Buoyancy Effects on Thermal Boundary Layer Over a Vertical Plate With a Convective Surface Boundary Condition

O. D. Makinde[1]
Senior Professor and Chair of Postgraduate Studies
e-mail: makinded@cput.ac.za

P. O. Olanrewaju
e-mail: oldapo_anu@yahoo.ie

Faculty of Engineering,
Cape Peninsula University of Technology,
P.O. Box 1906,
Bellville 7535, South Africa

This study aims to analyze the effects of thermal buoyancy on the laminar boundary layer about a vertical plate in a uniform stream of fluid under a convective surface boundary condition. Using a similarity variable, the governing nonlinear partial differential equations have been transformed into a set of coupled nonlinear ordinary differential equations, which are solved numerically by applying shooting iteration technique together with fourth-order Runge–Kutta integration scheme. The variations in dimensionless surface temperature and fluid-solid interface characteristics for different values of Prandtl number (Pr), local Grashof number Gr_x, and local convective heat transfer parameter Bi_x are graphed and tabulated. A comparison with previously published results on special case of the problem shows excellent agreement.
[DOI: 10.1115/1.4001386]

Keywords: vertical plate, boundary layer flow, heat transfer, local Grashof number, convective parameter

1 Introduction

Convective heat transfer studies are very important in processes involving high temperatures such as gas turbines, nuclear plants, thermal energy storage, etc. The classical problem (i.e., fluid flow along a horizontal, stationary surface located in a uniform freestream) was solved for the first time in 1908 by Blasius [1]; it is still a subject of current research [2,3] and, moreover, further study regarding this subject can be seen in most recent papers [4,5]. Moreover, Bataller [6] presented a numerical solution for the combined effects of thermal radiation and convective surface heat transfer on the laminar boundary layer about a flat-plate in a uniform stream of fluid (Blasius flow) and about a moving plate in a quiescent ambient fluid (Sakiadis flow). Recently, Aziz [7] investigated a similarity solution for laminar thermal boundary layer over a flat-plate with a convective surface boundary condition. Numerous studies such as Refs. [8–10] considered different variations in temperature and heat flux at the plate; no study appeared to have considered the combined effects of buoyancy force and a convective heat exchange at the plate surface on the boundary layer flow, which is the focus of this paper.

In this present paper, the recent work of Aziz [7] is extended to include the effect of buoyancy force. The numerical solutions of the resulting momentum and the thermal similarity equations are reported for representative values of thermophysical parameters characterizing the fluid convection process.

2 Mathematical Analysis

We consider a two-dimensional steady incompressible fluid flow coupled with heat transfer by convection over a vertical plate. A stream of cold fluid at temperature T_∞ moving over the right surface of the plate with a uniform velocity U_∞ while the left surface of the plate is heated by convection from a hot fluid at temperature T_f, which provides a heat transfer coefficient h_f (see Fig. 1). The density variation due to buoyancy effects is taken into account in the momentum equation (Boussinesq approximation). The continuity, momentum, and energy equations describing the flow can be written as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2}+g\beta(T-T_\infty) \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2} \tag{3}$$

where u and v are the x (along the plate) and the y (normal to the plate) components of the velocities, respectively, T is the temperature, υ is the kinematics viscosity of the fluid, and α is the thermal diffusivity of the fluid and β is the thermal expansion coefficient. The velocity boundary conditions can be expressed as

$$u(x,0)=v(x,0)=0 \tag{4}$$

$$u(x,\infty)=U_\infty \tag{5}$$

The boundary conditions at the plate surface and far into the cold fluid may be written as

$$-k\frac{\partial T}{\partial y}(x,0)=h_f[T_f-T(x,0)] \tag{6}$$

$$T(x,\infty)=T_\infty \tag{7}$$

Introducing a similarity variable η and a dimensionless stream function $f(\eta)$ and temperature $\theta(\eta)$ as

$$\eta=y\sqrt{\frac{U_\infty}{\upsilon x}}=\frac{y}{x}\sqrt{\mathrm{Re}_x},\quad \frac{u}{U_\infty}=f',\quad v=\frac{1}{2}\sqrt{\frac{U_\infty\upsilon}{x}}(\eta f'-f),$$

$$\theta=\frac{T-T_\infty}{T_f-T_\infty} \tag{8}$$

where prime symbol denotes differentiation with respect to η and $\mathrm{Re}_x=U_\infty x/\upsilon$ is the local Reynolds number. Equations (1)–(7) reduces to

$$f'''+\frac{1}{2}ff''+\mathrm{Gr}_x\theta=0 \tag{9}$$

$$\theta''+\frac{1}{2}\mathrm{Pr}\, f\theta'=0 \tag{10}$$

$$f(0)=f'(0)=0,\quad \theta'(0)=-\mathrm{Bi}_x[1-\theta(0)] \tag{11}$$

$$f'(\infty)=1,\quad \theta(\infty)=0 \tag{12}$$

where

[1]Corresponding author.

Contributed by the Fluids Engineering Division of ASME for publication in the JOURNAL OF FLUIDS ENGINEERING. Manuscript received December 23, 2009; final manuscript received January 30, 2010; published online April 16, 2010. Assoc. Editor: Paul Durbin.

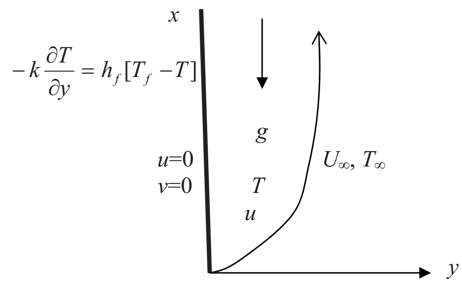

Fig. 1 Flow configuration and coordinate system

$$\mathrm{Bi}_x = \frac{h_f}{k}\sqrt{\frac{\upsilon x}{U_\infty}}, \quad \mathrm{Pr} = \frac{\upsilon}{\alpha}, \quad \mathrm{Gr}_x = \frac{\upsilon x g\beta(T_f - T_\infty)}{U_\infty^2} \tag{13}$$

For the momentum and energy equations to have a similarity solution, the parameters Gr_x and Bi_x must be constants and not functions of x as in Eq. (13). This condition can be met if the heat transfer coefficient h_f is proportional to $x^{-1/2}$ and the thermal expansion coefficient β is proportional to x^{-1}. We therefore assume

$$h_f = cx^{-1/2}, \quad \beta = mx^{-1} \tag{14}$$

where c and m are constants. Substituting Eq. (14) into Eq. (13), we have

$$\mathrm{Bi} = \frac{c}{k}\sqrt{\frac{\upsilon}{U_\infty}}, \quad \mathrm{Gr} = \frac{\upsilon m g(T_f - T_\infty)}{U_\infty^2} \tag{15}$$

With Bi and Gr defined by Eq. (15), the solutions of Eqs. (9)–(12) yield the similarity solutions, however, the solutions generated are the local similarity solutions whenever Bi_x and Gr_x are defined as in Eq. (13).

3 Numerical Solutions

The coupled nonlinear Eqs. (9) and (10) with the boundary conditions in Eqs. (11) and (12) are solved numerically using the fourth-order Runge–Kutta method with a shooting technique and implemented on Maple [11]. The step size 0.001 is used to obtain the numerical solution with seven-decimal place accuracy as the criterion of convergence.

4 Results and Discussion

Numerical calculations have carried out for different values of the thermophysical parameters controlling the fluid dynamics in the flow regime. The Prandtl number used are 0.72, 1, 3, and 7.1; the convective parameter Bi_x used are 0.05, 0.10, 0.20, 0.40, 0.60, 0.80, 1, 5, 10, and 20; and the Grashof number (Grashof number Gr_x) used are $\mathrm{Gr} > 0$ (which corresponds to the cooling problem). The cooling problem is often encountered in engineering applications; for example, in the cooling of electronic components and nuclear reactors. Comparisons of the present results with previously work is performed and excellent agreement has been obtained. We obtained the results as shown in Tables 1 and 2 and Figs. 2–6 below.

Table 1 shows the comparison of Aziz [7] work with the present work for Prandtl number ($\mathrm{Pr}=0.72$) and it is noteworthy that there is a perfect agreement in the absence of Grashof number. Table 2, illustrates the values of the skin-friction coefficient and the local Nusselt number in terms of $f''(0)$ and $-\theta'(0)$, respectively, for various values of embedded parameters. From Table 2, it is understood that the skin-friction and the rate of heat transfer at the plate surface increases with an increase in local Grashof number and convective surface heat transfer parameter. However, an increase in the fluid Prandtl number decreases the skin-friction but increases the rate of heat transfer at the plate surface. Figures 2 and 3 depict the fluid velocity profiles. Generally, the fluid velocity is zero at the plate surface and increases gradually away from the plate toward the freestream value satisfying the boundary conditions. It is interesting to note that an increase in the intensity of convective surface heat transfer (Bi_x) produces a slight increase in the fluid velocity within the boundary layer. Similar trend is observed with an increase in local Grashof number due to buoyancy effects. Moreover, the effect of local Grashof number on the velocity profiles (see Fig. 3) is more pronounced than the effect of convection parameter (Bi_x) (see Fig. 2). Figures 4–6 illustrate the fluid temperature profiles within the boundary layer. The fluid temperature is maximum at the plate surface and decreases exponentially to zero value far away from the plate satisfying the boundary conditions. From this figures, it is noteworthy that the thermal boundary layer thickness increases with an increase in Bi_x and decreases with and increase in the values of Gr_x and Pr. Hence, convective surface heat transfer enhances thermal diffu-

Table 1 Computations showing comparison with Aziz [7] results for $\mathrm{Gr}_x=0$ and $\mathrm{Pr}=0.72$

Bi_x	$\theta(0)$ Aziz [7]	$-\theta'(0)$ Aziz [7]	$\theta(0)$ Present	$-\theta'(0)$ Present
0.05	0.1447	0.0428	0.1447	0.0428
0.10	0.2528	0.0747	0.2528	0.0747
0.20	0.4035	0.1193	0.4035	0.1193
0.40	0.5750	0.1700	0.5750	0.1700
0.60	0.6699	0.1981	0.6699	0.1981
0.80	0.7302	0.2159	0.7302	0.2159
1.00	0.7718	0.2282	0.7718	0.2282
5.00	0.9441	0.2791	0.9442	0.2791
10.00	0.9713	0.2871	0.9713	0.2871
20.00	0.9854	0.2913	0.9854	0.2913

Table 2 Computation showing $f''(0)$, $\theta(0)$, and $\theta'(0)$ for different parameter values

Bi_x	Gr_x	Pr	$f''(0)$	$-\theta'(0)$	$\theta(0)$
0.1	0.1	0.72	0.36881	0.07507	0.24922
1.0	0.1	0.72	0.44036	0.23750	0.76249
10	0.1	0.72	0.46792	0.30559	0.96944
0.1	0.5	0.72	0.49702	0.07613	0.23862
0.1	1.0	0.72	0.63200	0.07704	0.22955
0.1	0.1	3.00	0.34939	0.08304	0.16954
0.1	0.1	7.10	0.34270	0.08672	0.13278

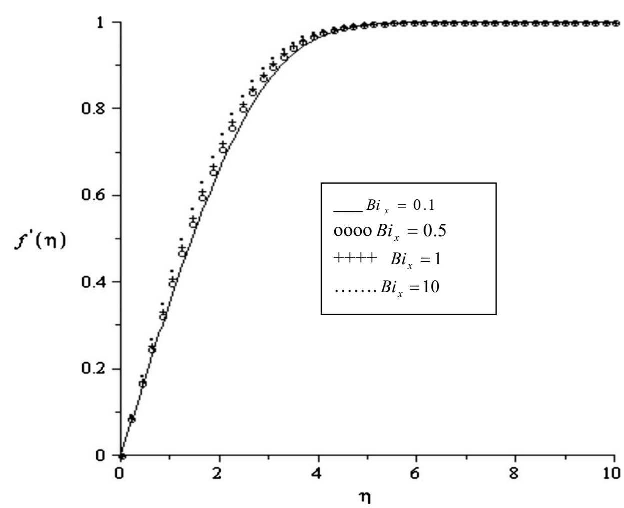

Fig. 2 Velocity profiles for $\mathrm{Pr}=0.72$, $\mathrm{Gr}_x=0.1$

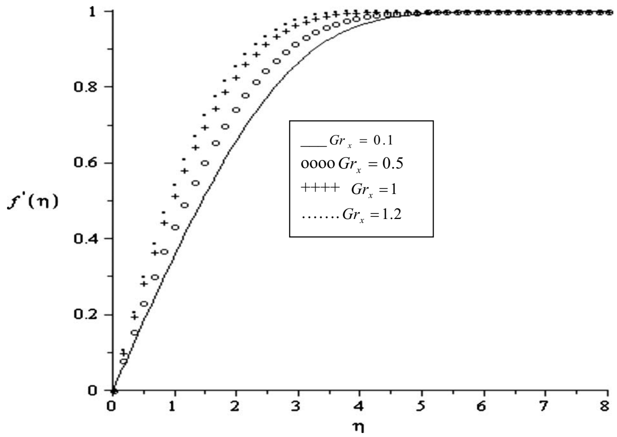

Fig. 3 Velocity profiles for Pr=0.72, Bi_x=0.1

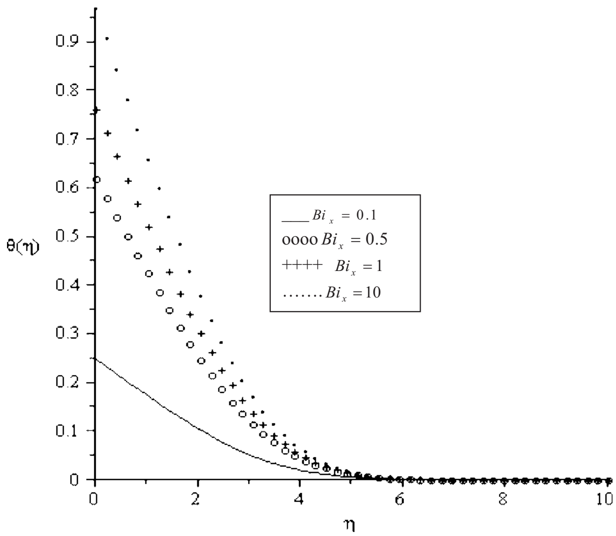

Fig. 4 Temperature profiles for Pr=0.72, Gr_x=0.1

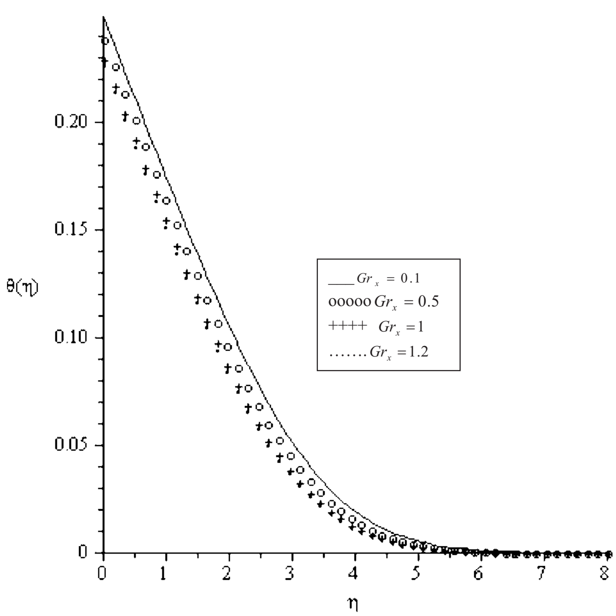

Fig. 5 Temperature profiles for Pr=0.72 and Bi_x=0.1

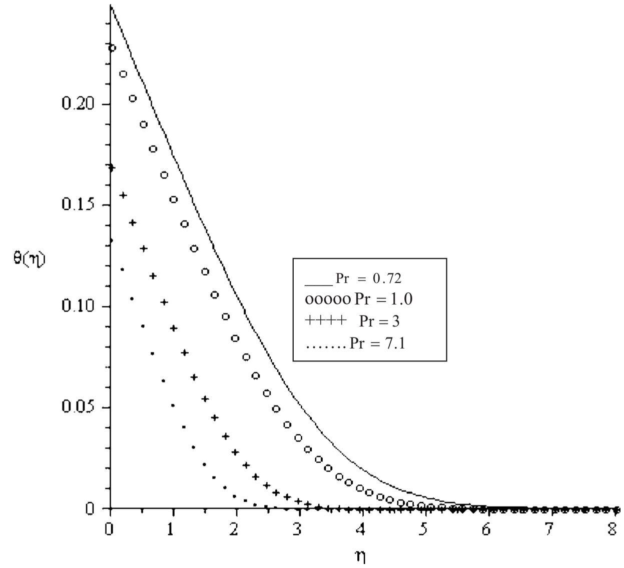

Fig. 6 Temperature profiles for Gr_x=0.1 and Bi_x=0.1

sion while an increase in the Prandtl number and the intensity of buoyancy force slows down the rate of thermal diffusion within the boundary layer.

5 Conclusion

Analysis has been carried out to study the boundary layer flow over a vertical plate with a convective surface boundary condition. A similarity solution for the momentum and the thermal boundary layer equations is possible if the convective heat transfer of the fluid heating the plate on its left surface is proportional to $x^{-1/2}$ and the thermal expansion coefficient β is proportional to x^{-1}. The numerical solutions of the similarity equations were reported for the various parameters embedded in the problem. The combined effects of increasing the Prandtl number and the Grashof number tends to reduce the thermal boundary layer thickness along the plate.

Acknowledgment

O.D.M. would like to thank the National Research Foundation (NRF) Thuthuka program for financial support. Dr. Olanrewaju of Convent University, Nigeria visited CPUT South Africa on Post-doctoral studies.

Nomenclature

(x,y) = Cartesian coordinates
(u,v) = velocity components
T_∞ = freestream temperature
T_f = hot fluid temperature
g = gravitational acceleration
T = fluid temperature
Pr = Prandtl number
U_∞ = freestream velocity
Gr_x = local Grashof number
Bi_x = local convective heat transfer parameter
k = thermal conductivity

Greek Symbols

α = thermal diffusivity of the fluid
β = thermal expansion coefficient
υ = kinematic viscosity

References

[1] Blasius, H., 1908, "Grenzschichten in Flussigkeiten mit kleiner reibung," Z. Math Phys., **56**, pp. 1–37.
[2] Weyl, H., 1942, "On the Differential Equations of the Simplest Boundary-Layer Problem," Ann. Math., **43**, pp. 381–407.
[3] Magyari, E., 2008, "The Moving Plate Thermometer," Int. J. Therm. Sci., **47**, pp. 1436–1441.
[4] Cortell, R., 2005, "Numerical Solutions of the Classical Blasius Flat-Plate Problem," Appl. Math. Comput., **170**, pp. 706–710.
[5] He, J. H., 2003, "A Simple Perturbation Approach to Blasius Equation," Appl. Math. Comput., **140**, pp. 217–222.
[6] Bataller, R. C., 2008, "Radiation Effects for the Blasius and Sakiadis Flows With a Convective Surface Boundary Condition," Appl. Math. Comput., **206**, pp. 832–840.
[7] Aziz, A., 2009, "A Similarity Solution for Laminar Thermal Boundary Layer Over a Flat Plate With a Convective Surface Boundary Condition," Commun. Nonlinear Sci. Numer. Simul., **14**, pp. 1064–1068.
[8] Makinde, O. D., and Sibanda, P., 2008, "Magnetohydrodynamic Mixed Convective Flow and Heat and Mass Transfer Past a Vertical Plate in a Porous Medium With Constant Wall Suction," ASME J. Heat Transfer, **130**, p. 112602.
[9] Makinde, O. D., 2009, "Analysis of Non-Newtonian Reactive Flow in a Cylindrical Pipe," ASME J. Appl. Mech., **76**, p. 034502.
[10] Cortell, R., 2008, "Similarity Solutions for Flow and Heat Transfer of a Quiescent Fluid Over a Nonlinearly Stretching Surface," J. Mater. Process. Technol., **2003**, pp. 176–183.
[11] Heck, A., 2003, *Introduction to Maple*, 3rd ed., Springer-Verlag, New York.