h}(\frac{1}{6}y_{n+1}' + \frac{3}{4}y_{n+2}' - \frac{1}{6}y_{n+3}' + \frac{1}{48}y_{n+4}' - \frac{37}{48}y_n') \quad (29)$$

$$f_{n+2} = \frac{1}{6}f_n + \frac{1}{h}(\frac{-4}{3}y_{n+1}' + \frac{1}{2}y_{n+2}' + \frac{4}{9}y_{n+3}' - \frac{1}{24}y_{n+4}' + \frac{31}{72}y_n') \quad (30)$$

$$f_{n+3} = \frac{-1}{41} f_n + \frac{1}{h}\left(\frac{3}{2} y'_{n+1} - \frac{9}{4} y'_{n+2} + \frac{7}{6} y'_{n+3} + \frac{3}{16} y'_{n+4} - \frac{29}{48} y'_n\right) \quad (31)$$

$$f_{n+4} = f_n + \frac{1}{h}\left(\frac{-16}{3} y'_{n+1} + 6y'_{n+2} - \frac{16}{3} y'_{n+3} + \frac{7}{3} y'_{n+4} + \frac{7}{3} y'_n\right) \quad (32)$$

Substituting (25)-(28) into the second derivative of (8) and comparing with (29)-(32) give

$$y'_{n+1} = y'_n - 0.222193 h f_n \quad (33)$$

$$y'_{n+2} = y'_n + 0.111768 h f_n \quad (34)$$

$$y'_{n+3} = y'_n + 1.188485 h f_n \quad (35)$$

$$y'_{n+4} = y'_n + 1.311393 h f_n \quad (36)$$

Analysis of the scheme

Convergence

The convergence analysis of this scheme is determined using the approach of Fatunla (1992), where each block integrator is represented as a single step block r- point multistep method of the form

$$A^0 Y_m = \sum_{i=1}^{k} A^{(i)} Y_{m-1} h^2 B^i F_{m-1}$$

where: $A^{(i)}, B^{(i)}, i = 0(1)k$ are r by r matrix respectively with element

$a^i{}_{i,j}, b^i{}_{i,j}\ for\ i,j = 1(1)r.$ specifically, A^0 is an $r \times r$ identity matrix,

$Y_m, Y_{m-1}, F_m\ and\ F_{m-1}$ are vectors of numerical estimate describe below. With r-vector

$Y_m\ and\ F_m\ (for\ n = mr, m = 0,1, \dots\dots)$

$$Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ \cdot \\ \cdot \\ y_{n+r} \end{bmatrix}, F_m = \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ \cdot \\ \cdot \\ f_{n+r} \end{bmatrix}, Y_{m-i} = \begin{bmatrix} y_{n-r} \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix}, F_{m-1} = \begin{bmatrix} f_{n-r} \\ \vdots \\ f_{n-1} \\ f_n \end{bmatrix}$$

Awoyemi (2009) for details

Analysis of the block

$$\det\left(\lambda\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) + \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix} = 0$$

$$\det\begin{pmatrix} \lambda & 0 & 0 & -1 \\ 0 & \lambda & 0 & -1 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & \lambda - 1 \end{pmatrix} = 0$$

$$\lambda^3(\lambda - 1) = 0$$

$$\lambda = 0,0,0,1$$

Hence the block is zero stable.

Numerical example

We test the efficiency of our scheme on linear and non linear second order differential equation.

Problem 1:

$$y'' - x(y')^2 = 0$$

$$y(0) = 1, y'(0) = \frac{1}{2}, h = 0.1/40$$

Exact solution $$y(x) = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$$

S/N	Expected solution	New method	error
1	1.00125000065104	1.001250000186	4.650D-10
2	1.00250000520835	1.0025000039971	1.211D-09
3	1.00375001757828	1.003750013474	4.030D-09
4	1.00500004166729	1.005000047982	6.314D-09
5	1.00625008138212	1.0062500803586	8.462D-09
6	1.00750014062974	1.00750002579038	1.148D-09
7	1.00875022331755	1.008750239662	1993D-09
8	1.01000033335333	1.0100000783829	2.550D-09
9	1.01125047464542	1.0112504890376	4.256D-09
10	1.01250065110271	1.0125006101014	4.100D-08

Problem 11

$y^{''} = y + xe^{3x}$

$$y(0) = \frac{-3}{32}, y^{'}(0) = \frac{-5}{32}, h = 0.1/40$$

Exact solution

$$y(x) = \frac{4x-3}{32\exp(-3x)}$$

S/N	Expected result y(x)	New method yn(x)	Error y(x)-yn(x)
1	-0.094140915761848	-0.0941409131568	2.61D-09
2	-0.094532404142338	-0.0945324074753	3.33D-09
3	-0.094924451608388	-0.0949244224215	2.92D-08
4	-0.095317044390700	-0.0953170247449	2.02D-08
5	-0.095710168480980	-0.0957101793552	1.08D-08
6	-0.096103809629113	-0.0961039982252	1.88D-07
7	-0.09649533403163	0.0964952920353	4.23D-08
8	-0.096892584872264	-0.0968923659413	2.18D-07
9	-0.097289689232184	-0.0972874625827	2.22D-06
10	-0.097683251173919	-0.0976830958236	1.55D-07

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(Manuscript received: 12th January, 2009; accepted: 22nd April, 2009)