Converse Variational Stability for Kurzweil Equations associated with Quantum Stochastic Differential Equations

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***Abstract*-- In analogous to classical ordinary differential equations, we study and establish results on converse variational stability of solution of quantum stochastic differential equations (QSDEs) associated with the Kurzweil equations. The results here generalize analogous results for classical initial value problems. The converse variational stability guaranteed the existence of a Lyapunov function when the solution is variationally stable.**

***Index Term*-- Non classical ODE; Kurzweil equations; Noncommutative stochastic processes, Converse Variational stability and Converse Asymptotic variational stability.**

1. Introduction

Results on kinds of variational stability of solution of the Kurzweil equations associated with quantum stochastic differential equations (QSDEs) have been established in [4]. In this work, we establish the converse variational stability of solution of equation of the results established in [4]. Because it is difficult to explicitly write the solution to the given equation, we employed Lyapunov's method [20] to establish results on converse variational stability of the trivial solution of the Kurzweil equations associated with QSDEs.

Lyapunov's method enables one to investigate stability of solution without explicitly solving the differential equation by making use of a real-valued function called the Lyapunov's function that satisfies some conditions such as positive definite, continuity, etc. Converse variational stability is more like a search for a Lyapunov's function [7-20]. It guarantees the existence of a Lyapunov function.

This paper is therefore devoted to the converse of results on variational stability established in [4], namely Theorems 5.3.3 and 5.3.4. The main goal here is to show that the variational stability and asymptotic variational stability imply the existence of Lyapunov functions with the properties described in Theorems 5.3.3 and 5.3.4, and hence strengthens our results on variational stability.

The rest of this paper is organized as follows. Section 2 will be devoted to some fundamental concepts, notations and structures of variational stability that are employed in subsequent sections. In sections, 3 we establish some concepts of converse variational stability within the context of QDES and the associated Kurzweil equation. In this same section we present some auxiliary results which will be used to establish the main results. Our main results will be established in section 4. We establish the main results on the converse of variational stability and asymptotic variational stability.

In what follows, as in [1, 2, 4] we employ the locally convex topological state space $\tilde{A}$ of noncommutative stochastic processes and we adopt the definitions and notations of the spaces $Ad(\tilde{A})$, $Ad(\tilde{A})_{wac}$, $L^{p}_{loc}(\tilde{A})$, $L^{\infty}_{loc}(\mathbb{R}_+)$, $BV(\tilde{A})$ and the integrator processes Λ_{Π}, A^{+}_{f}, A_g for $f, g \in L^{\infty}_{\gamma, loc}(\mathbb{R}_+)$, $\pi \in L^{\infty}_{B(\gamma), loc}(\mathbb{R}_+)$, and E, F, G, H lying in $Loc^{2}_{loc}(I \times \tilde{A})$.

We introduce the concept of converse variational stability of quantum stochastic differential equations driven by the Hudson - Parthasarathy [8] integrators $\Lambda_{\Pi}(t)$, $A^{+}_{f}(t)$, $A_g(t)$ given by

$$dX(t) = E(X(t), t)d\Lambda_{\Pi}(t) + F(X(t), t)dA^{+}_{f}(t) + G(X(t), t)dA_g(t) + H(X(t), t)dt$$
$$X(t_0) = X_0 , t \in [0; T] \qquad (1.1)$$

We shall consider the Kurzweil equation associated with the equivalent form of (1.1). As in the reference [4] solutions of (1.1) are $\tilde{A}$ - valued processes defined in [4]. For arbitrary , $\xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, the equivalent form of (1.1) is given by

$$\frac{d}{dt}\langle \eta, x(t)\xi \rangle = P(x,t)(\eta,\xi) \qquad (1.2)$$

Where the map $(x,t) \to P(x,t)\,(\eta,\xi)$ is as defined by equation (1. 4) in [4].

We employ the associated Kurzweil equation introduced in [4, 5] given by

$$\frac{d}{d\tau}\langle\eta, x(\tau)\xi\rangle = DF(x,t)(\eta,\xi) \tag{1.3}$$

Where

$$F(x, t)(\eta, \xi) = \int_0^t P(x,s)(\eta,\xi)ds \tag{1.4}$$

Next we present some fundamental concepts which we shall use in subsequent sections.

2. FUNDAMENTAL CONCEPTS AND DEFINITIONS OF VARIATIONAL STABILITY

In [4] it has been shown that the trivial process given by $X(s) \equiv 0$ for $s \in [0, T]$ is a solution of the Kurzweil equation (1.3).

Next we present some concepts of stability of the trivial solution $X(s) \equiv 0$, $s \in [0, T]$ of equation (1.3).

2.1 Definition: The trivial solution $X \equiv 0$ of equation (1.3) is said to be variationally stable if for every $\varepsilon > 0$, there exists $\delta(\eta, \xi, \varepsilon) := \delta_{\eta\xi} > 0$ such that if $Y : [0, T] \to \tilde{A}$ is a stochastic process lying in $Ad(\tilde{A})_{wac} \cap BV(\tilde{A})$ with

$$\|Y(0)\|_{\eta\xi} < \delta_{\eta\xi}$$

and

$$Var(\langle\eta, Y(s)\xi\rangle - \int_0^s DF(Y(\tau), t)(\eta,\xi)) < \delta_{\eta\xi}$$

then we have

$$\|Y(t)\|_{\eta\xi} < \varepsilon$$

For all $t \in [0, T]$ and for all $,\xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$.

2.2 Definition: The trivial solution $X \equiv 0$ of equation (1.3) is said to be variationally attracting if there exists $\delta_0 > 0$ and for every $\varepsilon > 0$, there exists $A = A(\varepsilon)$, $0 \le A(\varepsilon) < T$ and $B(\eta, \xi, \varepsilon) = B > 0$ such that if $Y \in Ad(\tilde{A})_{wac} \cap BV(\tilde{A})$ with $\|Y(0)\|_{\eta\xi} < \delta_0$ and

$$Var(\langle\eta, Y(s)\xi\rangle - \int_0^s DF(Y(\tau), t)(\eta,\xi)) < B$$

Then

$$\|Y(t)\|_{\eta\xi} < \varepsilon \quad \text{for all } t \in [A, T].$$

2.3 Definition: The trivial solution $X \equiv 0$ of equation (1.3) is called variationally asymptotically stable if it is variationally stable and variationally attracting.

Together with (1.1) we consider the perturbed QSDE

$$dX(t) = E(X(t), t)d\Lambda_\Pi(t) + F(X(t), t)dA_f^+(t) + G(X(t), t)dA_g(t) + (H(X(t), t) + p(t))dt$$

$$X(t_0) = X_0,\ t \in [0, T] \tag{2.1}$$

where $p \in Ad(\tilde{A})_{wac} \cap BV(\tilde{A})$. The perturbed equivalent form of (2.1) is given by

$$\frac{d}{dt}\langle\eta, x(t)\xi\rangle = P(x,t)(\eta,\xi) + \langle\eta, p(t)\xi\rangle \tag{2.2}$$

The Kurzweil equation associated with the perturbed QSDE (2.2) then becomes

$$\frac{d}{d\tau}\langle\eta, x(\tau)\xi\rangle = D[F(x,t)(\eta,\xi) + Q(t)(\eta,\xi)] \tag{2.3}$$

Where $Q : [0,T] \to \tilde{A}$ belongs to $Ad(\tilde{A})_{wac} \cap BV(\tilde{A})$ as well.

We remark here that the map given by equation (1.4) is of class $C(\tilde{A}\times[a, b], W)$ where

$$F(x, t)(\eta, \xi) + Q(t)(\eta, \xi) = \int_0^t [P(x,s)(\eta,\xi) + \langle\eta, p(s)\xi\rangle]ds$$

and

$$\langle \eta, p(s)\xi\rangle := Q(t)(\eta,\xi)$$

In [4], it has been shown that the right hand side $F(x,t)(\eta,\xi) + Q(t)(\eta,\xi)$ of equation (2.3) is of class $\mathcal{F}(\tilde{A} \times [a, b], *h_{\eta\xi}, W)$ where

$$*h_{\eta\xi}(t) = h_{\eta\xi}(t) + Var_{[0,T]}Q(t)(\eta,\xi)$$

and all fundamental results in [4] (e.g. the existence of solution) hold for equation (2.2) and hence (2.3).

2.4 Definition: The trivial solution X ≡ 0 of equation (1.3) is said to be variationally stable with respect to perturbations if for every ε > 0 there exists a $\delta = \delta_{\eta\xi} > 0$ such that if $\|Y_0\|_{\eta\xi} < \delta_{\eta\xi}$, $Y_0 \in \tilde{A}$ and the stochastic process Q belongs to the set $Ad(\tilde{A})_{wac} \cap BV(\tilde{A})$ such that

$$\mathrm{Var}(Q(t)(\eta,\xi)) < \delta\eta\xi, \text{ then } \|Y(t)\|_{\eta\xi} < \varepsilon$$

for t ∈ [0, T] where Y(t) is a solution of (2.3) with Y(0) = Y0.

2.5 Definition: The solution X ≡ 0 of (1.3) is called attracting with respect to perturbations if there exists $\delta_0 > 0$ and for every ε > 0, there is a

A = A(ε) ≥ 0 and B(η, ξ, ε) = B > 0 such that if

$$\|Y0\|_{\eta\xi} < \delta_0,\ Y_0 \in \tilde{A}$$

and $Q \in Ad(\tilde{A})_{wac} \cap BV(\tilde{A})$, satisfying $\mathrm{Var}(Q(t)(\eta,\xi)) < B$, then

$$\|Y(t)\|_{\eta\xi} < \varepsilon,$$

for all t ∈ [A, T], where Y(t) is a solution of (2.3).

2.6 Definition: The trivial solution X ≡ 0 of equation (1.3) is called asymptotically stable with respect to perturbations if it is stable and attracting with respect to perturbations.

2.7 Notation: Denote by $Ad(\tilde{A})_{wac} \cap BV(\tilde{A}) := A$ the set of all adapted stochastic processes $\varphi : [0; T] \to \tilde{A}$ that are weakly absolutely continuous and of bounded variation on [t0, T].

Next we establish some auxiliary results and definitions which we adopted from [4].

3. Auxiliary Results, Notations and Definitions

We introduce a modified notion of the variation of a stochastic process to suit the concept of converse variational stability.

3.1 Definition: Assume that $\Phi : [a, b] \to \tilde{A}$ is a given stochastic process. For a given decomposition

$$D : a = \alpha_0 < \alpha_1 < \ldots < \alpha_k = b$$

of the interval [a, b] ⊂ [0, T] and for every λ ≥ 0 define

$$u_\lambda(\Phi, D, \eta, \xi) = \sum_{j=1}^{k} e^{-\lambda(b-\alpha_{j-1})}\left\|\Phi(\alpha)_j - \Phi(\alpha)_{j-1}\right\|_{\eta\xi}$$

And set

$$e_\lambda Var_{[a,b]}\Phi_{\eta\xi} = Sup_D(\Phi, D, \eta, \xi)$$

where the supremum is taken over all decompositions D of the interval [a, b].

3.2 Definition: The number $e_\lambda Var_{[a,b]}\Phi_{\eta\xi}$ is called the e_λ-variation of the map $t \to \langle \eta, \Phi(t)\xi\rangle$ over the interval [a, b].

3.3 Definition: The real valued map $(x, t) \to V(x,t)(\eta,\xi)$ (x; t) is said to be positive definite if

(i) There exists a continuous nondecreasing function $b : [0, \infty) \to \mathbb{R}$ such that b(0) = 0 and

(ii) $V(x,t)(\eta,\xi) \geq b(\|x\|_{\eta\xi})$ for all $(x, t) \in \tilde{A} \times [0, T]$

(iii) $V(x,t)(\eta,\xi) = 0$, for all $(x, t) \in \tilde{A} \times [0, T]$.

3.1 Lemma: If -∞ < a < b < +∞ and $\Phi : [a, b] \to \tilde{A}$ is a stochastic process, then for every λ ≥ 0 we have

$$e^{-\lambda(b-a)}Var_{[a,b]}\Phi_{\eta\xi} \leq e_\lambda Var_{[a,b]}\Phi_{\eta\xi} \leq Var_{[a,b]}\Phi_{\eta\xi} \qquad (3.1)$$

If $a \le c \le b$, $\lambda \ge 0$ then the identity

$$e_\lambda Var_{[a,b]}\Phi_{\eta\xi} = e^{-\lambda(b-c)} e_\lambda Var_{[b,c]}\Phi_{\eta\xi} + e_\lambda Var_{[c,a]}\Phi_{\eta\xi} \tag{3.2}$$

holds.

Proof. For every $\lambda \ge 0$ and every decomposition D of [a, b] we have

$$e^{-\lambda(b-a)} \le e^{-\lambda(b-\alpha_{j-1})} \le e^0 = 1, j = 1, 2, \dots, k$$

Therefore

$$e^{-\lambda(b-a)} u_0(\Phi, D, \eta, \xi) \le u_\lambda(\Phi, D, \eta, \xi)$$
$$\le u_0(\Phi, D, \eta, \xi) = \sum_{j=1}^{k} \left|\Phi(\alpha_j)(\eta, \xi) - \Phi(\alpha_{j-1}(\eta, \xi)\right|$$

and passing to the supremum over all finite decomposition D of [a, b] we obtain the inequality (3.1)

$$e^{-\lambda(b-a)} Var_{[b,c]}\Phi_{\eta\xi} \le e_\lambda Var_{[b,c]}\Phi_{\eta\xi} \le Var_{[b,c]}\Phi_{\eta\xi}$$

The second statement can be established by restricting ourselves to the case of decomposition D which contain the point c as a node, i.e.

D : $a = \alpha_0 < \alpha_1 < \dots < \alpha_{l-1} < \alpha_l = c < \alpha_{l+1} < \dots < \alpha_k = b$

Then

$$u_\lambda(\Phi, D)(\eta, \xi) = \sum_{j=1}^{k} e^{-\lambda(b-\alpha_{j-1})} \left|\Phi(\alpha_j)(\eta, \xi) - \Phi(\alpha_{j-1}(\eta, \xi)\right|$$

$$= \sum_{j=1}^{l} e^{-\lambda(b-\alpha_{j-1})} \left|\Phi(\alpha_j)(\eta, \xi) - \Phi(\alpha_{j-1}(\eta, \xi)\right|$$

$$+ \sum_{j=l+k}^{k} e^{-\lambda(b-\alpha_{j-1})} \left|\Phi(\alpha_j)(\eta, \xi) - \Phi(\alpha_{j-1}(\eta, \xi)\right|$$

$$= e^{-\lambda(b-c)} \sum_{j=1}^{l} e^{-\lambda(b-\alpha_{j-1})} \left|\Phi(\alpha_j)(\eta, \xi) - \Phi(\alpha_{j-1}(\eta, \xi)\right|$$

$$+ \sum_{j=l+k}^{k} e^{-\lambda(b-\alpha_{j-1})} \left|\Phi(\alpha_j)(\eta, \xi) - \Phi(\alpha_{j-1}(\eta, \xi)\right|$$

$$= e^{-\lambda(b-c)} u_\lambda(\Phi, D_1, \eta, \xi) + u_\lambda(\Phi, D_2, \eta, \xi) \tag{3.3}$$

where

D1 : $a = \alpha_0 < \alpha_1 < \dots < \alpha_{l-1} < \alpha_l = c$

And

D_2 : $c = \alpha_l < \alpha_{l+1} < \dots < \alpha_k = b$

are decompositions of [a, c] and [c, b], respectively. On the other hand, any two such decompositions D1 and D2 form a decomposition D of the interval [a, b].

The equality

$$e_\lambda Var_{[a,b]}\Phi_{\eta\xi} \le e^{-\lambda(b-c)} e_\lambda Var_{[a,c]}\Phi_{\eta\xi} + e_\lambda Var_{[c,b]}\Phi_{\eta\xi}$$

now easily follows from (3.3) when we pass the corresponding suprema.

3.2 Corollary: Assume that the following hold.

(i) If $a \le c \le b$, and $\lambda \ge 0$ then

$$e_\lambda Var_{[a,c]}\Phi_{\eta\xi} \leq e_\lambda Var_{[a,b]}\Phi_{\eta\xi} \qquad (3.4)$$

(ii) Let (0) = 0, (t) = x and set $Sup_{s\in[a.t]}||\varphi(s)||_{\eta\xi} < a$, for a > 0, t > 0, $\varphi \in$ A.
(iii) For λ ≥ 0, s ≥ 0 and x ∈ Ã set

$$V(\lambda,\eta,\xi)(x,s) \coloneqq V_\lambda(x,s)(\eta,\xi) = \inf\{e_\lambda Var_{[0,s]}(\langle\eta,\varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi))\}$$

if s > 0 and

$$V_\lambda(x,s)(\eta,\xi) \coloneqq ||x||_{\eta\xi} \quad \text{if } s = 0 \qquad (3.5)$$

Note that the definition of $V_\lambda(x,s)(\eta,\xi)$ makes sense because for $\varphi \in$ A
the integral $\int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)$ is a function of bounded variation in the variable σ and therefore the function

$$\langle\eta,\varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)$$

is of bounded variation on [0, s] as well and the e_λ-variation of this function
is bounded. The trivial process $\varphi \equiv 0$ evidently belongs to A for x = 0 and therefore we
have

$$V_\lambda(0,s)(\eta,\xi) \qquad (3.6)$$

for every s ≥ 0 and λ ≥ 0 because

$$\langle\eta,\varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi) = 0$$

for σ > 0. Since

$$e_\lambda Var_{[0,s]}(\langle\eta,\varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)) \geq 0$$

for every $\varphi \in$ A, we have by the definition (3.5) also the inequality

$$|V_\lambda(x,s)(\eta,\xi)| \geq 0 \qquad (3.7)$$

for every s ≥ 0 and x ∈ Ã.
3.3 Lemma: For x, y ∈ Ã , s ∈ [0, T] and λ ≥ 0, the inequality

$$|V_\lambda(x,s)(\eta,\xi) - V_\lambda(y,s)(\eta,\xi)| \leq \left||x-y|\right|_{\eta\xi} \qquad (3.8)$$

holds.

Proof. Assume that s > 0 and 0 < β < s.

Let $\varphi \in$ A be arbitrary. Let $\varphi_\beta(\sigma) = \varphi(\sigma)$ for $\sigma \in$ [0, s-β], and set

$$\varphi_\beta(\sigma) = \varphi(\sigma-\beta) + \frac{1}{\beta}\big(y - \varphi(\sigma-\beta)\big)(\sigma - s - \beta)$$

For $\sigma \in [s-\beta, s]$.

The process φ_β coincides with φ on [0, s-β] and is linear with $\varphi_\beta(s) = y$ on $[s-\beta, s]$. By definition $\varphi_\beta \in A$ and by (3.2) from Lemma 3.1 we obtain

$$V_\lambda(y,s)(\eta,\xi) \leq e_\lambda Var_{[0,s]}(\langle\eta,\varphi_\beta(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi))$$

$$= e^{-\lambda\beta} e_\lambda Var_{[0,s-\beta]}(\langle\eta,\varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi))$$

$$+ e_\lambda Var_{[s-\beta,s]}(\langle\eta,\varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi))$$

$$\leq e_\lambda Var_{[0,s-\beta]}(\langle\eta,\varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)$$

$$+ Var_{[s-\beta,s]}(\langle\eta,\varphi(\sigma)\xi\rangle) - Var_{[s-\beta,s]}(\int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi))$$

$$\leq e_\lambda Var_{[0,s-\beta]}(\langle\eta,\varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)$$

$$+\|y-\varphi(s-\beta)\|_{\eta\xi} + h_{\eta\xi}(s) - h_{\eta\xi}(s-\beta), \varphi(s) = y$$

Since for every β > 0 we have

$$e^{-\lambda\beta} e_\lambda Var_{[0,s-\beta]}(\langle\eta,\varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi))$$

$$= e_\lambda Var_{[0,s]}(\langle\eta,\varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi))$$

$$- e_\lambda Var_{[s-\beta,s]}(\langle\eta,\varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi))$$

$$\leq e_\lambda Var_{[0,s]}(\langle\eta,\varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi))$$

by (3.6), we obtain for every β > 0 the inequality

$$V_\lambda(y,s)(\eta,\xi) \leq e_\lambda Var_{[0,s]}(\langle\eta,\varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi))$$

$$+\|y-\varphi(s-\beta)\|_{\eta\xi} + h_{\eta\xi}(s) - h_{\eta\xi}(s-\beta)$$

The function $h_{\eta\xi}$ is assumed continuous on [0, T] and the stochastic process φ is such that t ⟶ $\langle\eta,\varphi(\tau)\xi\rangle$ is continuous on [0, T] and therefore we have

$$\lim_{\tau\to s}\langle\eta,\varphi(\tau)\xi\rangle = \langle\eta,\varphi(s)\xi\rangle = \langle\eta,x\xi\rangle,$$

moreover the last inequality is valid for every β > 0 and consequently we can pass to the limit β⟶ 0 in order to obtain

$$V_\lambda(y,s)(\eta,\xi) \leq e_\lambda Var_{[0,s]}\left(\langle\eta,\varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right) + \|x-y\|_{\eta\xi}$$

for every $\varphi \in$ A. Taking the infimum for all $\varphi \in$ A on the right hand side of the last inequality we arrive at

$$V_\lambda(y,s)(\eta,\xi) \leq V_\lambda(x,s)(\eta,\xi) + \|x-y\|_{\eta\xi} \qquad (3.9)$$

Since this reasoning is fully symmetric with respect to x and y we similarly obtain also

$$V_\lambda(x,s)(\eta,\xi) \leq V_\lambda(y,s)(\eta,\xi) + \|x-y\|_{\eta\xi}$$

and this together with (3.9) yield (3.8) for s > 0.
If s = 0, then we have by definition

$$|V_\lambda(y,0)(\eta,\xi) - V_\lambda(x,0)(\eta,\xi)| = \left|\|y\|_{\eta\xi} - \|x\|_{\eta\xi}\right| \leq \|x-y\|_{\eta\xi}$$

this proves the Lemma.

3.4 Corollary: Since $V_\lambda(x,0)(\eta,\xi) = 0$ for every s > 0, we have by (3.6) and (3.8)

$$0 \leq V_\lambda(x,s)(\eta,\xi) \leq \|x\|_{\eta\xi} \qquad (3.10)$$

3.5 Lemma: For y ∈ Ã, s, r ∈ [0, T] and λ ≥ 0, the inequality

$$|V_\lambda(y,r)(\eta,\xi) - V_\lambda(y,s)(\eta,\xi)| \leq \left(1-e^{-\lambda|r-s|}\right)a + \left|h_{\eta\xi}(r) - h_{\eta\xi}(s)\right| \qquad (3.11)$$

holds.

Proof. Suppose that 0 ≤ s ≤ r and $\varphi \in$ A is given. Set $\|y\|_{\eta\xi} \leq a$. Then by Lemma 3.1 we have

$$e_\lambda Var_{[0,r]}\left(\langle\eta,\varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right)$$

$$= e^{-\lambda(r-s)} e_\lambda Var_{[0,s]}\left(\langle\eta,\varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right)$$

$$+ e_\lambda Var_{[s,r]}\left(\langle\eta,\varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right)$$

$$\geq e^{-\lambda(r-s)} V_\lambda(\varphi(s),s)(\eta,\xi) + e_\lambda Var_{[s,r]}\left(\langle\eta,\varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right)$$

$$\geq e^{-\lambda(r-s)}\left[V_\lambda(\varphi(s),s)(\eta,\xi) + Var_{[s,r]}\langle\eta,\varphi(\sigma)\xi\rangle - Var_{[s,r]}\left(\int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right)\right]$$

$$\geq e^{-\lambda(r-s)}[V_\lambda(\varphi(s),s)(\eta,\xi) + \|y-\varphi(s)\|_{\eta\xi} + h_{\eta\xi}(r) - h_{\eta\xi}(s)]$$

$$\geq e^{-\lambda(r-s)}\left[V_\lambda(y,s)(\eta,\xi) + h_{\eta\xi}(r) - h_{\eta\xi}(s)\right] \qquad (3.12)$$

The inequality (3.9) from Lemma 3.3 leads to

$$V_\lambda(\varphi(s),s)(\eta,\xi) + \|y-\varphi(s)\|_{\eta\xi} \geq V_\lambda(y,s)(\eta,\xi)$$

Taking the infimum over $\varphi \in$ A on the left hand side of (3.12) we have

$$V_\lambda(y,r)(\eta,\xi) \geq e^{-\lambda(r-s)}\left[V_\lambda(y,s)(\eta,\xi) + h_{\eta\xi}(r) - h_{\eta\xi}(s)\right]$$

$$\geq e^{-\lambda(r-s)}V_\lambda(y,s)(\eta,\xi) + \left(h_{\eta\xi}(r) - h_{\eta\xi}(s)\right) \qquad (3.13)$$

Now let $\varphi \in$ A be arbitrary. We define

$$\varphi^*(\sigma) = \begin{cases} \varphi(\sigma) & for\ \sigma \in [0,s], \\ y & for\ \sigma \in [s,r]. \end{cases}$$

We then have $\varphi^*(\sigma) = \varphi(s) = y,\ \varphi^* \in A$ and by (3.1), (3.6) we obtain

$$V_\lambda(y,\mathrm{r})(\eta,\xi) \leq e_\lambda Var_{[0,r]}\left(\langle\eta,\varphi^*(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi^*(\tau),t)(\eta,\xi)\right) =$$

$$= e^{-\lambda(r-s)}e_\lambda Var_{[0,s]}\left(\langle\eta,\varphi^*(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi^*(\tau),t)(\eta,\xi)\right) +$$

$$+e_\lambda Var_{[r,s]}\left(\langle\eta,\varphi^*(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi^*(\tau),t)(\eta,\xi)\right)$$

$$\leq e^{-\lambda(r-s)}e_\lambda Var_{[0,s]}\left(\langle\eta,\varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right) +$$

$$Var_{[s,r]}\langle\eta,\varphi^*(\sigma)\xi\rangle - Var_{[s,r]}\left(\int_0^\sigma DF(\varphi^*(\tau),t)(\eta,\xi)\right) \leq$$

$$= e^{-\lambda(r-s)}e_\lambda Var_{[0,s]}\left(\langle\eta,\varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right) + h_{\eta\xi}(r) - h_{\eta\xi}(s).$$

Taking the infimum over all $\varphi \in$ A on the right hand side of this inequality
we obtain

$$V_\lambda(y,\mathrm{r})(\eta,\xi) \leq e^{-\lambda(r-s)}V_\lambda(y,\mathrm{s})(\eta,\xi) + h_{\eta\xi}(r) - h_{\eta\xi}(s).$$

Together with (3.13) we have

$$|V_\lambda(y,\mathrm{r})(\eta,\xi) - e^{-\lambda(r-s)}V_\lambda(y,\mathrm{s})(\eta,\xi)| \leq h_{\eta\xi}(r) - h_{\eta\xi}(s)$$

Hence, by (3.10) we get the inequality

$$|V_\lambda(y,\mathrm{r})(\eta,\xi) - V_\lambda(y,s)(\eta,\xi)| \leq$$

$$|V_\lambda(y,\mathrm{r})(\eta,\xi) - e^{-\lambda(r-s)}V_\lambda(y,\mathrm{s})(\eta,\xi)| + |1 - e^{-\lambda(r-s)}|V_\lambda(y,\mathrm{s})(\eta,\xi) \leq$$

$$\leq |h_{\eta\xi}(r) - h_{\eta\xi}(s)| + (1 - e^{-\lambda(r-s)})\|\mathrm{y}\|_{\eta\xi} \leq$$

$$\leq |h_{\eta\xi}(r) - h_{\eta\xi}(s)| + (1 - e^{-\lambda(r-s)})\mathrm{a}$$

because $\|\mathrm{y}\|_{\eta\xi} \leq$ a . In this way we have obtained (3.11).
Assume that s = 0 and r > 0. Then by (3.10) and by the definition given in

(3.5) we get

$$V_\lambda(y,\mathrm{r})(\eta,\xi) - V_\lambda(y,\mathrm{s})(\eta,\xi) = V_\lambda(y,\mathrm{r})(\eta,\xi) - V_\lambda(y,0)(\eta,\xi) =$$

$$= V_\lambda(y,\mathrm{r})(\eta,\xi) - \|\mathrm{y}\|_{\eta\xi} \leq 0. \qquad (3.14)$$

We derive an estimate from below. Assume that $\varphi \in$ A. By (3.1) in Lemma 3.1 and Lemma 1.9.9 in [4], we have

$$e_\lambda Var_{[0,r]}\left(\langle \eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right)$$

$$\geq e_\lambda Var_{[0,r]}\langle \eta, \varphi(\sigma)\xi\rangle - e_\lambda Var_{[0,r]}\left(\int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right)$$

$$\geq e^{-\lambda r} Var_{[0,r]}\langle \eta, \varphi(\sigma)\xi\rangle - Var_{[0,r]}\left(\int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right)$$

$$\geq e^{-\lambda r}|\langle \eta, \varphi(r)\xi\rangle - \langle \eta, \varphi(0)\xi\rangle| - (h_{\eta\xi}(r) - h_{\eta\xi}(0))$$

$$= e^{-\lambda r}\|\mathrm{y}\|_{\eta\xi} - (h_{\eta\xi}(r) - h_{\eta\xi}(0))$$

By Lemma 3.1, Lemma 1.9.10 in [4] and (3.1). Passing again to the infimum for $\varphi \in$ A on the left hand side of this inequality we get

$$V_\lambda(y,\mathrm{r})(\eta,\xi) \geq e^{-\lambda r}\|\mathrm{y}\|_{\eta\xi} - (h_{\eta\xi}(r) - h_{\eta\xi}(s))$$

and

$$V_\lambda(y,\mathrm{r})(\eta,\xi) - V_\lambda(y,0)(\eta,\xi) = V_\lambda(y,\mathrm{r})(\eta,\xi) - \|\mathrm{y}\|_{\eta\xi}$$

$$\geq (e^{-\lambda r} - 1)\|\mathrm{y}\|_{\eta\xi} - (h_{\eta\xi}(r) - h_{\eta\xi}(0))$$

$$= (1 - e^{-\lambda r})\|\mathrm{y}\|_{\eta\xi} - (h_{\eta\xi}(r) - h_{\eta\xi}(0))$$

This together with (3.14) yields

$$|V_\lambda(y,\mathrm{r})(\eta,\xi) - V_\lambda(y,0)(\eta,\xi)| \leq (1 - e^{-\lambda r})\mathrm{a} - (h_{\eta\xi}(r) - h_{\eta\xi}(0))$$

and this means that the inequality (3.11) holds in this case too. The remaining case of r = s = 0 is evident because

$$|V_\lambda(y,\mathrm{r})(\eta,\xi) - V_\lambda(y,0)(\eta,\xi)| = 0 = (1 - e^{-\lambda r})\|\mathrm{y}\|_{\eta\xi} + (h_{\eta\xi}(r) - h_{\eta\xi}(0))$$

For the case when r < s we obtain

$$|V_\lambda(y,\mathrm{r})(\eta,\xi) - V_\lambda(y,\mathrm{s})(\eta,\xi)| \leq (1 - e^{-\lambda s})\|\mathrm{y}\|_{\eta\xi} + (h_{\eta\xi}(s) - h_{\eta\xi}(r)),$$

because the situation is symmetric in s and r. We have thus established results for the case when s ≥ 0, s and r.
By the previous Lemmas 3.1, 3.3 and (iii) of definition 3.3, we immediately conclude that the following holds.

3.6 Corollary: For x, y ∈ Ã, r, s ∈ [0,T] and λ ≥ 0 the inequality

$$|V_\lambda(x,r)(\eta,\xi) - V_\lambda(y,s)(\eta,\xi)| \le \|x-y\|_{\eta\xi}$$

$$+(1-e^{-\lambda|r-s|})a + |h_{\eta\xi}(r) - h_{\eta\xi}(s)| \qquad (3.15)$$

holds.

Next, we shall discuss the behaviour of the function $V_\lambda(x,r)(\eta,\xi)$ defined by (3.5) along the solutions of the Kurzweil equation

$$\frac{d}{d\tau}\langle \eta, x(\tau)\xi\rangle = DF(x,t)(\eta,\xi) \qquad (1.3)$$

We still assume that the assumptions given at the beginning of this section are satisfied for the right hand side *F(x, t)(η, ξ).* The next result will be employed in what follows.

3.7 Lemma: Assume that : [s, s + β(s)] → Ã is a solution of (1.3), s ≥ 0, β(s) > 0, then for every λ the inequality

$$\limsup_{\beta\to 0} \frac{V_\lambda(\psi(s+\beta), s+\beta)(\eta,\xi) + V_\lambda(\psi(s),s)(\eta,\xi)}{\beta} \le -\lambda V_\lambda(\psi(s),s)(\eta,\xi) \qquad (3.16)$$

Holds.

Proof: Let s ∈ [0, T] and x ∈ Ã be given. Let us choose a > 0 such that $a > \|x\|_{\eta\xi} + h_{\eta\xi}(1+s) - h_{\eta\xi}(s)$. Assume that φ ∈ A is given and let : [s, s + β(s)] → Ã be a solution of (1.3) on [s, s + β(s)] with ψ(s) = x where 0 < β(s) < 1. The existence of such a solution is guaranteed by the existence theorem in [4].

For 0 < β < β(s) define

$$\varphi_\beta(\sigma)(\eta,\xi) = \varphi(\sigma)(\eta,\xi) \text{ for } \sigma \in [0, s]$$

and

$$\varphi_\beta(\sigma)(\eta,\xi) = \psi(\sigma)(\eta,\xi) \text{ for } \sigma \in [s, s+\beta].$$

we have $\varphi(s) = \psi(s) = \varphi_\beta(s) = x.$

Then $\varphi_\beta \in A$, for β ∈ [s, s+β] and since ψ is weakly absolutely continuous and by the definition of a solution we have

$$|\langle \eta, \psi(\sigma)\xi\rangle| = |\langle \eta, x(s)\xi\rangle - \int_s^\sigma DF(\psi(\tau),t)(\eta,\xi)$$

$$\|x\|_{\eta\xi} + (h_{\eta\xi}(\sigma) - h_{\eta\xi}(s)) \le \|x\|_{\eta\xi} + (h_{\eta\xi}(1+s) - h_{\eta\xi}(s) \le a$$

For σ ∈ [s, s+β] and

$$V_\lambda(\psi(s+\beta), s+\beta)(\eta,\xi) \le$$

$$\le e_\lambda Var_{[0,s+\beta]}\left(\langle \eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right)$$

$$= e^{-\lambda\beta} e_\lambda Var_{[0,s]}\left(\langle \eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right)$$

$$+ e_\lambda Var_{[s,s+\beta]}\left(\langle \eta, \psi(\sigma)\xi\rangle - \int_0^s DF(\varphi(\tau),t)(\eta,\xi) \int_s^\sigma DF(\psi(\tau),t)(\eta,\xi)\right)$$

$$= e^{-\lambda\beta} e_\lambda Var_{[0,s]}\left(\langle \eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right)$$

$$+e_\lambda Var_{[s,s+\beta]}\left(\langle\eta, x\xi\rangle - \int_0^s DF(\varphi(\tau),t)(\eta,\xi)\right)$$

$$= e^{-\lambda\beta} e_\lambda Var_{[0,s]}\left(\langle\eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right)$$

Taking the infimum for all $\varphi \in$ A on the right hand side of this inequality
we obtain

$$V_\lambda(\psi(s+\beta), s+\beta)(\eta,\xi) \leq e^{-\lambda\beta} V_\lambda(x,s)(\eta,\xi) = e^{-\lambda\beta} V_\lambda(\psi(s),s)(\eta,\xi)$$

This inequality yields

$$V_\lambda(\psi(s+\beta), s+\beta)(\eta,\xi) - V_\lambda(\psi(s),s)(\eta,\xi) \leq (e^{-\lambda\beta}-1) V_\lambda(\psi(s),s)(\eta,\xi)$$

and also

$$\frac{V_\lambda(\psi(s+\beta), s+\beta)(\eta,\xi) + V_\lambda(\psi(s),s)(\eta,\xi)}{\beta} \leq \frac{e^{-\lambda\beta}-1}{\beta} V_\lambda(\psi(s),s)(\eta,\xi)$$

for every 0 < β < β(s).

Since $\lim_{\beta\to 0}\frac{e^{-\lambda\beta}-1}{\beta} \leq -\lambda$ we immediately obtain (3.16).

4. Converse Theorems

Now we establish the converse of Theorems 5.3.3 and 5.3.4 established in [4].

4.1 Theorem: Assume that the trivial solution x ≡ 0 of equation (1.3) is variationally stable then for every 0 < a < c, there exists a real-valued map $V(\mathrm{x}x, tt)(\eta\eta, \xi\xi)$ satisfying the following conditions:

(i) for every x ∈ Ã the function t → $(\mathrm{x}x, yt)(\eta\eta, \xi\xi)$ is of bounded variation in t and continuous in t,

(ii) $(0x, tt)(\eta\eta, \xi\xi)$ $V$and $|V(x,t)(\eta,\xi) - V(y,t)(\eta,\xi)| \leq \|x-y\|_{\eta\xi}$ for x, y ∈ Ã t ∈ [0, T],

(iii) the function $V(x,t)(\eta,\xi)$ is non-increasing along the solutions of the
equation (1.3),

(iv) the function $V(x,t)(\eta,\xi)$ is positive definite if there is a continuous
nondecreasing real-valued function b : [0,+∞) → R such that b(ρ) = 0
if and only if ρ = 0 and

$$b\left(\|x\|_{\eta\xi}\right) \leq V(x,t)(\eta,\xi).$$

for every x ∈ Ã, t ∈ [0, T].

Proof: The candidate for the function $V(x,t)(\eta,\xi)$ is the function $V_0(x,t)(\eta,\xi)$
defined by (3.5) in section 3.
For λ = 0, i.e. we take $V_\lambda(x,t)(\eta,\xi) = V_0(x,t)(\eta,\xi) = V(x,t)(\eta,\xi)$. Hypothesis (i) is established by Corollary 3.6. Hypothesis (ii) follow from (3.6) and from Lemma 3.3 i.e. The trivial process
x ≡ 0 evidently belongs to A for x = 0 and therefore we have

$$V(x,t)(\eta,\xi) = 0$$

for every s ≥ 0 and λ ≥ 0, because

$$\langle\eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi) = 0$$

for $\sigma > 0$. The inequality $|V(x,t)(\eta,\xi) - V(y,t)(\eta,\xi)| \leq \|x-y\|_{\eta\xi}$ follows from the proof of Lemma 3.3.

By Lemma 3.7 for every solution : [s, s + σ] → Ã of equation (1.3) we have

$$\lim_{\beta\to 0} sup \frac{V_\lambda(\psi(s+\beta), s+\beta)(\eta,\xi) + V_\lambda(\psi(s),s)(\eta,\xi)}{\beta} \leq 0$$

and therefore (iii) is also satisfied.
It remains to show that the function $V(x,t)(\eta,\xi)$ given in this way is positive definite. This is the only point where the variational stability of the solution $x \equiv 0$ of equation (1.3) is used.

Assume that there is an ε, 0 < ε < a and a sequence (x_k, t_k), k = 1, 2, . . . ,

$\varepsilon < \|x-y\|_{\eta\xi} < a$, $t_k \to \infty$ for $k \to \infty$ such that $V(x_k,t_k)(\eta,\xi) \to 0$ for $k \to \infty$. Let δ(ε) > 0 correspond to ε by Definition 2.4 of stability with respect to perturbations (the variational stability of $x \equiv 0$ is equivalent to the stability with respect to perturbations of this solution by Theorem 5.2.1 in [4]). Assume that k ∈ ℕ is such that for k > 0 we have $V(x_k,t_k)(\eta,\xi) < \delta(\varepsilon)$.Then there exists $\varphi_k \in A$ such that for every $t_k \in [0,T]$

$$Var_{[0,t_k]}\left(\langle \eta, \varphi_k(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi_k(\tau),t)(\eta,\xi)\right) < \delta(\varepsilon)$$

We set

$$\langle \eta, Q(\sigma)\xi\rangle = \langle \eta, \varphi_k(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi_k(\tau),t)(\eta,\xi) \text{ for } \sigma \in [0,t_k]$$

$$\langle \eta, Q(\sigma)\xi\rangle = \langle \eta, x_k(\sigma)\xi\rangle - \int_0^{t_k} DF(\varphi_k(\tau),t)(\eta,\xi) \text{ for } \sigma \in [t_k,T], t_k > 0$$

We then have

$$Var_{[0,T]}\langle \eta, Q(\sigma)\xi\rangle = Var_{[0,t_k]}\left(\langle \eta, \varphi_k(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi_k(\tau),t)(\eta,\xi)\right) < \delta(\varepsilon).$$

and the function $\langle \eta, Q(.)\xi\rangle$ is continuous on [0, T]. for $\sigma \in [0,t]$, we have

$$\langle \eta, \varphi_k(\sigma)\xi\rangle = \int_0^\sigma DF(\varphi_k(\tau),t)(\eta,\xi) + \langle \eta, \varphi_k(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi_k(\tau),t)(\eta,\xi)$$

$$= \int_0^\sigma DF(\varphi_k(\tau),t)(\eta,\xi) + \langle \eta, Q(\sigma)\xi\rangle - \langle \eta, Q(0)\xi\rangle$$

$$\langle \eta, \varphi_k(0)\xi\rangle - \int_0^\sigma D[F(\varphi_k(\tau),t)(\eta,\xi) + \langle \eta, Q(t)\xi\rangle]$$

because $\varphi_k(0) = 0$. Hence, φ_k is a solution of the equation

$$\frac{d}{d\tau}\langle \eta, y(\tau)\xi\rangle = D[F(y,t)(\eta,\xi) + \langle \eta, Q(t)\xi\rangle]$$

and therefore, by the variational stability we have $\|\varphi_k(s)\|_{\eta\xi} < \varepsilon$ for every $s \in [0, t_k]$. Hence we also have $\|\varphi_k(t_k)\|_{\eta\xi} = \|x_k\|_{\eta\xi} < \varepsilon$ and this contradicts our assumption. In this way we obtain that the function $V(x,t)(\eta,\xi)$ is positive definite and (iv) is also satisfied.
The next statement is the converse of Theorem 5.3.4 in [4] on variational asymptotic stability.

4.2 Theorem: Assume that the trivial solution x ≡ 0 of equation (1.3) is variationally asymptotically stable then for every 0 < a < c there exists a real-valued map
$U(x,t)(\eta,\xi)$: $\tilde{A} \times [0,T] \to \mathbb{R}$ satisfying the following conditions:

(i) For every x ∈ Ã the map $t \to U(x,t)(\eta,\xi)$ is continuous on [0, T] and is locally of bounded variation on [0, T],

(ii) $U(0,t)(\eta,\xi) = 0$ and

$|U(x,t)(\eta,\xi) - U(y,t)(\eta,\xi)| \le \|x-y\|_{\eta\xi}$ for x, y ∈ Ã, t ∈ [0, T],

(iii) For every solution (σ) of the equation (1.3) defined for $\sigma \ge$ t, where
(σ) = x ∈ Ã, the relation

$$\limsup_{\beta\to 0} \frac{U(\psi(t+\beta),t+\beta)(\eta,\xi) - U(x,t)(\eta,\xi)}{\beta} \le -U(x,t)(\eta,\xi)$$

holds,

(iv) the function $U(x,t)(\eta,\xi)$ is positive definite.

Proof: For x ∈ Ã, s ≥ 0 we set

$$U(x,s)(\eta,\xi) = V(x,s)(\eta,\xi)$$

Where $V_0(x,s)(\eta,\xi)$ is the function defined by (3.5) for λ = 1. In the same way as in the proof of Theorem 4.1 the map $U(x,s)(\eta,\xi)$ satisfies (i), (ii) and (iii). (The item (iii) is exactly the statement given in Lemma 3.7). It remains to show that (iv) is satisfied for this choice of the function $U(x,s)(\eta,\xi)$. Since the solution x ≡ 0 of equation (1.3) is assumed to be variationally attracting and by Theorem 5.2.1 in [4] it is also attracting with respect to perturbations and therefore there exists $\delta_0 > 0$ and for every ε > 0 there is a A = A(ε) ≥ 0 and B = B(ε) > 0 such that if $\|y_0\|_{\eta\xi} < \delta_0$, $y_0 \in \tilde{A}$ and
$Q \in BV(\tilde{A}) \cap (\tilde{A})_{wac}$ on $[t_0,t_1] \subset [0,T]$, and

$$Var_{[t_0,t_1]} Q < B(\varepsilon)$$

Then

$$\|y(t)\|_{\eta\xi} < \varepsilon$$

for all t ∈ [t0, t1] ∩ [t0 + A(ε), T] and t0 ≥ 0 where y(t) is a solution of

$$\frac{d}{d\tau}\langle \eta, y(\tau)\xi\rangle = D[F(y,t)(\eta,\xi) + \langle \eta, Q(t)\xi\rangle] \qquad (4.1)$$

with $\qquad y(t_0) = y_0$

Assume that the map U is not positive definite then there exists ε, 0 < ε < a = δ_0,
a > 0 and a sequence (x_k, t_k), k = 1, 2, . . . , assume also that $\varepsilon \le \|x_k\|_{\eta\xi} < a,\ t_k \to \infty$ for

t → ∞ such that $U(x_k,t_k) \to 0$ for t → ∞. Choose k0 ∈ ℕ such that for k ∈ ℕ , k > k0 we have

tk > A(ε) + 1 and

$$U(x_k,t_k)(\eta,\xi) < B(\varepsilon)e^{-(A(\varepsilon)+1)}, x_k \in \tilde{A}$$

According to the definition of the map U we choose $\varphi \subset$ A such that

$$e_1 Var_{[0,t_k]}\left(\langle \eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right) < B(\varepsilon)e^{-(A(\varepsilon)+1)}$$

Define t0 = tk − (A(ε) + 1). Then t0 > 0 because tk > A(ε) + 1 and also tk = t0 + A(ε) + 1 > t0 + A(ε).

Therefore,

$$e_1 Var_{[t_0,t_k]}\left(\langle \eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right) < B(\varepsilon)e^{-(\mathrm{A}(\varepsilon)+1)},$$

by inequality (3.1) in Lemma 3.1 also

$$e^{-(\mathrm{A}(\varepsilon)+1)} Var_{[t_0,t_k]}\left(\langle \eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right)$$

$$= e^{-(\mathrm{t_k}-\mathrm{t_0})} Var_{[t_0,t_k]}\left(\langle \eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right) < B(\varepsilon)e^{-(\mathrm{A}(\varepsilon)+1)}$$

and therefore, we get

$$Var_{[t_0,t_k]}\left(\langle \eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right) < B(\varepsilon). \quad (4.2)$$

For $\sigma \in$ [t0, tk] define

$$\langle \eta, Q(\sigma)\xi\rangle = \langle \eta, \varphi(t)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)$$

The function Q : [t0, tk] → Ã evidently lie in $BV(\tilde{\mathrm{A}}) \cap Ad(\tilde{\mathrm{A}})_{wac}$ and by the inequality (3.17) we have

$$Var_{[t_0,t_k]}\langle \eta, Q(\sigma)\xi\rangle = Var_{[t_0,t_k]}\left(\langle \eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)\right) < B(\varepsilon).$$

and

$$Var_{[t_0,t_k]}\langle \eta, Q(\sigma)\xi\rangle < B(\varepsilon).$$

Moreover,

$$\langle \eta, \varphi(\sigma)\xi\rangle = \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi) + \langle \eta, \varphi(\sigma)\xi\rangle - \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi)$$

$$= \int_0^\sigma DF(\varphi(\tau),t)(\eta,\xi) + \langle \eta, Q(\sigma)\xi\rangle$$

and also

$$\langle \eta, \varphi(s)\xi\rangle - \langle \eta, \varphi(t_0)\xi\rangle = \int_{t_0}^s DF(\varphi(\tau),t)(\eta,\xi) + \langle \eta, Q(s)\xi\rangle - \langle \eta, Q(t_0)\xi\rangle$$

$$= \int_{t_0}^s D[F(\varphi(\tau),t)(\eta,\xi) + \langle \eta, Q(t)\xi\rangle]$$

and this means that the function φ : [t0, tk] → Ã is a solution of the equations
(2.8) and (2.7) with

$$\|\varphi(t_0)\|_{\eta\xi} \leq a = \delta_0$$

because $\varphi \in$ A for each tk ∈ [0, T]. By the definition of variational attracting the inequality $\|\varphi(t_0)\|_{\eta\xi} < \varepsilon$ holds for every t > t0 + A(ε). This is of course valid also for the value
t = tk > t0 + A(ε), i.e. $\|\varphi(t_k)\|_{\eta\xi}$ = $\|x_k\|_{\eta\xi} < \varepsilon$ and this contradicts the assumption $\|x_k\|_{\eta\xi} \geq \varepsilon$. This yields the positive definiteness of the real-valued map U. And the result is established.

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