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TWO STEPS BLOCK METHOD FOR THE SOLUTION OF GENERAL SECOND ORDER INITIAL VALUE PROBLEMS OF ORDINARY DIFFERENTIAL EQUATION

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ABSTRACT

In this paper, an implicit block linear multistep method for the solution of ordinary differential equation was extended to the general form of differential equation. This method is self starting and does not need a predictor to solve for the unknown in the corrector. The method can also be extended to boundary value problems without additional cost. The method was found to be efficient after being tested with numerical problems of second order.

Keywords: Implicit, linear Multistep, Predictor, Corrector, Block Method.

INTRODUCTION

Due to remarkable improvement in computer technology, it was found necessary to develop and improve the existing method of numerical solution of general second order initial value problems which is given as:

$$y'' = f(x, y(x), y'(x)),\quad y(a) = \tau_0,\quad y'(a) = \tau_1 \tag{1}$$

Problem (1) is of interest because of its application in satellite tracking/warning systems, celestial mechanics, mass action kinetics, solar systems, molecular biology and spatial discretization of hyperbolic partial differential equation (Aladeselu, 2007).

Many scholars have done remarkable work in this area; the commonest among them is the reduction of the system to a first order equation and solving by an existing method suitable for first order equation. (Awoyemi, 1999; 1996). This method requires the development of separate sub-program for the starting values and functions arising from the system. The method requires too much time and effort in developing the computer programs. Awoyemi (1999), proposed a continuous scheme based on collocation which was found to have the following advantages:

i. it provides better error estimate
ii. it can be used for further analytical work in simpler forms than the discrete ones.
iii. it provides approximation at all interior points of the interval of consideration.

Above all, the main setback of the scheme proposed above is in the need to develop computer sub-programs needed to initialize the starting values hence, much time is lost and the cost of implementation is high.

The three factors to be considered in developing a numerical integrator are the following:
(i). Prudent management of time.
(ii). Cost of implementation
(iii). Effect as h (mesh size) decreases [4].

In solving these Adesalu (2007) proposed an improved family of block method for the special second order initial value problem in which 2 block-3 point numerical method was derived. The approach was found to be advantageous in many ways. Among this is that the continuous form can be used as interpolant for the computed numerical value for the dense output for analytical work at no extra cost of providing interpolant. Besides this, the continuous form can also be a big advantage in error control for choosing a step size adjustment strategy for the proposed block method.

(Yahaya, 2007) Constructed a Numerov method from a quadratic continuous polynomial solution. This process led to the Block method applied to both initial and boundary value problem for a more general second order problem.

In this paper, we want to extend the study of linear multistep method to the solution of the general second order initial value problem used in forming the block for calculating y_{n+1} and y_{n+2}, hy'_n will not be neglected as done in the special case of ordinary differential equation.

METHODOLOGY

We consider an approximate solution to (2) in power series

$$y(x) = \sum_{j=0}^{k} a_j \phi_j(x) \tag{3}$$

$\phi_j = \phi^j$ and a_j are constants to be determined. Consider a linear multistep method of the form

$$y(x) = \sum_{r=0}^{t-1} \phi_r(x) y_{n+r} + h^2 \sum_{r=0}^{m-1} \beta_r(x) f_{n+r} \tag{4}$$

where: $x = [x_n, x_{n+r}]$, k= step length, m=the distinct collocation point and t is the interpolation point. For our method, the step length k=2, while

$$\phi_r(x) = \sum_{i=0}^{i+m-1} \phi_{i+1,r}\, p_i(x) \quad , \quad r = 0,1,2,..,m-1 \tag{5}$$

$$h^2\beta(x) = \sum_{i=0}^{t+m-1} h^2\beta_{i+1,r}(x) p_i(x) \quad , \quad r = 0,1,2,..,m-1 \tag{6}$$

$$y(x_{n+r}) = y_{n+r} \;,\; r \in [0,1,2...,t-1] \;,\; y''(x) = f_{n+r}, \quad \text{r=0, 1, m-1} \tag{7}$$

To get $\phi_j(x)$ and $\varphi_j(x)$, According to Yahaya (2007), Onumanyi arrived at matrix of the form

DC = I

where: *I* is an identity matrix of dimension $(t+m)\times(t+m)$.

$$D = \begin{pmatrix} 1 & x_n & x_n{}^2 & . & x_n^{t+i-1} \\ 1 & x_{n+1} & x_{n+1}{}^2 & . & x_{n+1}^{t+i-1} \\ . & . & . & . & . \\ 1 & x_{n+t-1} & x_{n+t-1}{}^2 & . & x_{n+t+1}^{t+m-1} \\ 0 & 0 & 2 & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 2 & . & (t+m-1)(t+m-2)x_{m-2}^{t+m-2} \end{pmatrix}. \tag{8}$$

$$c = \begin{pmatrix} \alpha_{1,0} & \alpha_{1,1} & . & \alpha_{1,t-1} & h^2\varphi_{1,0} & . & h^2\varphi_{1,m-1} \\ \alpha_{2,0} & \alpha_{2,1} & . & \alpha_{2,t-1} & h^2\varphi_{2,0} & . & h^2\varphi_{2,m-1} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \alpha_{t+m,0} & \alpha_{t+m,1} & . & \alpha_{t+m,t-1} & h^2\varphi_{t+m,0} & . & h^2\varphi_{t+m,m-1} \end{pmatrix} \tag{9}$$

Development of the method

We note that when k=2, (4.4) reduces to

$$y(x) = \sum_{r=0}^{1} \phi_r y_{n+r}(x) + h^2 \sum_{r=0}^{2} \beta_r f_{n+r} \tag{10}$$

$$D = \begin{pmatrix} 1 & x_n & x_n^{\ 2} & x_n^{\ 3} & x_n^{\ 4} \\ 1 & x_{n+1} & x_{n+1}^{\ 2} & x_{n+1}^{\ 3} & x_{n+1}^{\ 4} \\ 0 & 0 & 2 & 6x_n & 12x_n^{\ 2} \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^{\ 2} \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^{\ 2} \end{pmatrix} \tag{11}$$

This gives a continuous scheme in which

$$\phi_0 = -(t-1)$$

$$\phi_1 = t$$

$$\beta_0 = \frac{h^2}{24}(t^4 - 6t^3 + 12t^2 - 7t)$$

$$\beta_1 = \frac{h^2}{12}(-t^4 + 4t^3 - 3t)$$

$$\beta_2 = \frac{h^2}{24}(t^4 - 2t^3 + t) \tag{12}$$

where: $t = \dfrac{x - x_n}{h}$

Substituting (12) in (10) and then evaluate at x_{n+2} i.e. when $t = 2$ gives

$$y_{n+2} - 2y_{n+1} + y_n = \frac{h^2}{12}(f_{n+2} + 10f_{n+1} + f_n) \tag{13}$$

(13) has order p=4 and error constant $cp^{+2} = -\dfrac{1}{240}$

Evaluating the first derivative of (12)

$$\phi_0' = -1$$

$$\phi_1' = -1$$

$$\beta_0' = \frac{h^2}{24}(4t^3 - 18t^2 + 24t - 7)$$

$$\beta_1' = \frac{h^2}{24}(-4t^3 + 12t^2 - 3)$$

$$\beta_0' = \frac{h^2}{24}(4t^3 - 6t^2 + 1) \tag{14}$$

Substituting (14) in (10) and then evaluate at t=0, 1 and 2, respectively, give

$$hy_n' - y_{n+1} + y_n = \frac{h^2}{24}(f_{n+2} - 6f_{n+1} - 7f_n) \tag{15}$$

(15) has order p=3 and error constant $cp^{+2} = -\frac{1}{45}$

$$hy_{n+1}' - y_{n+1} + y_n = \frac{h^2}{24}(-f_{n+2} + 10f_{n+1} + 3f_n) \tag{16}$$

(16) has order p=3 and error constant $cp^{+2} = \frac{7}{360}$

$$hy_{n+2}' - y_{n+1} + y_n = \frac{h^2}{24}(9f_{n+2} + 26f_{n+1} + f_n) \tag{17}$$

(17) has order p=3 and error constant $cp^{+2} = -\frac{1}{45}$

make $t = \frac{x_n - x}{h}$,

evaluate (14) when t=-1 and substituting in (10) gives

$$hy_{n+1}' - y_{n+1} + y_n = \frac{h^2}{12}\left(f_{n+2} - 2f_{n+1} + 13f_n\right) \tag{18}$$

Evaluate (14) when t=-2 and substitute in (10) gives

$$hy'_{n+2} - y_{n+1} + y_n = \frac{h^2}{4}\left(5f_{n+2} - 14f_{n+1} + 21f_n\right) \tag{19}$$

Solve (13), (15), (16) and (17) for y_{n+2} , y_{n+1} , y'_{n+2} and y'_{n+1} , give

$$y_{n+1} = y_n + hy'_n + h^2\left(\frac{1}{4}f_{n+1} - \frac{1}{24}f_{n+2} + \frac{7}{24}f_n\right) \tag{20}$$

$$y_{n+2} = y_n + 2hy'_n + h^2\left(\frac{2}{3}f_{n+1} + \frac{1}{3}f_n\right) \tag{21}$$

$$y'_{n+1} = y'_n + h^2\left(\frac{2}{3}f_{n+1} - \frac{1}{12}f_{n+2} + \frac{5}{12}f_n\right) \tag{22}$$

$$y'_{n+2} = y'_n + h^2\left(\frac{4}{3}f_{n+1} + \frac{1}{3}f_{n+2} + \frac{1}{3}f_n\right) \tag{23}$$

Solve (13), (15), (16) and (17) for y_{n+2} , y_{n+1} , f_{n+1} and f_{n+1} , give

$$y_{n+1} = y_n + \frac{1}{8}h^2 f_n + \frac{29}{48}hy'_n + \frac{5}{12}hy'_{n+1} - \frac{1}{48}hy'_{n+2} \tag{24}$$

$$y_{n+2} = y_n + \frac{1}{3}hy'_n + \frac{4}{3}hy'_{n+1} + \frac{1}{3}hy'_{n+2} \tag{25}$$

Solve (13), (15), (16), (17), (18) and (19) for y_{n+2} , y_{n+1} , y'_{n+1} and y'_{n+2} give

$$y_{n+1} = y_n + hy'_n + \frac{4}{9}h^2 f_n \tag{26}$$

$$y_{n+2} = y_n + 2hy_n' - \frac{10}{9}h^2 f_n \qquad (27)$$

$$y_{n+1}' = y_n' + \frac{2}{9}hf_n \qquad (28)$$

$$y_{n+2}' = y_n' - \frac{98}{9}hf_n \qquad (29)$$

Equations (26) and (27) give the starting value when the problem is a special case. Equations (28) and (29) give the starting value when the problem is a general second order ordinary differential equation. Equations (24) and (25) is used to predict the unknown parameters in the corrector when considering the special case. Equations (22) and (23) is used to predict the unknown parameters in the corrector when considering the general case. Equations (20) and (21) gives the actual value for two steps method.

Numerical Example

Problem I:

$$y'' - x(y')^2 = 0$$

$$y(0) = 1, y'(0) = \frac{1}{2}, h = 0.0025$$

Exact solution $y(x) = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$

Grid point	Expected result	Calculated result
0.0025	1.00125000065104	1.00125000065104
0.0050	1.00250000520835	1.002500000520835
0.0075	1.00375001757828	1.00375001757827
0.0100	1.00500004166729	1.00500004166729
0.0125	1.00625008138212	1.00625008138211
0.0150	1.00750014062974	1.00750014062974
0.0175	1.00875022331755	1.00875022331754
0.0200	1.01000033335333	1.0100003335332
0.0225	1.01125047464542	1.0112504746454
0.0250	1.01250065110271	0.01250065110269

Problem II

$$y'' = y + xe^{3x}$$

$$y(0) = \frac{-3}{32}, y'(0) = \frac{-5}{32}, h = 0.0025$$

$$y(x) = \frac{4x-3}{32\exp(-3x)}$$

Exact solution

Grid point	Expected result	Calculated result	Error
0.0025	-0.094140915761848	-0.0941409157619185	7.020D-14
0.0050	-0.094532404142338	-0.0945324041424606	1.217D-13
0.0075	-0.094924451608388	-0.094924451601872	3.396D-12
0.0100	-0.095317044390700	-0.095317044398913	8.122D-12
0.0125	-0.095710168480980	-0.0957101684955174	1.453D-11
0.0150	-0.096103809629113	-0.0961038096511447	2.233D-11
0.0175	-0.09649533403163	-0.096495333718854	3.156D-11
0.0200	-0.096892584872264	-0.0968925849144715	4.220D-11
0.0225	-0.097289689232184	-0.097286892863963	5.421D-11
0.0250	-0.097683251173919	-0.0976832512414662	6.754D-11

CONCLUSION

In this paper a two-step self starting implicit linear multistep method solution of general second order ordinary differential equation has been developed. The method is cheap and reliable. It encourages prudent management of time and is very efficient from the result of the numerical test displayed above. This method gives an encouraging result despite the low order of 1.5. The desirable property of a numerical solution is to behave like the theoretical solution of the problem which can be seen in the result above. It is therefore recommended to solve general type of second order ordinary differential equation.

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