Buoyancy Effects on Thermal Boundary Layer Over a Vertical Plate With a Convective Surface Boundary Condition

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This study aims to analyze the effects of thermal buoyancy on the laminar boundary layer about a vertical plate in a uniform stream of fluid under a convective surface boundary condition. Using a similarity variable, the governing nonlinear partial differential equations have been transformed into a set of coupled nonlinear ordinary differential equations, which are solved numerically by applying shooting iteration technique together with fourth-order Runge-Kutta integration scheme. The variations in dimensionless surface temperature and fluid-solid interface characteristics for different values of Prandtl number (Pr), local Grashof number Grx, and local convective heat transfer parameter Bi are graphed and tabulated. A comparison with previously published results on special case of the problem shows excellent agreement.

Keywords: vertical plate, boundary layer flow, heat transfer, local Grashof number, convective parameter

1 Introduction

Convective heat transfer studies are very important in processes involving high temperatures such as gas turbines, nuclear plants, thermal energy storage, etc. The classical problem (i.e., fluid flow along a horizontal, stationary surface located in a uniform freestream) was solved for the first time in 1908 by Blasius [1]; it is still a subject of current research [2,3] and, moreover, further study regarding this subject can be seen in most recent papers [4,5]. Moreover, Bataller [6] presented a numerical solution for the combined effects of thermal radiation and convective surface heat transfer on the laminar boundary layer about a flat-plate in a uniform stream of fluid (Blasius flow) and about a moving plate in a quiescent ambient fluid (Sakiadis flow). Recently, Aziz [7] investigated a similarity solution for laminar thermal boundary layer over a flat-plate with a convective surface boundary condition. Numerous studies such as Refs. [8–10] considered different variations in temperature and heat flux at the plate; no study appeared to have considered the combined effects of buoyancy force and a convective heat exchange at the plate surface on the boundary layer flow, which is the focus of this paper.

In this present paper, the recent work of Aziz [7] is extended to include the effect of buoyancy force. The numerical solutions of the resulting momentum and the thermal similarity equations are reported for representative values of thermophysical parameters characterizing the fluid convection process.

2 Mathematical Analysis

We consider a two-dimensional steady incompressible fluid flow coupled with heat transfer by convection over a vertical plate. A stream of cold fluid at temperature $T_\infty$ moving over the right surface of the plate with a uniform velocity $U_\infty$ while the left surface of the plate is heated by convection from a hot fluid at temperature $T_f$, which provides a heat transfer coefficient $h_f$ (see Fig. 1). The density variation due to buoyancy effects is taken into account in the momentum equation (Boussinesq approximation). The continuity, momentum, and energy equations describing the flow can be written as

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

\[
\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{\alpha} \frac{\partial T}{\partial y} + g \beta (T - T_\infty)
\]

\[
\frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}
\]

where $u$ and $v$ are the $x$ (along the plate) and the $y$ (normal to the plate) components of the velocities, respectively, $T$ is the temperature, $v$ is the kinematics viscosity of the fluid, and $\alpha$ is the thermal diffusivity of the fluid and $\beta$ is the thermal expansion coefficient.

The velocity boundary conditions at the plate surface and far into the cold fluid may be written as

\[
u(x,0) = v(x,0) = 0
\]

\[
u(x,\infty) = U_\infty
\]

The boundary conditions at the plate surface and far into the cold fluid may be written as

\[-k \frac{\partial T}{\partial y}(x,0) = h_f[T_f(T,x,0) - T_\infty]
\]

\[T(x,\infty) = T_\infty
\]

Introducing a similarity variable $\eta$ and a dimensionless stream function $f(\eta)$ and temperature $\theta(\eta)$ as

\[
\eta = \sqrt{\frac{U_\infty}{U_\infty}} \frac{y}{x} \sqrt{Re_x}, \quad \frac{u}{U_\infty} = f', \quad v = \frac{1}{2} \sqrt{\frac{U_\infty}{x}} (\eta f' - f),
\]

\[
\theta = \frac{T - T_\infty}{T_f - T_\infty}
\]

where prime symbol denotes differentiation with respect to $\eta$ and $Re_x = U_\infty x / v$ is the local Reynolds number. Equations (1)–(7) reduces to

\[
f'' + \frac{1}{2} \beta \frac{ff'' + Gr_x \theta}{Re_x} = 0
\]

\[
\theta' + \frac{1}{2} \frac{Pr}{Re_x} f \theta'' = 0
\]

\[
f(0) = f'(0) = 0, \quad \theta'(0) = -Bi \left[1 - \theta(0)\right]
\]

\[f'(\infty) = 1, \quad \theta(\infty) = 0
\]

where
For the momentum and energy equations to have a similarity solution, the parameters Gr and Bi must be constants and not functions of x as in Eq. (13). This condition can be met if the heat transfer coefficient \( h_x \) is proportional to \( x^{1/2} \) and the thermal expansion coefficient \( \beta \) is proportional to \( x^{-1} \). We therefore assume

\[
h_x = cx^{1/2}, \quad \beta = mx^{-1}
\]

where \( c \) and \( m \) are constants. Substituting Eq. (14) into Eq. (13), we have

\[
Bi = \frac{c}{k} \sqrt{\frac{\nu}{U_w}}, \quad Gr = \frac{\nu g \beta (T_f - T_w)}{U_w^2}
\]

With Bi and Gr defined by Eq. (15), the solutions of Eqs. (9)–(12) yield the similarity solutions, however, the solutions generated are the local similarity solutions whenever \( Bi \) and \( Gr \) are defined as in Eq. (13).

3 Numerical Solutions

The coupled nonlinear Eqs. (9) and (10) with the boundary conditions in Eqs. (11) and (12) are solved numerically using the fourth-order Runge–Kutta method with a shooting technique and implemented on Maple [11]. The step size 0.001 is used to obtain the numerical solution with seven-decimal place accuracy as the criterion of convergence.

4 Results and Discussion

Numerical calculations have carried out for different values of the thermophysical parameters controlling the fluid dynamics in the flow regime. The Prandtl number used are 0.72, 1, 3, and 7.1; the convective parameter \( Bi_a \) used are 0.05, 0.10, 0.20, 0.40, 0.60, 0.80, 1, 5, 10, and 20; and the Grashof number (Grashof number \( Gr \)) used are \( Gr > 0 \) (which corresponds to the cooling problem). The cooling problem is often encountered in engineering applications; for example, in the cooling of electronic components and nuclear reactors. Comparisons of the present results with previously work is performed and excellent agreement has been obtained. We obtained the results as shown in Tables 1 and 2 and Figs. 2–6 below.

Table 1: Computations showing comparison with Aziz [7] results for \( Gr_x = 0 \) and \( Pr = 0.72 \)

<table>
<thead>
<tr>
<th>( Bi )</th>
<th>( \theta(0) ) Aziz [7]</th>
<th>( -\theta(0) ) Aziz [7]</th>
<th>Present</th>
<th>( \theta(0) ) Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.1447</td>
<td>0.0428</td>
<td>0.1447</td>
<td>0.0428</td>
</tr>
<tr>
<td>0.10</td>
<td>0.2528</td>
<td>0.0747</td>
<td>0.2528</td>
<td>0.0747</td>
</tr>
<tr>
<td>0.20</td>
<td>0.4035</td>
<td>0.1193</td>
<td>0.4035</td>
<td>0.1193</td>
</tr>
<tr>
<td>0.40</td>
<td>0.5750</td>
<td>0.1700</td>
<td>0.5750</td>
<td>0.1700</td>
</tr>
<tr>
<td>0.60</td>
<td>0.6699</td>
<td>0.1981</td>
<td>0.6699</td>
<td>0.1981</td>
</tr>
<tr>
<td>0.80</td>
<td>0.7302</td>
<td>0.2159</td>
<td>0.7302</td>
<td>0.2159</td>
</tr>
<tr>
<td>1.00</td>
<td>0.7718</td>
<td>0.2282</td>
<td>0.7118</td>
<td>0.2282</td>
</tr>
<tr>
<td>5.00</td>
<td>0.9441</td>
<td>0.2791</td>
<td>0.9441</td>
<td>0.2791</td>
</tr>
<tr>
<td>10.00</td>
<td>0.9713</td>
<td>0.2871</td>
<td>0.9713</td>
<td>0.2871</td>
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<tr>
<td>20.00</td>
<td>0.9854</td>
<td>0.2913</td>
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<td>0.2913</td>
</tr>
</tbody>
</table>

Table 2: Computations showing \( f'(0), \theta(0), \) and \( -\theta(0) \) for different parameter values

<table>
<thead>
<tr>
<th>( Bi )</th>
<th>( Gr )</th>
<th>( Pr )</th>
<th>( f'(0) )</th>
<th>( -\theta(0) )</th>
<th>( \theta(0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>0.72</td>
<td>0.36881</td>
<td>0.07507</td>
<td>0.24922</td>
</tr>
<tr>
<td>1.0</td>
<td>0.1</td>
<td>0.72</td>
<td>0.44036</td>
<td>0.23750</td>
<td>0.76249</td>
</tr>
<tr>
<td>10</td>
<td>0.1</td>
<td>0.72</td>
<td>0.46792</td>
<td>0.30559</td>
<td>0.96944</td>
</tr>
<tr>
<td>0.1</td>
<td>0.5</td>
<td>0.72</td>
<td>0.49702</td>
<td>0.07613</td>
<td>0.23862</td>
</tr>
<tr>
<td>0.1</td>
<td>1.0</td>
<td>0.72</td>
<td>0.63200</td>
<td>0.07044</td>
<td>0.22955</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>3.00</td>
<td>0.34939</td>
<td>0.08304</td>
<td>0.16954</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>7.10</td>
<td>0.34270</td>
<td>0.08672</td>
<td>0.13278</td>
</tr>
</tbody>
</table>

Fig. 2: Velocity profiles for \( Pr = 0.72 \), \( Gr_x = 0.1 \)
sion while an increase in the Prandtl number and the intensity of buoyancy force slows down the rate of thermal diffusion within the boundary layer.

5 Conclusion

Analysis has been carried out to study the boundary layer flow over a vertical plate with a convective surface boundary condition. A similarity solution for the momentum and the thermal boundary layer equations is possible if the convective heat transfer of the fluid heating the plate on its left surface is proportional to $x^{-1/2}$ and the thermal expansion coefficient $\beta$ is proportional to $x^{-1}$. The numerical solutions of the similarity equations were reported for the various parameters embedded in the problem. The combined effects of increasing the Prandtl number and the Grashof number tends to reduce the thermal boundary layer thickness along the plate.

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Nomenclature

- $(x,y)$ = Cartesian coordinates
- $(u,v)$ = velocity components
- $T_e$ = freestream temperature
- $T_f$ = hot fluid temperature
- $g$ = gravitational acceleration
- $T$ = fluid temperature
- $Pr$ = Prandtl number
- $U_e$ = freestream velocity
- $Gr_e$ = local Grashof number
- $Bi_k$ = local convective heat transfer parameter
- $k$ = thermal conductivity

Greek Symbols

- $\alpha$ = thermal diffusivity of the fluid
- $\beta$ = thermal expansion coefficient
- $\nu$ = kinematic viscosity
References